INTRODUCTION TO ANALYSIS LECTURE NOTES

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SET THEORY

Set — a collection of objects = elements (or **members** of the set). It is a basic concept and we will accept it without rigorous definition. If m is a member of the set A, write $m \in A$ (m **belongs** to A). If is not a member of the set A, write $m \notin A$.

Examples.

(1) Number sets
   (a) $\mathbb{N}$ — natural numbers, $1, 2, 3, \ldots$. Our convention: $0 \notin \mathbb{N}$
   (b) $\mathbb{Z}$ — integers, $\ldots, -2, -1, 0, 1, 2, \ldots$
   (c) $\mathbb{Q}$ — rationals, numbers of the form $\frac{m}{n}$ with $m \in \mathbb{Z}$, $n \in \mathbb{N}$
   (d) $\mathbb{R}$ — real numbers. How do you define them?
(2) Intervals: $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$, $(-\infty, 5] = \{x \in \mathbb{R} : x \leq 5\}$
(3) Empty set: $\emptyset$ — the set with no elements
(4) Finite sets: have finitely many elements, e.g. $A = \{1, 2, x\}$
(5) Sets containing sets: $A = \{\{1, 2\}, \{3\}, 2\}$, $1 \notin A$, $3 \notin A$, $\{3\} \in A$.

Subsets.

Definition 1. *A is a* **subset** *of B if every element of A is an element of B*

Example. *$A \subseteq A$ for any set A ($\forall A$); $\emptyset \subseteq A$ for any set A ($\forall A$); $\{1, 2, 3\} \subseteq \mathbb{N} \subseteq \mathbb{Q}$.*

Definition 2. *Two sets, A and B, are* **equal** *if $A \subseteq B$ and $B \subseteq A$. In this case A and B have exactly the same elements.*

Example. *Let $A := \{x \in \mathbb{Z} : x > 0\}$ and $B := \mathbb{N}$. Then $A = B$.*

Definition 3. *A is a* **proper subset of** *B if $A \subseteq B$ and $A \neq B$. Write $A \subsetneq B$.*

Example. *$\mathbb{N} \subsetneq \mathbb{Z}$; $\mathbb{Z} \subsetneq \mathbb{Q}$.*

Set operations

(1) **Intersection**: $A \cap B = \{x : x \in A \text{ and } x \in B\}$. If $A \cap B = \emptyset$ we say that A and B are **disjoint**.

   Example. $\{-1, 3\} \cap \mathbb{N} = \{3\}$; $\{0, 1, 2, 3\} \cap \{\{0\}\} = \emptyset$

(2) **Union**: $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

   Example. $\{\ldots, -2, -1, 0\} \cup \mathbb{N} = \mathbb{Z}$; $\{0, 1, 2, 3\} \cup \{\{0\}\} = \{\{0\}, 0, 1, 2, 3\}$

(3) **Complement**: the **complement of** A **in** B is $B\backslash A = \{x \in B : x \notin A\}$.

Example. $\mathbb{Z}\backslash\mathbb{N} = \{\ldots, -2, -1, 0\}$

If $B = U$ — a **universal set** that contains all sets of current interest, we write $A^c := U\backslash A$.

Example. $[0,1)^c = \mathbb{R}\backslash[0,1) = (-\infty, 0) \cup [1, \infty)$

Examples.

a) Prove that for any sets A, B, C we have $(A \cap B) \cap C = A \cap B \cap C$

Proof. We need to show: $(A \cap B) \cap C \subseteq A \cap B \cap C$ and $(A \cap B) \cap C \supseteq A \cap B \cap C$.
1. Let $x \in (A \cap B) \cap C \Rightarrow x \in (A \cap B)$ and $x \in C \Rightarrow x \in A$ and $x \in B$, and $x \in C \Rightarrow x \in A \cap B \cap C \Rightarrow (A \cap B) \cap C \subseteq A \cap B \cap C$.
2. Let $x \in A \cap B \cap C \Rightarrow x \in A$ and $x \in B$ and $x \in C \Rightarrow x \in (A \cap B)$ and $x \in C$ $\Rightarrow x \in (A \cap B) \cap C \Rightarrow A \cap B \cap C \subseteq (A \cap B) \cap C$.
1.,2. $\Rightarrow (A \cap B) \cap C = A \cap B \cap C$. □

b) Prove that for any sets A, B, C we have $C\backslash(A \cap B) = (C\backslash A) \cup (C\backslash B)$

Proof. 1. Let $x \in C\backslash(A \cap B) \Rightarrow x \in C$ and $x \notin A \cap B \Rightarrow x \in C$ and x not in both A and $B \Rightarrow x \in C$ and ($x \notin A$ or $x \notin B$) $\Rightarrow$ ($x \in C$ and $x \notin A$) or ($x \in C$ and $x \notin B$) $\Rightarrow x \in (C\backslash A) \cup (C\backslash B) \Rightarrow C\backslash(A \cap B) \subseteq (C\backslash A) \cup (C\backslash B)$
2. Let $x \in (C\backslash A) \cup (C\backslash B) \Rightarrow$ ($x \in C$ and $x \notin A$) or ($x \in C$ and $x \notin B$) $\Rightarrow$ $x \in C$ and ($x \notin A$ or $x \notin B$) $\Rightarrow x \in C$ and x not in both A and $B \Rightarrow x \in C$ and $x \notin A \cap B \Rightarrow x \in C\backslash(A \cap B) \Rightarrow (C\backslash A) \cup (C\backslash B) \subseteq C\backslash(A \cap B)$ □

(4) **Intersections and unions of arbitrary families**
Let $I \subseteq \mathbb{R}$ be an index set, then

$$\bigcap_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for all } \alpha \in I\}; \quad \bigcup_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for some } \alpha \in I\}$$

If $I = \mathbb{N}$ write $\cap_{k=1}^{\infty} A_k,\ \cup_{k=1}^{\infty} A_k$

Examples.

a) Let $A_k = [k, \infty),\ k \in \mathbb{N}$. Prove that $\cap_{k \in \mathbb{N}} A_k = \emptyset$ and $\cup_{k \in \mathbb{N}} A_k = [1, \infty)$

Proof. We will show $\cap_{k \in \mathbb{N}} A_k = \emptyset$, the second part is an **exercise**.
1. For any set A we have $\emptyset \subseteq A$, thus $\emptyset \subseteq \cap_{k \in \mathbb{N}} A_k$
2. Let $x \in \cap_{k \in \mathbb{N}} A_k \Rightarrow x \in A_k\ \forall k \in \mathbb{N} \Rightarrow x \in [k, \infty)\ \forall k \in \mathbb{N} \Rightarrow x \geq k\ \forall k \in \mathbb{N}$ which is impossible for any $x \in \mathbb{R}$, thus $x \in \emptyset \Rightarrow \cap_{k \in \mathbb{N}} A_k \subseteq \emptyset$. □

b) Let $B_r = [0, |r|],\ r \in \mathbb{R}$ (note that in this case the index set $I = \mathbb{R}$). Prove that $\cap_{r \in \mathbb{R}} B_r = \{0\}$ and $\cup_{r \in \mathbb{R}} B_r = [0, \infty)$

Proof. We will show $\cup_{r\in\mathbb{R}} B_r = [0,\infty)$, the first part is an **exercise**.
1. Let $x \in \cup_{r\in\mathbb{R}} B_r \Rightarrow \exists\ r \in \mathbb{R}$ such that $x \in B_r \Rightarrow \exists\ r \in \mathbb{R}$ such that $x \in [0,|r|]$ $\Rightarrow \exists\ r \in \mathbb{R}$ such that $0 \le x \le |r| \Rightarrow x \ge 0 \Rightarrow x \in [0,\infty)$.
2. Let $x \in [0,\infty) \Rightarrow x \ge 0$. Since we always have $x \le |x|$ (we will prove this fact later) this gives $0 \le x \le |x| \Rightarrow x \in [0,|x|] \Rightarrow x \in B_x \Rightarrow x \in B_r$ for some $r \in \mathbb{R}$ (namely, $r = x$) $\Rightarrow x \in \cup_{r\in\mathbb{R}} B_r$. □

Cartesian product

Let (a,b) denote an **ordered pair**, i.e. $(a,b) = (c,d)$ iff (= if and only if) $a = c$ and $b = d$.

Remark 4. $\{a,b\} = \{b,a\} \Rightarrow$ *the set* $\{a,b\}$ *is* **not ordered**

Definition 5. *For any sets, A and B, the* **Cartesian product** $A \times B$ *is the set*

$$A \times B := \{(a,b) : a \in A, b \in B\}$$

of all ordered pairs where the first component varies over all elements of A, and the second — over all elements of B.

Definition 6. n-fold Cartesian product *of n sets $A_1, A_2 \ldots, A_n$ is the set of all ordered n-tuples*

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1,\ldots,a_n) : a_k \in A_k,\ \forall k = 1,\ldots,n\}.$$

Exercise 7. *Prove that $A \times B \times C$ is equivalent to $(A \times B) \times C$ and $A \times (B \times C)$. More precisely, two elements in $A \times B \times C$ are equal iff the corresponding elements of $(A \times B) \times C$ are equal. Similarly with $A \times (B \times C)$.*

Example 1. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}\}$ *— the real (Cartesian) plane* $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$

Example 2. $A = \{1,2,3,4\}$, $B = \{a,b,c\}$ *then*

$$A \times B = \{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c),(3,a),(3,b),(3,c),(4,a),(4,b),(4,c)\},$$

and

$$B \times A = \{(a,1),(a,2),(a,3),(a,4),(b,1),(b,2),(b,3),(b,4),(c,1),(c,2),(c,3),(c,4)\}.$$

Note $A \times B \neq B \times A$!

Example 3. *Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$*

Proof. 1. Let $(x,y) \in A \times (B \cup C) \Rightarrow x \in A$ and $y \in B \cup C \Rightarrow x \in A$ and $y \in B$ or $C \Rightarrow (x \in A$ and $y \in B)$ or $(x \in A$ and $y \in C) \Rightarrow (x,y) \in A \times B$ or $(x,y) \in A \times C$ $\Rightarrow (x,y) \in (A \times B) \cup (A \times C)$
2. Let $(x,y) \in (A \times B) \cup (A \times C) \Rightarrow (x,y) \in A \times B$ or $(x,y) \in A \times C \Rightarrow (x \in A$ and $y \in B)$ or $(x \in A$ and $y \in C) \Rightarrow x \in A$ and $y \in B$ or $C \Rightarrow x \in A$ and $y \in B \cup C \Rightarrow$ $(x,y) \in A \times (B \cup C)$ □

Equivalence relations

Intuition: Consider two real numbers, 2 and 4. If we were asked "what is the relation between them"? One natural response would be that the second number is twice the first number. And there are a lot of other real numbers satisfying this condition: 3 and 6, $-\pi$ and -2π, 2.2 and 4.4, etc. So we might want to define a "relation" on $\mathbb{R}$ by saying aRb if $b = 2a$. Note that all the points on the line $y = 2x$ in $\mathbb{R}^2$ satisfy this relation. Here is the formal definition

Definition 8. *A (binary)* ***relation*** *on a set A is a subset $R \subseteq A \times A$. If $(a, b) \in R$ write aRb.*

Examples.

(1) $A = \mathbb{R}$, $R := \{(x, y) \in \mathbb{R}^2 : x \leq y\}$. $(x, y) \in R \iff x \leq y$.
(2) $A = \mathbb{N}$, $R := \{(n, m) \in \mathbb{N}^2 : n^2 < m + 1\}$.
(3) Given a set S, the *power set* of S is the set $\mathcal{P}(S)$ of all subsets of S:
$$\mathcal{P}(S) = \{B \colon B \subseteq S\}.$$
Define the relation R on $\mathcal{P}(S)$ by ARB if $A \cap B = \emptyset$. Note that this is a subset of $\mathcal{P}(S) \times \mathcal{P}(S)$ consisting of all pairs of disjoint subsets of S.

Definition 9. *A relation R on a set A is called an* **equivalence relation** *(write $a \sim b$ if $(a, b) \in R$) if $\forall a, b, c \in A$:*

(1) $a \sim a$ — reflexivity
(2) $a \sim b \Longrightarrow b \sim a$ — symmetry
(3) $a \sim b$, $b \sim c \Longrightarrow a \sim c$ — transitivity

Examples of equivalence relations.

(1) $A = \mathbb{Z}$, $R = \{(a, b) \in \mathbb{Z} \colon a = b\}$ or $a \sim b$ if $a = b$
(2) $A = \mathbb{Z}$, $a \sim b$ if $a - b$ is divisible by 3
(3) $A = \mathbb{R} \times \mathbb{R} \backslash \{0\}$, $(a, b) \sim (c, d)$ if $\frac{a}{b} = \frac{c}{d}$. Reflexivity: $(a, b) \sim (a, b)$ since $\frac{a}{b} = \frac{a}{b}$; symmetry: if $(a, b) \sim (c, d)$ then $\frac{a}{b} = \frac{c}{d}$, which implies $\frac{c}{d} = \frac{a}{b}$ which implies $(c, d) \sim (a, b)$; transitivity: if $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$, then $\frac{a}{b} = \frac{c}{d}$ and $\frac{c}{d} = \frac{e}{f} \Rightarrow \frac{a}{b} = \frac{e}{f} \Rightarrow (a, b) \sim (e, f)$.
Exercise: write this relation as a subset of $\mathbb{R}^4$.
(4) $A = \mathbb{S}^2$, a sphere in $\mathbb{R}^3$, $a \sim b$ if $a = b$ or a and b are antipodal

Non-examples.

(1) $A = \mathbb{R}^2$, $(a, b)R(c, d)$ if $ac = -bd$: symmetric, but not reflexive and not transitive
(2) $A = \mathbb{Z}$, aRb if $a = 2b$: all three properties fail
(3) The relation on the power set $\mathcal{P}(S)$ given above is symmetric, but not reflexive and not transitive

Equivalence classes. Let A be any set, and $\sim$ an equivalence relation on it.

Definition 10. *The* ***equivalence class*** *of $a \in A$ is the set*

$$[a] := \{b \in A : b \sim a\}$$

of all elements $b \in A$ s.t. $b \sim a$.

Example. *$A = \mathbb{R} \times \mathbb{R} \setminus \{0\}$. Define $\sim$ by $(a, b) \sim (c, d)$ if $\frac{a}{b} = \frac{c}{d}$ then*

$$[(1, 2)] = \{(x, y) \in \mathbb{R} \times \mathbb{R} \setminus \{0\} : y = 2x\} \subseteq \mathbb{R}^2.$$

Theorem 11. *For any $a, b \in A$ either $[a] = [b]$ or $[a] \cap [b] = \emptyset$.*

Proof. Let $a, b \in A$ and suppose $\exists c \in [a] \cap [b]$. We want to show the equality of sets $[a] = [b]$.
1. Let $x \in [a] \Rightarrow x \sim a$. Since $c \in [a]$ we also have $c \sim a$ and by symmetry $a \sim c$. By transitivity from $x \sim a$ and $a \sim c$ we obtain $x \sim c$. Similarly, since $c \in [b] \Rightarrow c \sim b$ $\Rightarrow$(by transitivity again) $x \sim b \Rightarrow x \in [b]$.
2. Now assume $x \in [b]$. In the same way as in part 1. we can show that $x \in [a]$.
1.,2. $\Rightarrow [a] = [b]$ provided $[a] \cap [b] \neq \emptyset$ □

Definition 12. ***Partition*** *of A is the collection of all distinct equivalence classes, i.e. A can be written as a union $A = \cup_{\alpha \in I} A_\alpha$ of pairwise disjoint equivalence classes of A.*

Theorem 13. *If $A = \cup_{\alpha \in I} A_\alpha$ where A_α's are pairwise disjoint, then $R = \{(a, b) \in A \times A : a, b \in A_\alpha$ for some $\alpha \in I\}$ defines an equivalence relation on A.*

Proof. **Exercise**. □

FUNCTIONS

A **function** $f : A \to B$ is a rule that assigns to each element of A a unique element of B. The set A is the **domain** of f, and B is the **co-domain** of f.

Definition 1. *Let $f : A \to B$, the* **range** *of f is the following subset of B*

$$R(f) = \{b \in B : \exists a \in A \text{ s.t. } b = f(a)\}$$

Also denoted as $f(A)$.

Examples.

(1) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$. $R(f) = [-1, 1]$
(2) $f : \mathbb{N} \to \mathbb{Z}$

$$f(n) = \begin{cases} \frac{n-1}{2}, & \text{if n is odd,} \\ -\frac{n}{2}, & \text{if n is even.} \end{cases}$$

(3) Identity function: $id_A : A \to A$, $id_A(x) = x \ \forall x \in A$.
(4) Indicator function. Let $B \subseteq A$, $\chi_B : A \to \mathbb{R}$ is defined by

$$\chi_B(x) = \begin{cases} 1, & \text{if } x \in B, \\ 0, & \text{if } x \notin B. \end{cases}$$

Definition 2. *Two functions f and g are* **equal** *if they have the same domain, same co-domain, and $f(x) = g(x)$ for every point x in the domain.*

Example.

(1) Define

$$f : \mathbb{R} \to \mathbb{R}, \ f(x) = x^2$$
$$g : \mathbb{R} \to [0, \infty), \ g(x) = x^2$$

Then $f \neq g$ since they have different co-domains
(2) Define

$$f : \mathbb{R} \to \mathbb{R}, \ f(x) = |x|$$
$$g : \mathbb{R} \to \mathbb{R}, \ g(x) = \sqrt{x^2}$$

Then $f = g$ since they have the same domain, same co-domain, and $\sqrt{x^2} = |x|$ $\forall x \in \mathbb{R}$.

Functions acting on sets

Let $f : A \to B$, and $E \subseteq A$. The **image of E under f** is the following subset of B

$$f(E) := \{f(e) : e \in E\} \subseteq B.$$

Let $D \subseteq B$. The **inverse image (or preimage) of D under f** is the following subset of A

$$f^{-1}(D) = \{a \in A : f(a) \in D\} \subseteq A.$$

Remark 3. *$x \in f^{-1}(D)$ iff $f(x) \in D$.*

Remark 4. *The notation $f^{-1}(set)$ does not imply that f has an inverse.*

Examples.

(1) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.

$$f([0,2]) = [0,4], \quad f^{-1}([-1,3]) = [-\sqrt{3}, \sqrt{3}], \quad f^{-1}(\{1\}) = \{-1,1\}$$

(2) Let $f : A \to B$ and $E \subseteq A$. Then $E \subseteq f^{-1}(f(E))$.
Proof. By definition $f^{-1}(f(E)) = \{x \in A : f(x) \in f(E)\}$. Let $e \in E$, this implies $f(e) \in f(E) \Rightarrow e \in f^{-1}(f(E))$. □

(3) Back to the first example: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.

$$f([0,\infty)) = [0,\infty), \text{ and } f^{-1}([0,\infty)) = \mathbb{R} \;\Rightarrow\; f^{-1}(f([0,\infty))) = \mathbb{R} \neq [0,\infty)$$

In fact we have $[0,\infty) \subsetneq \mathbb{R}$.

Composition of functions

Let A, B, C, D be any sets such that $B \subseteq C$. And let

$$f : A \to B, \quad \text{and} \quad g : C \to D.$$

The **composition** of g and f is the function $g \circ f : A \to D$ defined by $g \circ f(x) = g(f(x))$

Examples.

(1) Let

$$f : \mathbb{R} \to \mathbb{R}, \; f(x) = (x+1)^2$$
$$g : \mathbb{R} \to \mathbb{R}, \; g(x) = x^3 - 1$$

Then

$$g \circ f(x) = g(f(x)) = g((x+1)^2) = \left[(x+1)^2\right]^3 - 1 = (x+1)^6 - 1$$
$$f \circ g(x) = f(g(x)) = f(x^3 - 1) = (x^3 - 1 + 1)^2 = x^6.$$

Note that $g \circ f \neq f \circ g$.

(2) Let

$$f : [0,\infty) \to [0,\infty),\ f(x) = \sqrt{x}$$
$$g : \mathbb{R} \to [0,\infty),\ g(x) = x^2$$

Then

$$g \circ f : [0,\infty) \to [0,\infty),\ g \circ f(x) = (\sqrt{x})^2 = |x|$$
$$f \circ g : \mathbb{R} \to [0,\infty),\ f \circ g(x) = \sqrt{x^2} = |x|.$$

Injective, surjective, and bijective functions.

Definition 5. *$f : A \to B$ is* **injective***, or* **one-to-one** *$(1-1)$, if $\forall a, a' \in A$, $f(a) = f(a')$ implies $a = a'$ (equivalently $a \neq a'$ implies $f(a) \neq f(a')$).*
$f : A \to B$ is **surjective***, or* **onto***, if $\forall b \in B\ \exists a \in A$ such that $f(a) = b$ (equivalently $f(A) = B$).*
$f : A \to B$ is **bijective** *if it is both injective and surjective.*

Inverse. Let $f : A \to B$ be a bijective function. Then by surjectivity for every $b \in B$ we have an element $a \in A$ such that $f(a) = b$. Moreover, by injectivity this element is unique. Thus we have the following definition

Definition 6. *Let $f : A \to B$ be a bijective function. The* **inverse** *of f is the function $g : B \to A$ defined by $g(b) = a$ iff $f(a) = b$. In this case the function f is also called* **invertible***, and we denote the inverse $f^{-1} = g$.*

Example. *Let $f(x) = x^2$.*
1. $f : \mathbb{R} \to \mathbb{R}$. Not injective: $f(1) = f(-1)$, but $1 \neq -1$. Not surjective: $\nexists x \in \mathbb{R}$ s.t. $f(x) = -1$
2. $f : \mathbb{R} \to [0,\infty)$. Not injective: $f(1) = f(-1)$, but $1 \neq -1$. Surjective
3. $f : [0,\infty) \to [0,\infty)$. Injective and surjective $\Rightarrow$ bijective.
The inverse $f^{-1} : [0,\infty) \to [0,\infty)$ is given by $f^{-1}(y) = \sqrt{y}$.

Theorem 7. *(1) Let $f : A \to B$ and $g : C \to D$ be* injective *functions, and $B \subseteq C$. Then $g \circ f : A \to D$ is injective.*
(2) If $f : A \to B$ and $g : B \to D$ are surjective *functions, then so is $g \circ f : A \to D$.*
(3) If $f : A \to B$ and $g : B \to D$ are bijective *functions, then so is $g \circ f : A \to D$*

Proof. (1) Let $f : A \to B$ and $g : C \to D$ be *injective* functions, and $B \subseteq C$. WTS: $g \circ f : A \to D$ is injective, i.e. if $g \circ f(a) = g \circ f(a')$ then $a = a'$.
Suppose $g \circ f(a) = g \circ f(a')$, which means $g(f(a)) = g(f(a'))$. Since g is injective, this implies $f(a) = f(a')$. Since f is injective, this implies $a = a'$. We showed $g \circ f(a) = g \circ f(a') \Rightarrow a = a'$, which means $g \circ f$ is injective.

(2) Let $f : A \to B$ and $g : B \to D$ be *surjective* functions. WTS: $g \circ f : A \to D$ is surjective, i.e. $\forall d \in D \ \exists a \in A$ s.t. $d = g \circ f(a)$.
Let $d \in D$, since $g : B \to D$ is surjective this implies $\exists b \in B$ s.t. $d = g(b)$. Since $f : A \to B$ is surjective, we have that for that b there exists an $a \in A$ s.t. $b = f(a)$. Therefore, $d = g(b) = g(f(a)) = g \circ f(a)$ which shows $g \circ f$ is surjective.
(3) Follows directly from (1) and (2).

□

Example. *Consider the functions*

$$\begin{aligned} f &: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ &(x, y) \mapsto x \end{aligned}$$

and

$$\begin{aligned} g &: \mathbb{R} \to \mathbb{R} \times \mathbb{R} \\ &x \mapsto (x, 0) \end{aligned}$$

Then f is not injective, but it is surjective. g is injective, but not surjective. We can also calculate $f \circ g : \mathbb{R} \to \mathbb{R}$

$$f(g(x)) = f((x, 0)) = x + 0 = x$$

so $f \circ g(x) = x$. Thus, it is both surjective and injective, and therefore bijective. For $g \circ f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ we get

$$g(f(x, y)) = g(x) = (x, 0).$$

Thus, $g \circ f$ is not injective ($g \circ f(1, 1) = g \circ f(1, 0) = 1$) and not surjective (nothing maps to $(1, 1)$).

FIELDS

Field axioms

A **binary operation** on a set A is a function $\circ : A \times A \to A$.

Example. *Operations of $+, -, \cdot$ on $\mathbb{R}$; $\cup$ and $\cap$ on the power set of any set.*

Definition 1. *A set F is a* **field** *if it has two binary operations $\oplus$ (addition) and $\otimes$ (multiplication) on it, together with two special elements, $\mathbf{0}, \mathbf{1} \in F$, satisfying the 10 field axioms:*

F1 *Addition is commutative: for all $x, y \in F$*

$$x \oplus y = y \oplus x$$

F2 *Addition is associative: for all $x, y, z \in F$*

$$(x \oplus y) \oplus z = x \oplus (y \oplus z)$$

F3 *There is an additive identity: an element $\mathbf{0}$ such that for all $x \in F$*

$$x \oplus \mathbf{0} = x$$

F4 *Additive inverse: for every $x \in F$ there is an element $y \in F$ such that*

$$x \oplus y = \mathbf{0}$$

The element y is denoted $-x$. **Subtraction** *is then defined by $x - z := x \oplus (-z)$*

F5 *Multiplication is commutative: for all $x, y \in F$*

$$x \otimes y = y \otimes x$$

F6 *Multiplication is associative: for all $x, y, z \in F$*

$$(x \otimes y) \otimes z = x \otimes (y \otimes z)$$

F7 *There is a multiplicative identity: an element $\mathbf{1}$ such that for all $x \in F$*

$$\mathbf{1} \otimes x = x$$

F8 *Multiplicative inverse: for every $x \in F$, $x \neq \mathbf{0}$, there is an element $y \in F$ such that*

$$x \otimes y = \mathbf{1}$$

The element y is denoted x^{-1} or $1/x$. **Division** *is then defined by $x/z := x \otimes z^{-1}$ for $z \neq \mathbf{0}$.*

F9 *Distributivity: for all $x, y, z \in F$*

$$x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$$

F10

$$\mathbf{1} \neq \mathbf{0}$$

Examples.

(1) The set of rationals, $\mathbb{Q}$, with $\oplus = +$, $\otimes = \cdot$, $\mathbf{0} = 0$, and $\mathbf{1} = 1$, is a field.
(2) $\mathbb{N}$ is not a field, there is no additive identity
(3) $\mathbb{Z}$ is not a field, no multiplicative inverse. E.g. $2 \in \mathbb{Z}$, but $\nexists x \in \mathbb{Z}$ s.t. $x \cdot 2 = 1$.

Field properties.

(1) The additive identity is unique

Proof. Let $\mathbf{0_1}, \mathbf{0_2} \in F$ satisfy

(a) $$x \oplus \mathbf{0_1} = x, \quad \forall x \in F$$
(b) $$x \oplus \mathbf{0_2} = x, \quad \forall x \in F$$

Let $x = \mathbf{0_2}$, then from equation (a) we get

$$\begin{aligned} \mathbf{0_2} \oplus \mathbf{0_1} &= \mathbf{0_2} && \text{by (a)} \\ \Rightarrow \mathbf{0_2} &= \mathbf{0_2} \oplus \mathbf{0_1} \\ &= \mathbf{0_1} \oplus \mathbf{0_2} && \text{by commutativity of addition} \\ &= \mathbf{0_1} && \text{by (b)} \end{aligned}$$

□

(2) Cancellation law. Let $x, y, z \in F$. Then

$$x \oplus y = x \oplus z \;\Rightarrow\; y = z$$

Proof. Adding $-x$ to both sides of the equation we have

$$\begin{aligned} -x \oplus (x \oplus y) &= -x \oplus (x \oplus z) \\ (-x \oplus x) \oplus y &= (-x \oplus x) \oplus z && \text{by associativity of addition} \\ \mathbf{0} \oplus y &= \mathbf{0} \oplus z && \text{by additive inverse} \\ y \oplus \mathbf{0} &= z \oplus \mathbf{0} && \text{by commutativity of addition} \\ y &= z && \text{by additive identity} \end{aligned}$$

□

(3) Let $x \in F$, then

$$x \otimes \mathbf{0} = \mathbf{0}$$

Proof.

$$\begin{aligned} x \otimes \mathbf{0} &= (x \otimes \mathbf{0}) \oplus \mathbf{0} && \text{by additive identity} \\ &= (x \otimes \mathbf{0}) \oplus ((x \otimes \mathbf{0}) - (x \otimes \mathbf{0})) && \text{by additive inverse} \\ &= ((x \otimes \mathbf{0}) \oplus (x \otimes \mathbf{0})) - (x \otimes \mathbf{0}) && \text{by associativity of addition} \\ &= x \otimes (\mathbf{0} \oplus \mathbf{0}) - x \otimes \mathbf{0} && \text{by distributivity} \\ &= x \otimes \mathbf{0} - x \otimes \mathbf{0} && \text{by additive identity} \\ &= \mathbf{0} && \text{by additive inverse} \end{aligned}$$

□

(4) Let $x, y \in F$. Then

$$x \otimes y = \mathbf{0} \iff x = \mathbf{0} \text{ or } y = \mathbf{0}$$

Proof. For if and only if statement we need to prove both ways. Start with assuming $x \otimes y = \mathbf{0}$. If $x \neq \mathbf{0}$ by multiplicative inverses there exists x^{-1}, and thus

$$\begin{aligned} x^{-1} \otimes (x \otimes y) &= x^{-1} \otimes \mathbf{0} && \text{multiplying both sides by } x^{-1} \\ (x^{-1} \otimes x) \otimes y &= \mathbf{0} && \text{by associativity of multiplication and property (3)} \\ \mathbf{1} \otimes y &= \mathbf{0} && \text{by multiplicative inverses} \\ y &= \mathbf{0} && \text{by multiplicative identity} \end{aligned}$$

This shows that either $x = \mathbf{0}$ or $y = \mathbf{0}$. Now assume that $x = \mathbf{0}$ or $y = \mathbf{0}$, then from (3) and commutativity of multiplication we immediately get $x \otimes y = \mathbf{0}$. □

(5) Let $x \in F$, then

$$(-\mathbf{1}) \otimes x = -x$$

Proof. We have

$$\begin{aligned} x \oplus ((-\mathbf{1}) \otimes x) &= (\mathbf{1} \otimes x) \oplus ((-\mathbf{1}) \otimes x) && \text{by multiplicative identity} \\ &= (x \otimes \mathbf{1}) \oplus (x \otimes (-\mathbf{1})) && \text{by commutativity of multiplication} \\ &= x \otimes (\mathbf{1} \oplus (-\mathbf{1})) && \text{by distributivity} \\ &= x \otimes \mathbf{0} && \text{by additive inverse} \\ &= \mathbf{0} && \text{by (3)} \end{aligned}$$

We showed $x \oplus (-\mathbf{1}) \otimes x = \mathbf{0}$ which means $(-\mathbf{1}) \otimes x$ is the additive inverse of x, in other words $(-\mathbf{1}) \otimes x = -x$. □

(6) Using field axioms explain mathematically why "you cannot divide by zero"
SOLUTION: From part (3) we see that any element of the field multiplied by $\mathbf{0}$ gives $\mathbf{0}$, and since $\mathbf{0} \neq \mathbf{1}$ no element can be a multiplicative inverse of $\mathbf{0}$. Thus, $\mathbf{0}$ does not have a multiplicative inverse and the operation of division by zero is not defined.

Modular arithmetic

Let $n \in \mathbb{N}$, define a relation on $\mathbb{Z}$ by $a \sim b$ if $a - b = nz,\ z \in \mathbb{Z}$, i.e. $a - b$ is divisible by n. This relation is called **congruence modulo** n. Notation

$$a = b \mod n$$

(a is congruent (equal) to b modulo n). It is straightforward to check that this is an equivalence relation
Reflexivity: $a - a = 0$ divisible by n
Symmetry: $a - b = nz \Rightarrow b - a = n(-z)$
Transitivity: $a-b = nz$, $b-c = nx$ with $z, x \in \mathbb{Z} \Rightarrow a-c = a-b+b-c = n(z+x)$ and $z + x \in \mathbb{Z}$.
We can then partition $\mathbb{Z}$ into disjoint equivalence classes. How many classes do we get? First note that the classes

$$[0], [1], [2], \ldots, [n-1]$$

are all disjoint, so we have *at least* n distinct equivalence classes. On the other hand $[n] = [0]$, $[n+1] = [1]$ etc. So there are *at most* n distinct equivalence classes. Thus, there are exactly n distinct equivalence classes, and we denote the set of all of them

$$\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n = \{[0], [1], [2], \ldots, [n-1]\}.$$

Examples.

(1) $n = 2, \mathbb{Z}_2 = \{[0], [1]\}$
(2) clock = mod 12, week days = mod 7, months = mod 12
(3) Let $n \in \mathbb{Z}$ be a (k+1)-digit number with digits $n_k n_{k-1} \ldots n_1 n_0$, i.e.

$$n = n_0 + n_1 \cdot 10 + \cdots + n_k \cdot 10^k.$$

Show that $n = n_0 + n_1 + \cdots + n_k \mod 9$. In particular, if $n_0 + n_1 + \cdots + n_k = 0 \mod 9$ (i.e. it is divisible by 9), then $n = 0 \mod 9$, i.e. n is divisible by 9.

Proof. Working modulo 9 we have

$$\begin{aligned} n &= n - n_1 \cdot 9 - n_2 \cdot 99 - \cdots - n_k \cdot 99\ldots9 \mod 9 \\ &= n_0 + n_1 \cdot 1 + \cdots + n_k \cdot 1 \mod 9 \\ &= n_0 + n_1 + \cdots + n_k \mod 9 \end{aligned}$$

□

Goal: Define operations of $+$ and $\cdot$ on $\mathbb{Z}_n$. Can we make $\mathbb{Z}_n$ into a field?

Proposition 2. *If $a = a' \mod n$ and $b = b' \mod n$, then*

(1) $a + b = a' + b' \mod n$
(2) $ab = a'b' \mod n$

Proof. Write $\sim$ for $= \mod n$.

(1) We have $a \sim a'$ which implies by definition that there exists $k \in \mathbb{Z}$ s.t. $a = a' + kn$. Similarly from $b \sim b'$ we have $\exists l \in \mathbb{Z}$ s.t. $b = b' + ln$. Adding the two equalities gives
$$a + b = a' + b' + kn + ln = a' + b' + (k + l)n,\ k + l \in \mathbb{Z}$$
which implies $a + b \sim a' + b'$.

(2) With the notation from part (1) we have
$$ab = (a' + kn)(b' + ln) = a'b' + (a'l + b'k + kln)n, \text{ where } a'l + b'k + kln \in \mathbb{Z}$$
since $a', l, b', k, n \in \mathbb{Z}$.

Thus, $ab \sim a'b'$.

□

Denote
$$\bar{a} := [a].$$

Definition 3. *For $\bar{a}, \bar{b} \in \mathbb{Z}_n$ define*

(1) $\bar{a} \oplus \bar{b} = \overline{a + b}$. Denote $\bar{a} + \bar{b} = \bar{a} \oplus \bar{b}$.
(2) $\bar{a} \otimes \bar{b} = \overline{ab}$ and let $\mathbf{1} = \bar{1}, \mathbf{0} = \bar{0}$. Denote $\bar{a} \cdot \bar{b} = \bar{a} \otimes \bar{b}$.

Addition and multiplication tables.

$\mathbb{Z}_4$

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

•	1	2	3
1	1	2	3
2	2	0	2
3	3	2	1

$\mathbb{Z}_5$

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

•	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Remark 4. *Notice that in $\mathbb{Z}_4$ we have $\bar{2} \otimes \bar{2} = \bar{0}$ since 2 is a factor in 4. Therefore, $\mathbb{Z}_4$ is not a field (recall property (4)). We also notice that $\bar{2}$ does not have a multiplicative inverse.*
In $\mathbb{Z}_5$ however, no two nonzero elements multiply to zero (no zero divisors), and every nonzero element has a multiplicative inverse.

Theorem 5. *(1) $\mathbb{Z}_p$ is a field for any prime integer p*

(2) $\mathbb{Z}_n$ is not a field if $n > 1$ is not a prime

Proof of (2). Let $n = kl$ with $k, l \in \mathbb{Z}$. Since $k < n$ we have $\bar{k} \neq \bar{n} = \bar{0}$, and similarly $\bar{l} \neq \bar{0}$. On the other hand,

$$\bar{k} \otimes \bar{l} = \overline{kl} = \overline{n} = \bar{0}$$

This contradicts property (4) of a field, and thus $\mathbb{Z}_n$ is not a field. □

Idea for the proof of (1). To show that each element has an additive inverse define $-\bar{a} = \overline{p - a}$. To show each nonzero element has a multiplicative inverse proceed as follows

- Show that there are no $\bar{a}, \bar{b} \in \mathbb{Z}_p \backslash \{\bar{0}\}$ such that $\bar{a} \otimes \bar{b} = \bar{0}$
- Show that if $\bar{a}, \bar{b}, \bar{c} \in \mathbb{Z}_p \backslash \{\bar{0}\}$ and $\bar{b} \neq \bar{c}$, then $\bar{a} \otimes \bar{b} \neq \bar{a} \otimes \bar{c}$
- Show that for any $\bar{a} \neq \bar{0}$ there holds $\bar{a} \otimes \bar{0} \neq \bar{a} \otimes \bar{1} \neq \cdots \neq \bar{a} \otimes \overline{p-1}$
- Conclude that there must exist $\bar{b} \in \mathbb{Z}_p$ such that $\bar{a} \otimes \bar{b} = \bar{1}$.

□

Examples.

(1) What is $\bar{6}^{37}$ in $\mathbb{Z}_5$?
Since $\bar{6} = \bar{1}$ we have

$$\bar{6}^{37} = \bar{6} \cdot \bar{6} \cdot \cdots \cdot \bar{6} = \bar{1} \cdot \bar{1} \cdot \cdots \cdot \bar{1} = \bar{1}.$$

(2) What is $5^{5^5} \mod 3$?
Since $5 = 2 \mod 3$ we have $2^{5^5} \mod 3$. To further simplify this expression, note

$$\begin{aligned} 2^0 &= 1 \\ 2^1 &= 2 \\ 2^2 &= 4 = 1 \mod 3 \\ 2^3 &= 8 = 2 \mod 3 \\ &\cdots \end{aligned}$$

We thus observe that $2^n = 2^{(n \mod 2)} \mod 3$. Indeed suppose $n = l + 2k$, then

$$2^n = 2^{l+2k} = 4^k 2^l = 1^k 2^l = 2^l \mod 3, \text{ since } 4 = 1 \mod 3.$$

Therefore, we need to calculate $5^5 \mod 2$. Since $5 = 1 \mod 2$, we have $5^5 = 1 \mod 2$, and finally

$$\begin{aligned} 5^{5^5} &= 2^{5^5} \mod 3 \\ &= 2^{(5^5 \mod 2)} \mod 3 \\ &= 2^1 = 2 \mod 3 \end{aligned}$$

Order

Definition 6. *An* **ordered field** *is a field F with a relation, denoted $<$, satisfying four axioms*

O1 *(Trichotomy) For all $x, y \in F$, exactly one of the following statements is true:*

$$x < y, \quad y < x, \quad x = y.$$

O2 *(Transitivity) The relation $<$ is transitive, i.e. for all $x, y, x \in F$,*

$$x < y \quad \textit{and} \quad y < z \quad \Rightarrow \quad x < z.$$

O3 *(Additive translation) For all $x, y, z \in F$,*

$$x < y \quad \Rightarrow \quad x \oplus z < y \oplus z$$

O4 *(Multiplicative translation) For all $x, y, x \in F$,*

$$x < y \quad \textit{and} \quad \mathbf{0} < z \quad \Rightarrow \quad x \otimes z < y \otimes z$$

Remark 7. *The relation $x > y$ is defined by $x > y \iff y < x$. And the relation $x \leq y$ is defined by $x \leq y \iff x < y$ or $x = y$.*

Order properties.

(1) $x < y \Rightarrow -x > -y$
(2) $x < y$ and $z < \mathbf{0} \Rightarrow x \otimes z > y \otimes z$
(3) $x \neq \mathbf{0} \Rightarrow x \otimes x > \mathbf{0}$
(4) $\mathbf{1} > \mathbf{0}$ and $-\mathbf{1} < \mathbf{0}$
(5) $w < x$ and $y < z \Rightarrow w \oplus y < x \oplus z$

Proof. (1)

$$\begin{aligned}
x &< y \\
-x \oplus x &< -x \oplus y && \text{by additive translation} \\
-y \oplus (-x \oplus x) &< -y \oplus (-x \oplus y) && \text{by additive translation} \\
-y \oplus \mathbf{0} &< -y \oplus (y \oplus -x) && \text{by additive inverse, and commutativity of addition} \\
-y &< (-y \oplus y) \oplus (-x) && \text{by additive identity, and associativity of addition} \\
-y &< -x && \text{by additive inverse and additive identity}
\end{aligned}$$

(2) Since $z < \mathbf{0}$ we have $-z > -\mathbf{0}$ by (1). We also note that since $\mathbf{0}+\mathbf{0} = \mathbf{0}$, it implies that $\mathbf{0}$ is the additive inverse of itself, i.e. $-\mathbf{0} = \mathbf{0}$. Thus, $-z > \mathbf{0}$. Together with

$x < y$ by O4 it implies

$$\begin{aligned} x \otimes (-z) &< y \otimes (-z) \\ -(x \otimes z) &< -(y \otimes z) && \text{by homework question } (-x) \otimes y = -(x \otimes y) \\ -(-(x \otimes z)) &> -(-(y \otimes z)) && \text{by (1)} \\ x \otimes z &> y \otimes z && \text{by Exercise 8 below} \end{aligned}$$

Exercise 8. *Show that for any $x \in F$ we have $-(-x) = x$, i.e. the additive inverse of the additive inverse of x is x.*

(3) Since $x \neq \mathbf{0}$, by O1 we have either $x > \mathbf{0}$ or $x < \mathbf{0}$. If $x > \mathbf{0}$ we have by O4 applied to $\mathbf{0} < x$

$$\begin{aligned} \mathbf{0} \otimes x &< x \otimes x \\ \mathbf{0} &< x \otimes x && \text{by property (3) of the fields we proved in class} \end{aligned}$$

On the other hand, if $x < \mathbf{0}$ we apply (2) to $x < \mathbf{0}$ to obtain

$$x \otimes x > \mathbf{0} \otimes x = \mathbf{0}$$

(4) By (3) we have $\mathbf{1} \otimes \mathbf{1} > \mathbf{0}$ and by F7 $\mathbf{1} \otimes \mathbf{1} = \mathbf{1}$, so $\mathbf{1} > \mathbf{0}$. Then by (1) we get $-\mathbf{1} < -\mathbf{0} = \mathbf{0}$ where the last equality has been shown above in the proof of (2).

(5) Since $w < x$, then by O3 we get

$$w \oplus y < x \oplus y.$$

Similarly, $y < z \Rightarrow y \oplus x < z \oplus x$, or by F1

$$x \oplus y < x \oplus z.$$

Using transitivity (O4) from these two inequalities we conclude $w \oplus y < x \oplus z$.

□

Example 1. *Let p be a prime, then $\mathbb{Z}/p\mathbb{Z}$ is a field. Can it be ordered?*
Suppose it can be done, and let $<$ be the order relation. Then from what we proved before (property (4)), we must have $\mathbf{0} < \mathbf{1}$, or $\bar{0} < \bar{1}$. By additive translation (O3) and the definition of $\oplus$ in $\mathbb{Z}/p\mathbb{Z}$, we have

$$\bar{1} < \bar{2},\ \bar{2} < \bar{3},\ \ldots,\ \overline{p-1} < \bar{p} = \bar{0}.$$

Then applying transitivity (O2) p times we obtain $\bar{1} < \bar{0}$, which is a contradiction. Therefore, we conclude that $\mathbb{Z}/p\mathbb{Z}$ cannot be ordered.

Number sets.

(1) $\mathbb{N}$ — natural numbers (with addition and multiplication) can be constructed intuitively. However, they do not contain additive or multiplicative inverses.

(2) $\mathbb{Z}$ ="$\mathbb{N} \cup \{0\} \cup$ additive inverses". Can be formally constructed as the set of equivalence classes on $\mathbb{N}_0 \times \mathbb{N}_0$ under the relation $(a,b) \sim (c,d)$ if $a+d = b+c$, with addition $[(a,b)] \oplus [(c,d)] = [(a+c, b+d)]$ and multiplication $[(a,b)] \otimes [(c,d)] = [(ac+bd, ad+bc)]$. Additive and multiplicative identities can then be defined as $\mathbf{0} = [(0,0)]$ and $\mathbf{1} = [(1,0)]$. We can also introduce oder by defining $[(a,b)] < [(c,d)] \iff a+d < b+c$.
$\mathbb{Z}$, however, is still missing multiplicative inverses.

(3) $\mathbb{Q}$ is an ordered field. It can be constructed from $\mathbb{Z}$ as follows. Define a relation $(a,b) \sim (c,d)$ on $\mathbb{Z} \times \mathbb{Z} \backslash \{0\}$ as follows: $(a,b) \sim (c,d)$ if $a \cdot d = b \cdot c$. On the set of equivalence classes under this relation define addition and multiplication by

$$[(a,b)] \oplus [(c,d)] = [(ad+bc, bd)], \ [(a,b)] \otimes [(c,d)] = [(ac, bd)].$$

With $\mathbf{1} = [(1,1)]$ and $\mathbf{0} = [(0,1)]$ this set satisfies ten field axioms. The order relation can then be defined as $[(a,b)] < [(c,d)]$ if $bd > 0$ and $ad < bc$, or $bd < 0$ and $ad > bc$. One can check that the four order axioms hold for this relation.

(4) $\mathbb{R}$: why do we need a bigger field? What is missing from $\mathbb{Q}$? Consider the set $A := \{x \in \mathbb{Q} : x^2 < 2\}$. It can be shown that for all $x \in A$ we have $x < 2$, so we say that the set A is *bounded above*. However, for every $x \in A$ we can find another $y \in A$ satisfying $y > x$. In other words there is no "exact" upper bound on A. Another way to capture this is to consider a sequence of rational numbers

$$1, \ 1+1, \ 1+\frac{1}{1+1}, \ 1+\frac{1}{1+\frac{1}{1+1}}, \ \dots$$

The members of this sequence become "closer and closer" to each other, but there is no rational number that it converges to. The property that is missing is called **completeness**, and we will make it precise in the next section.

Completeness

Bounds. Let F be an ordered field, $X \subseteq F$.

Definition 9. *(1) $b \in F$ is an* **upper bound** *of X if $x \leq b \ \forall x \in X$. It is a* **least upper bound (supremum)** *of X if (i) it is an upper bound of X; and (ii) for every upper bound c of X there holds: $c \geq b$. Notation: $b = \sup X$.*

(2) $a \in F$ is a **lower bound** *of X if $x \geq a \ \forall x \in X$. It is a* **greatest lower bound (infimum)** *of X if (i) it is a lower bound of X; and (ii) for every lower bound c of X there holds: $c \leq b$. Notation: $a = \inf X$.*

Proposition 10. *If $\sup X$ exists, it is unique. If $\inf X$ exists, it is unique.*

Proof. We will consider the case of supremum, infimum left as an **exercise**. Let b_1 and b_2 be suprema of X. We want to show $b_1 = b_2$.
We have (1) b_1 is an upper bound of X; (2) $\forall$ upper bound b of X there holds $b \geq b_1$; (3) b_2 is an upper bound of X; (4) $\forall$ upper bound b of X there holds $b \geq b_2$.

Properties (1) and (2) hold because $b_1 = \sup X$, and properties (3) and (4) hold because $b_2 = \sup X$. We have

$$(2), (3) \;\Rightarrow\; b_2 \geq b_1; \;\; (1), (4) \;\Rightarrow\; b_1 \geq b_2.$$

By trichotomy this gives $b_1 = b_2$. □

Definition 11. *(1) The set X is* **bounded above** *if it has an upper bound. It is* **bounded below** *if it has a lower bound. X is bounded if it has an upper bound and a lower bound.*

(2) If $\sup X \in X$ it is called the **maximum** *of X, and if $\inf X \in X$ it is called the* **minimum** *of X.*

Examples.

(1) Let $F = \mathbb{Q}$, $\mathbb{N} \subseteq \mathbb{Q}$. $\mathbb{N}$ is bounded below, but not bounded above (thus $\sup \mathbb{N}$ and $\max \mathbb{N}$ DNE). $\inf \mathbb{N} = 1$, and since $1 \in \mathbb{N}$, $\min \mathbb{N} = 1$.

(2) $X = (0, 1) \subseteq \mathbb{R}$. $\inf X = 0$, $\sup X = 1$, $\min, \max$ DNE.

(3) Let $X = \{1/n : n \in \mathbb{N}\} \subseteq \mathbb{Q}$. Then $\sup X = \max X = 1$, $\inf X = 0$, $\min X$ DNE

Proof. We will prove that $\inf X = 0$. First, $1/n > 0 \; \forall n \in \mathbb{N}$, thus $x > 0 \; \forall x \in X$ and 0 is a lower bound. Second, any negative lower bound is < 0, so it cannot be a *greatest* lower bound. Finally, we show that no positive rational can be a lower bound for X and this will conclude that $\inf X = 0$. Let $r \in \mathbb{Q}$, $r > 0$, define $n = \lceil 1/r \rceil + 1 \in \mathbb{N}$. Then $n > 1/r \Rightarrow r > 1/n$, and since $1/n \in X$, r is not a lower bound for X. Thus, 0 is a lower bound for X and no positive rational can be a lower bound i.e. $b \leq 0 \; \forall$ lower bound b. Therefore, $0 = \inf X$. □

(4) $X = \{(-1)^n : n \in \mathbb{N}\} \subseteq \mathbb{Q}$. We can see that $X = \{-1, 1\}$ so $\inf X = \min X = -1$ and $\sup X = \max X = 1$.

Completeness. An ordered field F is **complete** if every non-empty subset of F that is bounded above, has a supremum in F.

Example. *Let $F = \mathbb{Q}$, recall $A = \{x \in \mathbb{Q} : x^2 < 2\}$. Claim: A is bounded above. Proof: we will show $b = 3$ is an upper bound for A. Indeed, if $x > 3$, then by transitivity $x > 0$, and by multiplicative translation $x^2 > 3x$ and $3x > 3^2$ (since we also have $3 > 0$). So by transitivity $x^2 > 3^2 = 9$, and thus $x \notin A$. We showed $x > 3 \Rightarrow x \notin A$ which by contrapositive shows $x \in A \Rightarrow x \leq 3$. So for every $x \in A$ we must have $x \leq 3$ and therefore 3 is an upper bound.*

We now claim that A does not have a supremum in $\mathbb{Q}$. The proof is rather technical, and we will not give all the details; rather we sketch the idea. Suppose $\exists c \in \mathbb{Q}$ s.t. $c = \sup A$. It can be shown that $c^2 = 2$ (this is the technical part), and to arrive at a contradiction we will show that c cannot be rational. Assume it is, and write it as an irreducible fraction

$$c = \frac{m}{n}, \;\; m \in \mathbb{Z}, \;\; n \in \mathbb{N}$$

Since $c^2 = 2$, we have $m^2 = 2n^2$, so m^2 is even. This implies that m is even, so $m = 2k$ for some $k \in \mathbb{Z}$. Then $4k^2 = m^2 = 2n^2 \Rightarrow n^2 = 2k^2$, thus n^2 is even and so is n. Since m and n are both even, the fraction m/n is reducible which contradicts our assumption. This concludes that the set $A \subseteq \mathbb{Q}$, although bounded above, does not have a supremum, and thus $\mathbb{Q}$ **is not complete**.

Definition 12. *Two ordered fields, F_1 and F_2 are* **isomorphic** *if there exists a bijection $f : F_1 \to F_2$ s.t.*

$$f(a \oplus b) = f(a) \oplus f(b),\ f(a \otimes b) = f(a) \otimes f(b),\ \text{and}\ a < b \iff f(a) < f(b),\ \forall a, b \in F_1$$

In other words, f is a bijection that preserves arithmetic operations and order.

Theorem 13. *There exists a complete ordered field, $\mathbb{R}$. Moreover, $\mathbb{R}$ contains a subfield that is isomorphic to $\mathbb{Q}$. Finally, any two complete ordered fields are isomorphic.*

Idea of proof. The field $\mathbb{R}$ will be constructed from $\mathbb{Q}$. The members of $\mathbb{R}$ will be certain subsets of $\mathbb{Q}$, called *cuts*. A cut is, by definition, any set $\alpha \subseteq \mathbb{Q}$ with the following three properties:

(1) $\alpha \neq \emptyset$ and $\alpha \neq \mathbb{Q}$
(2) If $p \in \alpha$, $q \in \mathbb{Q}$, and $q < p$, then $q \in \alpha$
(3) If $p \in \alpha$, then $p < r$ for some $r \in \alpha$ (i.e. α has no largest member)

Define "$\alpha < \beta$" to mean $\alpha \subsetneq \beta$. Define addition as

$$\alpha \oplus \beta = \{r + s : r \in \alpha, s \in \beta\}.$$

Define $\mathbf{0} = \{r \in \mathbb{Q} : r < 0\}$. The additive inverse of α will be

$$-\alpha = \{p \in \mathbb{Q} : -p - r \notin \alpha \text{ for some } r > 0\}.$$

For multiplication first consider the set $\mathbb{R}^+ = \{\alpha \in \mathbb{R} : \alpha > \mathbf{0}\}$. For $\alpha, \beta \in \mathbb{R}^+$ define

$$\alpha \otimes \beta = \{p \in \mathbb{Q} : p < rs \text{ for some } r \in \alpha, s \in \beta, r > 0, s > 0\}.$$

Define $\mathbf{1} = \{q \in \mathbb{Q} : q < 1\}$. To complete the definition of multiplication, set $\alpha \otimes \mathbf{0} = \mathbf{0} \otimes \alpha = \mathbf{0}$ for all cuts α, and

$$\alpha \otimes \beta = \begin{cases} (-\alpha) \otimes (-\beta), & \text{if } \alpha < \mathbf{0}, \beta < \mathbf{0}, \\ -((-\alpha) \otimes \beta), & \text{if } \alpha < \mathbf{0}, \beta > \mathbf{0}, \\ -(\alpha \otimes (-\beta)), & \text{if } \alpha > \mathbf{0}, \beta < \mathbf{0}. \end{cases}$$

It then remains to verify 10 field axioms, 4 order axioms, and the completeness axiom. Rational numbers are then isomorphic to "rational cuts" defined by

$$r^* = \{p \in \mathbb{Q} : p < r\}.$$

□

Example. *A familiar real number $\sqrt{2}$ can be defined as the following cut*

$$\sqrt{2} = \{r \in \mathbb{Q} : r^2 < 2\} \cup \{r \in \mathbb{Q} : r < 0\}.$$

Absolute value

Let F be an ordered field (not necessarily complete). The **absolute value** of $x \in F$ is

$$|x| := \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$$

Remark 14. *We write* 0 *for* **0** *(the additive identity of the field* F*), and* $-x$ *stands for the additive inverse of* x*.*

Remark 15. $|x| \in F \ \forall x \in F$.

Properties. Let F be an ordered field, then $\forall x, y, a \in F$

(1) $|x| \geq 0$; and $|x| = 0$ iff $x = 0$
(2) $|x| = |-x|$
(3) $-|x| \leq x \leq |x|$
(4) $|x| \leq a$ iff $-a \leq x \leq a$. Similarly, $|x| < a$ iff $-a < x < a$
(5) $|xy| = |x||y|$ (here xy stands for $x \otimes y$).

Proof. We will prove (3) and (4), the rest is an **exercise**.
(3) Recall that by trichotomy we have $x = 0$, $x > 0$, or $x < 0$.
Case 1: $x = 0 \Rightarrow$(by (1)) $|x| = 0 \Rightarrow -|x| = 0$. We thus have $-|x| = x = |x|$, which in particular implies $-|x| \leq x \leq |x|$.
Case 2: $x > 0 \Rightarrow$ (by definition) $|x| = x$, and in particular $x \leq |x|$. Moreover, $-|x| = -x < 0$ (by the property of order proved in class), thus by transitivity combining with $x > 0$ we obtain $-|x| < x$. In particular, $-|x| \leq x$.
Case 3: $x < 0 \Rightarrow$ (by definition) $|x| = -x \Rightarrow -|x| = -(-x) = x$ (see Exercise 8), and in particular $-|x| \leq x$. Moreover, $|x| = -x > 0$ (by the property of order proved in class), thus by transitivity combining with $x < 0$ we obtain $|x| > x$. In particular, $x \leq |x|$.
(4) $\Rightarrow$: let $|x| \leq a$, then (3) and transitivity $x \leq |x| \leq a$. Again by property (1) of order we have $-|x| \geq -a$, and thus (by (3) and transitivity) $-a \leq -|x| \leq x$.
$\Leftarrow$: let $-a \leq x \leq a$. Then if $x \geq 0$ we have by definition $|x| = x \leq a$. On the other hand, if $x < 0$ we have $|x| = -x$, and since $x \geq -a \Rightarrow -x \leq -(-a) = a$. Thus, $|x| = -x \leq a$ in this case as well. By trichotomy, there are no other cases.
The proof for the strict inequality proceeds in exactly the same way. □

Triangle inequality. Let F be an ordered filed, and $x, y \in F$. Then

$$|x + y| \leq |x| + |y| \tag{1}$$

Remark 16. *Here* $+$ *stands for the addition operation* $\oplus$ *in* F*.*

Proof of (1). By property (3) we have $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$. Adding the two inequalities (property (5) of order proved in class) we obtain

$$-|x| - |y| \leq x + y \leq |x| + |y|$$

Next, recall that we proved $-x = (-1) \otimes x$, which implies by distributivity $-|x| - |y| = (-1)|x| + (-1)|y| = (-1)(|x| + |y|) = -(|x| + |y|)$. Therefore,

$$-(|x| + |y|) \leq x + y \leq |x| + |y|,$$

which by property (4) (with $a = |x| + |y|$ and $x + y$ in place of x) implies (1). □

Complex numbers

Note that there is no element $x \in \mathbb{R}$ s.t. $x^2 = -1$. Goal: construct the smallest possible field containing $\mathbb{R}$ and an element x s.t. $x^2 = -1$.

Definition 17. *The set of complex numbers is defined as*

$$\mathbb{C} := \mathbb{R}^2 = \{(a, b) : a, b \in \mathbb{R}\}$$

with addition $+$ and multiplication $\cdot$ defined by

$$(a, b) + (c, d) = (a + c, b + d)$$
$$(a, b) \cdot (c, d) = (ac - bd, ad + bc)$$

$\forall (a, b), (c, d) \in \mathbb{C}$.

Theorem 18. *$\mathbb{C}$ is a field with $\mathbf{0} = (0, 0)$ and $\mathbf{1} = (1, 0)$.*

Proof. **Exercise.** Hint: verify that $-(a, b) = (-a, -b)$ and $(a, b)^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$ are the additive and multiplicative inverses respectively. □

Remark 19. *If we defined "multiplication" by simply assigning $(a, b) \cdot (c, d) = (ac, bd)$, then we would have $(1, 0) \cdot (0, 1) = (0, 0) = \mathbf{0}$ while $(1, 0) \neq \mathbf{0}$ and $(0, 1) \neq \mathbf{0}$ which is impossible in the field.*

We now note that $\forall (a, b) \in \mathbb{C}$ we can write $(a, b) = (a, 0) + (b, 0) \cdot (0, 1)$. All the elements of the form $(a, 0)$ form a subset of $\mathbb{C}$ that can be put in a bijective correspondence with the set of real numbers $(a, 0) \mapsto a$. Moreover, this bijection preserves the arithmetic operations $(a, 0) + (b, 0) = (a + b, 0)$, $(a, 0) \cdot (b, 0) = (ab, 0)$. From now on we will identify the subset $\{(a, 0) \in \mathbb{C}\} \subseteq \mathbb{C}$ with the set of real numbers. We next establish an important property of the special element $(0, 1)$.

Definition 20. *Define $i = (0, 1)$.*

Proposition 21. *$i^2 = -1$, i.e. $(0, 1) \cdot (0, 1) = (-1, 0)$.*

Proof. The result follows by direct verification using the definition of multiplication from Definition 17. □

Definition 22. *The* **real part** *of a complex number* (a, b) *is* $\mathrm{Re}(a, b) = a$. *The* **imaginary part** *of a complex number* (a, b) *is* $\mathrm{Im}(a, b) = b$. *We then write* $(a, b) = a + bi$, *and note that both real and imaginary parts of a complex number are real numbers.*

Example 2. *Consider* $2 + 3i, 4 + 5i \in \mathbb{C}$. *We then have*

(1) $(2 + 3i) + (4 + 5i) = 6 + 8i$
(2) $(2 + 3i)(4 + 5i) = 8 + 10i + 12i + 15i^2 = -7 + 22i$
(3) $\dfrac{1}{2 + 3i} = (2, 3)^{-1} = \left(\dfrac{2}{2^2 + 3^2}, \dfrac{-3}{2^2 + 3^2}\right) = \dfrac{2}{13} - \dfrac{3}{13}i$

Solutions to polynomials in different number fields. With identification of $\mathbb{R}$ with a subset of $\mathbb{C}$, the field of complex numbers is the largest number set we have constructed so far. In other words we have

$$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$$

At every step in this chain of inclusions as we are adding new elements, we enlarge the class of polynomials that have roots in a particular number set. For example,

$x + 1$ has no solution in $\mathbb{N}$, but has a solution in $\mathbb{Z}$, $x = -1$

$2x + 1$ has no solution in $\mathbb{Z}$, but has a solution in $\mathbb{Q}$, $x = -\dfrac{1}{2}$

$x^2 - 2$ has no solution in $\mathbb{Q}$, but has a solution in $\mathbb{R}$, $x = \sqrt{2}$ (and $x = -\sqrt{2}$)

$x^2 + 1$ has no solution in $\mathbb{R}$, but has a solution in $\mathbb{C}$, $x = i$ (and $x = -i$)

This may serve as an additional motivation for constructing the field of complex numbers.

Complex conjugate and modulus.

Definition 23. *The* **complex conjugate** *of* $z = a + ib \in \mathbb{C}$ *is*

$$\bar{z} = a - ib \in \mathbb{C}.$$

The **modulus** *(or the length, or the absolute value) of* $z = a + ib \in \mathbb{C}$ *is*

$$|z| = \sqrt{z \cdot \bar{z}} = \sqrt{a^2 + b^2} \in \mathbb{R}_{\geq 0}.$$

Remark 24. *If* $x \in \mathbb{R}$, *i.e.* $\mathrm{Re}(x) = x$ *and* $\mathrm{Im}(x) = 0$, *then the modulus of* x *coincides with the absolute value defined previously:* $|x| = \sqrt{x^2}$.

Examples.

(1) Write in the form $a + ib$

$$\frac{4 - i}{3 + 2i} = \frac{(4 - i)(3 + 2i)}{9 + 4} = \frac{8 - 12i}{13} = \frac{8}{13} - \frac{12}{13}i$$

(2) $\dfrac{1}{2+3i}+\dfrac{1-i}{2-3i}=\dfrac{2-3i}{13}+\dfrac{(1-i)(2+3i)}{13}=\dfrac{7}{13}-\dfrac{2}{13}i.$

Properties.

(1) $\overline{\bar{z}}=z$
(2) $\overline{z+w}=\bar{z}+\bar{w}$
(3) $z+\bar{z}=2\,\mathrm{Re}(z)$ and $z-\bar{z}=2i\,\mathrm{Im}(z)$
(4) $\overline{z\cdot w}=\bar{z}\cdot\bar{w}$
(5) if $w\neq 0$ then $\bar{w}\neq 0$ and
$$\overline{\left(\frac{z}{w}\right)}=\frac{\bar{z}}{\bar{w}}$$
(6) $z=\bar{z}$ iff $z\in\mathbb{R}$
(7) $z=0$ iff $\bar{z}=0$
(8) $|z|=|\bar{z}|$
(9) $z\cdot\bar{z}=|z|^2$
(10) $|zw|=|z||w|$
(11) $|\,\mathrm{Re}(z)|\leq|z|$ and $|\,\mathrm{Im}(z)|\leq|z|$
(12) Triangle inequality: $|z+w|\leq|z|+|w|$

Proof. (1) Let $z=a+ib$, then $\bar{z}=a-ib=a+i(-b)\Rightarrow\overline{\bar{z}}=a-i(-b)=a+ib=z$
(2) Let $z=a+ib$ and $w=c+id$, then
$$\overline{z+w}=\overline{a+c+i(b+d)}=a+c-i(b+d)=(a-ib)+(c-id)=\bar{z}+\bar{w}$$
(3) Let $z=a+ib$, then
$$z+\bar{z}=a+ib+a-ib=2a=2\,\mathrm{Re}(z);\ z-\bar{z}=a+ib-(a-ib)=2ib=2i\,\mathrm{Im}(z)$$
(4) Let $z=a+ib$ and $w=c+id$, then
$$\overline{z\cdot w}=\overline{(a+ib)(c+id)}=\overline{ac-bd+i(ad+bc)}=ac-bd-i(ad+bc)=(a-ib)(c-id)=\bar{z}\cdot\bar{w}$$
(5) Let $w=c+id$, if $w\neq 0$ then either $c\neq 0$ or $d\neq 0$. Since $\bar{w}=c-id$ we thus obtain that $\bar{w}\neq 0$. Next, using property (4) we have
$$\overline{\left(\frac{z}{w}\right)}\cdot\bar{w}=\overline{\left(\frac{z}{w}\cdot w\right)}=\bar{z}.$$
Dividing both sides by $\bar{w}\neq 0$ gives the result.
(6) We need to show two implications. First, assume $z=\bar{z}$. Denoting $z=a+ib$ we obtain
$$a+ib=a-ib\ \Rightarrow\ 2ib=0\ \Rightarrow\ b=0\ \Rightarrow\ z=a\in\mathbb{R}.$$
On the other hand, assume $z=a\in\mathbb{R}$, i.e. $z=a+0i\Rightarrow\bar{z}=a-0i=a=z$.
(7) If $z=a+ib=0$, then $a=b=0\Rightarrow\bar{z}=a-ib=0$. If $\bar{z}=0$, then by property (1) $z=\overline{\bar{z}}=0$ by what we just showed.
(8) Let $z=a+ib$, then $|z|=\sqrt{a^2+b^2}$, and $\bar{z}=a-ib\Rightarrow|\bar{z}|=\sqrt{a^2+(-b)^2}=\sqrt{a^2+b^2}=|z|$.

(9) Let $z = a + ib$, then $\bar{z} = a - ib$ so

$$z \cdot \bar{z} = (a + ib)(a - ib) = a^2 + b^2 + iab - iab = a^2 + b^2 = |z|^2.$$

(10) By (9)

$$|zw|^2 = (zw)\overline{(zw)} = (by\ (4)) = zw\bar{z}\bar{w} = z\bar{z}w\overline{w} = (by\ (9)\ again) = |z|^2|w|^2.$$

Taking the square root on both sides, and using the fact that modulus is always nonnegative, we obtain $|zw| = |z||w|$.

(11) Let $z = a + ib$, then $|z|^2 = a^2 + b^2$, and $|\operatorname{Re} z|^2 = a^2$. This gives $|\operatorname{Re} z|^2 \leq |z|^2$, which implies by taking the square root and using nonnegativity of modulus, $|\operatorname{Re} z| \leq |z|$. By the property of absolute value ($x \leq |x|\ \forall x \in \mathbb{R}$) we conclude $\operatorname{Re} z \leq |z|$. The second inequality for the imaginary part is shown in the same way.

(12) For the triangle inequality, first recall from your homework that $z\bar{w} + \bar{z}w = 2\operatorname{Re}(\bar{z}w)\ \forall z, w \in \mathbb{C}$. Then using (9) we have

$$\begin{aligned} |z + w|^2 &= (z + w)\overline{(z + w)} = (by\ (2)) = (z + w)(\bar{z} + \bar{w}) = z\bar{z} + w\bar{w} + z\bar{w} + \bar{z}w \\ &= (by\ (9)\ and\ hw) = |z|^2 + |w|^2 + 2\operatorname{Re}(\bar{z}w) \leq (by\ (11)) \leq |z|^2 + |w|^2 + 2|\bar{z}w| \\ &= (by\ (10)\ and\ (8)) = |z|^2 + |w|^2 + 2|z||w| \end{aligned}$$

Finally, using nonnegativity of the modulus and extracting the square root completes the proof.

□

Proposition 25. $\mathbb{C}$ *cannot be ordered.*

Proof. Suppose that it can, and let $<$ be the order relation. Since $i \neq 0$ we have by property (3) of order proved in class: $i^2 > 0$. Thus, $-1 > 0 \Rightarrow 1 < 0$, which is a contradiction since in every ordered field we must have $1 > 0$ (order property (4)). □

Polar form of a complex number

We showed that for any nonzero $z \in \mathbb{C}$, $z/|z|$ is a unit vector. Another way to think about it, is that it is a point on the unit circle centered at $(0, 0)$. In turn, every point on the unit circle has the form $(\cos\theta, \sin\theta)$ where θ is the angle between the line through the origin containing the point, and the x-axis. This motivates the following definition

Definition 26. *The* **argument** *of $z \in \mathbb{C}\backslash\{0\}$ is the angle θ between z and the x-axis,* $\arg(z) = \theta$. *The* **polar form** *of z is*

$$z = |z|(\cos\theta + i\sin\theta).$$

The argument is usually taken to be in $(-\pi, \pi]$ or $[0, 2\pi)$.

Remark 27. *We have* $z = |z|(\cos\theta + i\sin\theta) \Rightarrow \operatorname{Re} z = |z|\cos\theta$, $\operatorname{Im} z = |z|\sin\theta$ $\Rightarrow \tan\theta = \dfrac{\operatorname{Im} z}{\operatorname{Re} z}$.

Examples.

(1) $z = 7 + 7i$

$$|z| = \sqrt{7^2 + 7^2} = 7\sqrt{2};\ \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$$

$z = 7\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$.

(2) $z = \dfrac{\sqrt{3}}{2} + \dfrac{i}{2}$

$$|z| = \frac{1}{2}\sqrt{3+1} = 1;\ \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$$

$z = 1\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$.

(3) $z = 3 - 5i$

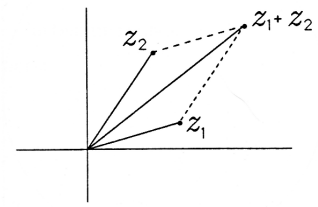
$$|z| = \sqrt{3^2 + 5^2} = \sqrt{34};\ \tan\theta = -\frac{5}{3}$$

Moreover, $\cos\theta = 3/\sqrt{34} > 0$ and $\sin\theta = -5/\sqrt{34} \Rightarrow \theta \in [3\pi/2, 2\pi)$. Thus

$$z = \sqrt{34}\left(\cos\theta + i\sin\theta\right),$$

where θ is the unique angle such that $\theta \in [3\pi/2, 2\pi)$ and $\tan\theta = -5/3$.

Geometry of addition and multiplication. For addition we can use the parallelogram law



For multiplication, let $z = |z|(\cos\theta + i\sin\theta)$ and $w = |w|(\cos\psi + i\sin\psi)$. Then using trigonometric identities we obtain

$$zw = |z||w|(\cos\theta + i\sin\theta)(\cos\psi + i\sin\psi) = |z||w|(\cos(\theta+\psi) + i\sin(\theta+\psi))$$

That is when multiplying two complex numbers, their length multiply, and the arguments add.

Example 3. *Let $z = 7+7i$, we showed $z = 7\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$. We then can easily calculate different powers of z*

$$z^2 = z \cdot z = 49 \cdot 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$
$$z^3 = (7\sqrt{2})^3\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$
$$\dots$$

Bounded, open, closed sets

Let $F = \mathbb{R}$ or $\mathbb{C}$.
Recall that if $x \in \mathbb{R}$ we defined $|x|$ the absolute value of x. If $z \in \mathbb{C}$, then $|z|$ is the modulus of z (which coincides with the absolute value, if z happens to be real).

Definition 28. *A subset $A \subseteq F$ is* **bounded** *if $\exists M \in \mathbb{R}$, $M > 0$ s.t. $|z| \leq M$ $\forall z \in A$.*

Examples.

(1) $\{z \in \mathbb{C} : |z| \leq 5\} \subseteq \mathbb{C}$ is bounded
(2) $\{i^n : n \in \mathbb{N}\} \subseteq \mathbb{C}$ is bounded since $|i^n| = |i|^n = 1 \leq 1$ $\forall n \in \mathbb{N}$
(3) $\{x \in \mathbb{R} : x \leq 1\}$ is unbounded, since $\forall M > 0$ there is $x = -M - 1 \leq 1$ s.t. $|x| = M + 1 > M$.

Definition 29. *The* **distance** *between $z, w \in F$ is*

$$d(z, w) = |z - w|.$$

Proposition 30 (Triangle inequality for the distance)**.** *$\forall z, w, u \in F$ we have*

$$d(z, w) \leq d(z, u) + d(u, w).$$

Example 4. *Let $F = \mathbb{C}$, $z = a+ib$, $w = c+id \Rightarrow d(z, w) = |z-w| = \sqrt{(a-c)^2 + (b-d)^2}$.*

Definition 31. *The* **open ball** *centered at $z \in F$ with radius $r \in \mathbb{R}$, $r > 0$ is the following subset of F*

$$B(z, r) := \{w \in F : d(w, z) < r\}.$$

Examples.

(1) $F = \mathbb{R}$

$$B(a, r) = (a - r, a + r) \quad \text{open interval of } \mathbb{R}$$

(2) $F = \mathbb{C}$

$B(a+ib, r) = \{x+iy \in \mathbb{C} : |(x+iy)-(a+ib)| < r\} = B(a+ib, r) = \{(x, y) \in \mathbb{R}^2 : (x-a)^2+(y-b)^2 < r^2$

open disk centered at $a + ib$, radius r.

Definition 32. *A subset $U \subseteq F$ is* **open** *if $\forall x \in U$ $\exists \varepsilon_x > 0$ s.t. $B(x, \varepsilon_x) \subseteq U$. That is every point in U has a ball around it that is contained entirely in U.*

TOPOLOGY OF $\mathbb{R}$ AND $\mathbb{C}$

Let $F = \mathbb{R}$ or $\mathbb{C}$.
Recall that if $x \in \mathbb{R}$ we defined $|x|$ the absolute value of x. If $z \in \mathbb{C}$, then $|z|$ is the modulus of z (which coincides with the absolute value, if z happens to be real).

Definition 1. *A subset $A \subseteq F$ is* **bounded** *if $\exists M \in \mathbb{R}$, $M > 0$ s.t. $|z| \leq M$ $\forall z \in A$.*

Examples.

(1) $\{z \in \mathbb{C} : |z| \leq 5\} \subseteq \mathbb{C}$ is bounded
(2) $\{i^n : n \in \mathbb{N}\} \subseteq \mathbb{C}$ is bounded since $|i^n| = |i|^n = 1 \leq 1$ $\forall n \in \mathbb{N}$
(3) $\{x \in \mathbb{R} : x \leq 1\}$ is unbounded, since $\forall M > 0$ there is $x = -M - 1 \leq 1$ s.t. $|x| = M + 1 > M$.

Definition 2. *The* **distance** *between $z, w \in F$ is*

$$d(z, w) = |z - w|.$$

Proposition 3 (Triangle inequality for the distance)**.** *$\forall z, w, u \in F$ we have*

$$d(z, w) \leq d(z, u) + d(u, w).$$

Example 1. *Let $F = \mathbb{C}$, $z = a+ib$, $w = c+id \Rightarrow d(z, w) = |z-w| = \sqrt{(a-c)^2 + (b-d)^2}$.*

Definition 4. *The* **open ball** *centered at $z \in F$ with radius $r \in \mathbb{R}$, $r > 0$ is the following subset of F*

$$B(z, r) := \{w \in F : d(w, z) < r\}.$$

Examples.

(1) $F = \mathbb{R}$, $B(3, 1) = (2, 4)$
(2) $F = \mathbb{C}$, $B(1 + i, 1) = \{w \in \mathbb{C} : |w - (1 + i)| < 1\}$ — an open disk centered at $1 + i$, radius 1.

Definition 5. *A subset $U \subseteq F$ is* **open** *if $\forall x \in U$ $\exists \varepsilon_x > 0$ s.t. $B(x, \varepsilon_x) \subseteq U$. That is every point in U has a ball around it that is contained entirely in U.*

Proposition 6. *$U \subseteq F$ is open iff it can be written as a union of open balls.*

Proof. $\Rightarrow$: suppose $U \subseteq F$ is open, which means $\forall x \in U$ $\exists \varepsilon_x > 0$ s.t. $B(x, \varepsilon_x) \subseteq U$.
Claim: $U = \cup_{x \in U} B(x, \varepsilon_x)$.
Proof of claim: let $y \in U \Rightarrow y \in B(x, \varepsilon_x)$ for $x = y \in U \Rightarrow y \in \cup_{x \in U} B(x, \varepsilon_x)$.
On the other hand, if $y \in \cup_{x \in U} B(x, \varepsilon_x)$, then $\exists x \in U$ s.t. $y \in B(x, \varepsilon_x)$. But since $B(x, \varepsilon_x) \subseteq U$ this gives $y \in U$.

$\Leftarrow$: suppose U can be written as a union of open balls, i.e. there exists a subset $S \subseteq F$ s.t. $U = \cup_{a\in S} B(a, r_a)$, where $r_a > 0\ \forall a$. Now let $x \in U$, we want to show that there is a ball around x that is entirely in U. From our assumption we have $\exists a \in S$ s.t. $x \in B(a, r_a) \Rightarrow d(x, a) < r_a$. This shows in particular that $r_a - d(x, a) > 0$, so we denote $\varepsilon_x = r_a - d(x, a)$. We now claim that $B(x, \varepsilon_x) \subseteq U$. Indeed, let $y \in B(x, \varepsilon_x)$ i.e. $d(y, x) < \varepsilon_x$. Then by the triangle inequality $d(y, a) \leq d(y, x) + d(x, a) < \varepsilon_x + d(x, a) = r_a \Rightarrow y \in B(a, r_a) \subseteq U$. □

Examples.

(1) $\emptyset$ is open
(2) F is open
(3) An open ball is open
(4) $F = \mathbb{R}$, $(a, b) \subseteq \mathbb{R}$ is open since $(a, b) = B\left(\frac{a+b}{2}, \frac{b-a}{2}\right)$
(5) $F = \mathbb{C}$, $A = (a, b) \times (c, d) = \{z \in \mathbb{C} : a < \operatorname{Re} z < b, c < \operatorname{Im} z < d\}$ is open. For any $z \in A$ let $\varepsilon_z = \min\{\operatorname{Re} z - a, b - \operatorname{Re} z, \operatorname{Im} z - c, d - \operatorname{Im} z\} > 0$, then $B(z, \varepsilon_z) \subseteq A$ (**exercise**).
(6) $F = \mathbb{R}$, $[a, b] \subseteq \mathbb{R}$ is not open. We have $a \in [a, b]$, but for any $\varepsilon > 0$ $B(a, \varepsilon) = (a - \varepsilon, a + \varepsilon) \not\subseteq [a, b]$.

Definition 7. *A subset $X \subseteq F$ is* **closed** *if its complement $X^c = F \backslash X$ is open.*

Remark 8. *There are sets that are open, closed, both, neither. This means that the properties of open and closed should always be checked* **separately**, *as they are not mutually exclusive.*

Examples.

(1) $\emptyset$ is closed, since $\emptyset^c = F$ which is open
(2) F is closed since $F^c = \emptyset$ which is open
(3) $F = \mathbb{R}$, $[a, b] \subseteq \mathbb{R}$ is closed since $[a, b]^c = (-\infty, a) \cup (-\infty, b)$ which is open (**exercise**).
(4) $F = \mathbb{R}$ or $\mathbb{C}$, the closed disk defined as

$$D(a, r) = \{z \in F : d(z, a) \leq r\}$$

is closed.

Proof. $D(a, r)^c = \{x \in F : d(x, a) > r\} = \cup_{z\in D^c} B(z, d(z, a) - r) \Rightarrow D(a, r)^c$ is open. □

Topology.

Definition 9. *A collection τ of subsets of F forms a* **topology** *on F if*

(1) $\emptyset, F \in \tau$
(2) $U_\alpha \in \tau \ \forall \alpha \in I \Rightarrow \cup_{\alpha \in I} U_\alpha \in \tau$
(3) $U_\alpha \in \tau \ \forall \alpha \in I$ and I is finite $\Rightarrow \cap_{\alpha \in I} U_\alpha \in \tau$

Proposition 10. *The collection of open sets on $F = \mathbb{R}$ or $\mathbb{C}$ forms a topology on F. More precisely,*

(1) $\emptyset$ and F are open
(2) arbitrary unions of open sets are open
(3) finite intersections of open sets are open

Proof. We will show part (3), the rest is an **exercise**. Let $U_1, \ldots, U_n$ be open, and consider $U := \cap_{k=1}^n U_k$. To show U is open we will show that for any point $a \in U$ $\exists r_a > 0$ s.t. $B(a, r_a) \subseteq U$. Let $a \in U \Rightarrow a \in U_k \ \forall k = 1, \ldots, n$. Since each U_k is open we have $\forall k = 1, \ldots, n \ \exists r_k > 0$ s.t. $B(a, r_k) \subseteq U_k$. Define $r_a = \min\{r_1, r_2, \ldots, r_n\} > 0$, then $B(a, r_a) \subseteq B(a, r_k) \subseteq U_k \ \forall k = 1, \ldots, n \Rightarrow B(a, r_a) \subseteq \cap_{k=1}^n U_k = U$, which shows U is open. □

Question. *Why did we have to require that there are only finitely many sets? What happens to the proof if we consider an infinite collection of sets $U_1, U_2, \ldots$?*

INDUCTION

Suppose we want to prove that some statement P is true for all integers n that are greater or equal to some fixed integer n_0 (often $n_0 = 0$ or 1). (P_n $\forall n \geq n_0$). The method of proof, known as **mathematical induction** consists of two steps:

Step 1 (base case): Show that P holds for n_0 (P_{n_0}).
Step 2 (inductive step): Assume that P holds for some $n \geq n_0$ (P_n for some $n \geq n_0$). Show that P holds for $n+1$ (P_{n+1}).

Why does it prove P for all $n \geq n_0$? First of all, in step 1 you show P_{n_0} and then step 2 gives you P_{n_0+1}. Then you can apply step 2 again for $n = n_0 + 1$ and conclude P_{n_0+2}. You can continue this way to conclude P_n for all $n \geq n_0$.

Examples.

(1) Let $s_1 = 1$ and define recursively $s_{n+1} = 2s_n$ $\forall n \geq 1$. What is a general formula for s_n in terms of n? First we calculate a few first terms: $s_2 = 2s_1 = 2$, $s_3 = 2s_2 = 2^2$, $s_4 = 2s_3 = 2^3$, ... This suggests $s_n = 2^{n-1}$ $\forall n \geq 1$, which we now prove by induction. We denote P_n the statement that $s_n = 2^{n-1}$.
Base case: $n_0 = 1$, we know $s_1 = 1 = 2^0 = 2^{1-1}$, so the formula holds for $n = 1$.
Induction step: assume $s_n = 2^{n-1}$ for some $n \geq 1$. We then have $s_{n+1} = 2s_n$ (by definition), thus by induction assumption $s_{n+1} = 2s_n = 2 \cdot 2^{n-1} = 2^n = 2^{(n+1)-1}$ which is P_{n+1}.

(2) Prove that $\forall n \geq 1$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Proof. Base case: let $n_0 = 1$. Then

$$\sum_{k=1}^{1} k = 1 = \frac{1(1+1)}{2}$$

which is P_1.
Inductive step: assume $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ (P_n) for some $n \geq 1$. Then

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1) = (by\ P_n) = \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2},$$

which is P_{n+1}. □

(3) Prove that $\forall x \in \mathbb{R}$ and $\forall n \geq 1$

$$(1-x)\sum_{k=0}^{n} x^k = 1 - x^{n+1}$$

Proof. Base case: Let $n_0 = 1$, then

$$(1-x)\sum_{k=0}^{1} x^k = (1-x)(1+x) = 1 - x^2 = 1 - x^{1+1},$$

which is P_1.
Inductive step: Assume that for some $n \geq 1$ we have P_n, i.e.

$$(1-x)\sum_{k=0}^{n} x^k = 1 - x^{n+1}.$$

Then we have

$$(1-x)\sum_{k=0}^{n+1} x^k = (1-x)\sum_{k=0}^{n} x^k + (1-x)x^{n+1} = (by\ P_n) = 1 - x^{n+1} + x^{n+1} - x^{n+2} = 1 - x^{n+2},$$

which is P_{n+1}. □

(4) Prove that $2^{n-1} \leq n! \ \ \forall n \geq 1$

Proof. Base case: Let $n_0 = 1$, then

$$2^{n_0 - 1} = 2^0 = 1 \leq 1 = 1!,$$

which is P_1. Inductive step: assume $2^{n-1} \leq n!$ for some $n \geq 1$, then

$$\begin{aligned} 2^n = 2^{n-1} \cdot 2 &\leq n! \cdot 2 && \text{by } P_n \\ &\leq n! \cdot (n+1) && \text{since } n \geq 1 \Rightarrow n+1 \geq 2 \\ &= (n+1)! && \text{which is } P_{n+1}. \end{aligned}$$

□

(5) Prove that $\prod_{k=2}^{n}(1 - 1/k) = 1/n \ \ \forall n \geq 2$.

Proof. Base case: $n_0 = 2$, then

$$\prod_{k=2}^{2}\left(1 - \frac{1}{k}\right) = 1 - \frac{1}{2} = \frac{1}{2},$$

which is P_2. Inductive step: suppose $\prod_{k=2}^{n}(1 - 1/k) = 1/n$ (P_n) for some $n \geq 2$. Then

$$\prod_{k=2}^{n+1}\left(1 - \frac{1}{k}\right) = \prod_{k=2}^{n}\left(1 - \frac{1}{k}\right) \cdot \left(1 - \frac{1}{n+1}\right) = \frac{1}{n}\left(1 - \frac{1}{n+1}\right) = \frac{1}{n} \cdot \frac{n}{n+1} = \frac{1}{n+1},$$

which is P_{n+1}. □

(6) There are exactly n people at a gathering. Everybody shakes everybody else's hand exactly once. How many handshakes are there?

SOLUTION: Let a_n be the number of handshakes between n people. We first need to come up with a formula for a_n, and then prove it by induction. We start by calculating first few values: $a_1 = 0$, $a_2 = 1$, $a_3 = 3$, $a_4 = 6$. The pattern is not immediately clear, but we can reason in the following way: suppose there are n people who all shook each other's hands. If another person enters the room and shakes everyone hands, then there are extra n handshakes. This gives a recursive formula

$$a_{n+1} = a_n + n. \tag{$\star$}$$

Using this formula repeatedly we obtain

$$a_{n+1} = a_n + n = a_{n-1} + n - 1 + n = a_{n-2} + (n-2) + (n-1) + n = \cdots = a_1 + 1 + 2 + \cdots + n = \sum_{k=1}^{n} k.$$

The result in example (2) thus leads to the following assumption

$$a_{n+1} = \frac{n(n+1)}{2},$$

or

$$a_n = \frac{(n-1)n}{2} \quad (P_n).$$

We will now prove P_n $\forall n \geq 1$. Base case: $n_0 = 1$, then $a_1 = 0 = (1-1)1/2$ so P_{n_0} is true. Inductive step: assume P_n is true for some $n \geq 1$. Then using $(\star)$ we have

$$a_{n+1} = a_n + n = \frac{(n-1)n}{2} + n = \frac{(n-1)n + 2n}{2} = \frac{n(n+1)}{2},$$

which is P_{n+1}.

SEQUENCES

Definition 1. *An* **infinite sequence** *is a function* $s : \mathbb{N} \to \mathbb{C}$ *(domain* $= \mathbb{N}$*, co-domain* $= \mathbb{C}$*). Notation: write* $s_n = s(n)$ *— the* n^{th} *term of the sequence,* $s = \{s_1, s_2, \dots\} = \{s_n\}_{n \geq 1}$ *— the sequence.*

Examples.

(1) $\{c\}$, $c_n = c \ \forall n \in \mathbb{N}$ — constant sequence
(2) $\{1/n\} = \{1, 1/2, 1/3, \dots\}$
(3) $\{(-1)^n\} = \{-1, 1, -1, 1, \dots\}$
(4) $\{i^n\} = \{i, -1, -i, 1, i, -1, -i, 1, \dots\}$
(5) Recursively defined $s_1 = 1$, $s_{n+1} = 2s_n \ \forall n \geq 1$. We have shown that $s_n = 2^{n-1}$.
(6) $\{k\}_{k \in \mathbb{Z}}$ is not a sequence, since the domain is $\mathbb{Z}$ not $\mathbb{N}$.

Arithmetic operations. Let $\{s_n\}, \{t_n\}$ infinite sequences, and $c \in \mathbb{C}$. Define

(1) $\{s_n\} \pm \{t_n\} = \{s_n \pm t_n\}$
(2) $\{s_n\} \cdot \{t_n\} = \{s_n \cdot t_n\}$
(3) $c\{s_n\} = \{cs_n\}$
(4) $\{s_n\}/\{t_n\} = \{s_n/t_n\}$ if $t_n \neq 0 \ \forall n \in \mathbb{N}$.

Limits of sequences

Definition 2. *Let* $\{a_n\}$ *be a sequence of complex numbers, and* $a \in \mathbb{C}$*. We say that* $\{a_n\}$ **converges to** a*, or* $\lim_{n\to\infty} a_n = a$ *(*$a_n \to a$ *as* $n \to \infty$*), if*

$$\forall \varepsilon > 0 \ \ \exists N \in \mathbb{N} \ \ s.t. \ \ |a - a_n| < \varepsilon \ \ \forall n \geq N.$$

a is called a **limit** *of* $\{a_n\}$*. The sequence is called* **convergent** *if it has a limit. Otherwise, it is called* **divergent***.*

Remarks.

(1) N depends on ε. Usually, for smaller ε you need a larger N
(2) $|a_n - a| < \varepsilon \iff a_n \in B(a, \varepsilon)$. Thus the condition $\lim_{n\to\infty} a_n = a$ can be re-written as

$$\forall \varepsilon > 0 \ \ \exists N \in \mathbb{N} \ \text{ s.t. } \ a_n \in B(a, \varepsilon) \ \ \forall n \geq N.$$

(3) If $\lim_{n\to\infty} a_n = a$, then given $\varepsilon > 0$ all but finitely many a_n's are in the ε-ball around a.

Examples.

(1) $\{c\}$ converges to c

Proof. Let $\varepsilon > 0$, choose $N = 1$. Then if $n \geq 1$ we have $|c_n - c| = |c - c| = 0 < \varepsilon$. □

(2) $\lim_{n\to\infty} 1/n = 0$

Proof. Let $\varepsilon > 0$, choose N to be any integer satisfying $N > 1/\varepsilon$. Then if $n \geq N$ we have $n > 1/\varepsilon$, and therefore

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon.$$

□

(3) $\lim_{n\to\infty} c/n = 0 \ \forall c \in \mathbb{C}$.

Proof. If $c = 0$ then we have a constant sequence $\{0\}$ and by example (1) it converges to 0. If $c \neq 0$, then $|c| > 0$. Let $\varepsilon > 0$ and choose any integer $N > |c|/\varepsilon$. If $n \geq N$ we have

$$\left|\frac{c}{n} - 0\right| = \frac{|c|}{n} \leq \frac{|c|}{N} < \frac{|c|}{|c|/\varepsilon} = \varepsilon.$$

□

Theorem 3. *If a sequence converges, its limit is unique.*

Proof. Suppose $s_n \to s$ and $s_n \to s'$. This means that for every $\varepsilon > 0$ we can find natural numbers N_1 and N_2 s.t.

$$|s_n - s| < \frac{\varepsilon}{2}, \ \ \forall n \geq N_1$$
$$|s_n - s'| < \frac{\varepsilon}{2}, \ \ \forall n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$, then we have both $N \geq N_1$ and $N \geq N_2$. Therefore, using the triangle inequality we have

$$|s - s'| = |(s - s_N) + (s_N - s')| \leq |s - s_N| + |s_N - s'| < \varepsilon.$$

This shows $|s - s'| < \varepsilon \ \forall \varepsilon > 0$. We now claim that this implies $|s - s'| = 0$. Indeed, assume the contrary $|s - s'| > 0$ and choose $\varepsilon = |s - s'|/2$. Then $|s - s'| < |s - s'|/2$ $\Rightarrow |s - s'|/2 < 0 \Rightarrow |s - s'| < 0$ which is a contradiction since the modulus is always non-negative. Thus, $|s - s'| = 0 \Rightarrow s = s'$. □

Theorem 4. *Let $\alpha \in \mathbb{C}$, and consider the sequence $\{\alpha^n\}$. There are three cases*

(1) If $\alpha = 1$, then $\alpha^n \to 1$ as $n \to \infty$
(2) If $|\alpha| < 1$, then $\alpha^n \to 0$ as $n \to \infty$
(3) If $\alpha \neq 1$ and $|\alpha| \geq 1$, then $\{\alpha^n\}$ diverges.

Proof. (1) Let $\alpha = 1$, then $\alpha^n = 1\ \forall n \in \mathbb{N}$, and therefore $\{\alpha^n\} = \{1\}$ a constant sequence which converges to 1 by Example (1).

(2) Let $|\alpha| < 1$. If $\alpha = 0$ then $\{\alpha^n\} = \{0\}$ is again a constant sequence, which converges to 0. If $\alpha \neq 0$, then $|\alpha| > 0$. Moreover, since $|\alpha| < 1$, we have $\ln|\alpha| < 0$. Now let $\varepsilon > 0$. Assume first that $\varepsilon < 1$, which in particular implies $\ln\varepsilon < 0$, so $\frac{\ln\varepsilon}{\ln|\alpha|} > 0$. Take N to be any integer strictly bigger than $\frac{\ln\varepsilon}{\ln|\alpha|}$, $N > \frac{\ln\varepsilon}{\ln|\alpha|} > 0$. Then if $n \geq N$ we have

$$\begin{aligned} n > \frac{\ln\varepsilon}{\ln|\alpha|} \ &\Rightarrow\ n\ln|\alpha| < \ln\varepsilon \ (\text{since } \ln|\alpha| < 0) \\ e^{n\ln|\alpha|} &< n^{\ln\varepsilon} \\ |\alpha|^n &< \varepsilon \\ \big||\alpha|^n - 0\big| &< \varepsilon. \end{aligned}$$

In the case $\varepsilon \geq 1$ it is sufficient to take $N = 1$ and is left as an **exercise**.

(3) Let $\alpha \neq 1$ and $|\alpha| \geq 1$. We will prove that $\{\alpha^n\}$ diverges by contradiction. Suppose that $\{\alpha^n\}$ converges, i.e. $\exists a \in \mathbb{C}$ s.t. $\lim_{n\to\infty}\alpha^n = a$, which means

$$(*) \qquad \forall\varepsilon > 0\ \ \exists N \in \mathbb{N}\ \text{ s.t. }\ |\alpha^n - a| < \varepsilon\ \ \forall n \geq N.$$

Let $\varepsilon = \frac{|\alpha-1|}{2}$ which is strictly positive since $\alpha \neq 1$. Then by (*) there exists $N \in \mathbb{N}$ s.t.

$$|\alpha^n - a| < \frac{|\alpha-1|}{2}\ \ \forall n \geq N.$$

We also have $|\alpha^{n+1} - a| < \frac{|\alpha-1|}{2}$ for such n, since $n+1 > n$. Using the triangle inequality we then obtain

$$|\alpha^{n+1} - \alpha^n| = |\alpha^{n+1} - a + a - \alpha^n| \leq |\alpha^{n+1} - a| + |a - \alpha^n| < |\alpha - 1|.$$

On the other hand, since $|\alpha| \geq 1$ we have

$$|\alpha^{n+1} - \alpha^n| = |\alpha^n(\alpha - 1)| = |\alpha|^n|\alpha - 1| \geq |\alpha - 1|.$$

We thus showed

$$|\alpha - 1| \leq |\alpha^{n+1} - \alpha^n| < |\alpha - 1|,$$

which is a contradiction.

□

Examples.

(1) $\{(-1)^n\}$ diverges. Indeed, we have $\alpha = -1$, so $\alpha \neq 1$ and $|\alpha| = 1$ which is case (3). Thus $\{(-1)^n\}$ diverges.

(2) $\{(1/2+i/4)^n\}$ converges to 0. In this case we have

$$\alpha = \frac{1}{2} + \frac{i}{4} \quad \Rightarrow \quad |\alpha| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2} = \sqrt{\frac{5}{16}} < 1,$$

which is case (2) of the Theorem.

Divergence. Recall that $a_n \to a$ if

$$\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\text{ s.t. }\; \forall n \geq N \;\; |\alpha^n - a| < \varepsilon.$$

Therefore to say $a_n \not\to a$ we need to negate the above statement to obtain

$$\exists \varepsilon > 0 \;\; \forall N \in \mathbb{N} \;\; \exists n \geq N \;\text{ s.t. }\; |\alpha^n - a| \geq \varepsilon.$$

This motivates the following definition

Definition 5. *The sequence* $\{a_n\}$ **diverges** *if*

$$\forall a \in \mathbb{C} \;\; \exists \varepsilon > 0 \;\; \forall N \in \mathbb{N} \;\; \exists n \geq N \;\; \textit{s.t.} \;\; |\alpha^n - a| \geq \varepsilon.$$

In other words, it does not converge to any number $a \in \mathbb{C}$.

Example. *Consider* $\{n\}$ *we will show that the sequence diverges. Let* $a \in \mathbb{C}$ *be any complex number, choose* $\varepsilon = 1$. *For every* $N \in \mathbb{N}$ *choose an integer* $n \geq N + 1 + |a|$, *then we have using the triangle inequality*

$$n = |n| = |n - a + |a|| \leq |n-a| + |a| \quad \Rightarrow \quad |n-a| \geq n - |a| \geq N + 1 + |a| - |a| = N + 1 > 1 = \varepsilon.$$

Bounded sequences.

Definition 6. *A sequence* $\{s_n\}$ *of complex numbers is* **bounded** *if*

$$\exists M > 0 \;\; \textit{s.t.} \;\; |s_n| \leq M \;\; \forall n \in \mathbb{N} \;\; (\textit{equivalently} \;\; s_n \in B(0, M) \;\; \forall n \in \mathbb{N}).$$

Theorem 7. $\{s_n\}$ *converges* $\Rightarrow$ $\{s_n\}$ *is bounded.*

Proof. Suppose $s_n \to s$ as $n \to \infty$. Let $\varepsilon = 1$, there exists $N \in \mathbb{N}$ s.t. $s_n \in B(s, 1)$ $\forall n \geq N$ (*). Let

$$M = \max\{|s_1|, |s_2|, \ldots, |s_N|, 1 + |s|\}.$$

Then

$|s_n| \leq M \;\; \forall n \leq N$ by definition of M,

$|s_n| \leq |s_n - s| + |s| < 1 + |s| \leq M \;\; \forall n \geq N$ using the triangle inequality, (*), and definition of M

Thus, $|s_n| \leq M$ $\forall n \in \mathbb{N}$. □

Remark 8. *Note that the reverse implication is not true, for example* $(-1)^n$ *is bounded but does not converge.*

Limit theorems.

Theorem 9. *Suppose that* $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$, *and* $c \in \mathbb{C}$. *Then*

(1) $\lim(s_n + t_n) = \lim s_n + \lim t_n = s + t$
(2) $\lim(s_n t_n) = \lim s_n \cdot \lim t_n = st$
(3) If $t_n \neq 0\ \forall n \in \mathbb{N}$ *and* $t \neq 0$, *then* $\lim \dfrac{s_n}{t_n} = \dfrac{\lim s_n}{\lim t_n} = \dfrac{s}{t}$
(4) $\lim(cs_n) = cs$

Proof. We will show (2), the rest is an **exercise**. Let $\varepsilon > 0$, we WANT TO SHOW:

(WTS) $$\exists N \in \mathbb{N} \text{ s.t. } |s_n t_n - st| < \varepsilon \ \ \forall n \geq N.$$

Define

$$\eta = \min\left\{1, \frac{\varepsilon}{1 + |s| + |t|}\right\},$$

and note that from the definition $0 < \eta \leq 1$. We have

$$\begin{aligned} s_n \to s &\ \Rightarrow\ \exists N_1 \in \mathbb{N} \text{ s.t. } |s_n - s| < \eta \ \ \forall n \geq N_1 \\ t_n \to t &\ \Rightarrow\ \exists N_2 \in \mathbb{N} \text{ s.t. } |t_n - t| < \eta \ \ \forall n \geq N_2. \end{aligned}$$

Let $N = \max\{N_1, N_2\}$, then $\forall n \geq N$ $|s_n - s| < \eta$ and $|t_n - t| < \eta$ (*). Next it is an **exercise** to check that

$$s_n t_n - st = (s_n - s)(t_n - t) - s(t - t_n) - t(s - s_n).$$

Therefore, $\forall n \geq N$

$$\begin{aligned} |s_n t_n - st| &\leq |s_n - s||t_n - t| + |s||t - t_n| + |t||s - s_n| && \text{by triangle inequality} \\ &< \eta^2 + |s|\eta + |t|\eta && \text{by (*)} \\ &= \eta(\eta + |s| + |t|) \\ &\leq \eta(1 + |s| + |t|) && \text{since } \eta < 1 \\ &\leq \varepsilon && \text{since } \eta \leq \frac{\varepsilon}{1 + |s| + |t|}. \end{aligned}$$

□

Examples.

(1) $\lim \dfrac{n^3}{3n^3 - n} = \dfrac{1}{3}$

Proof.

$$\begin{aligned}
\lim \frac{n^3}{3n^3 - n} = \lim \frac{\frac{1}{n^3}}{\frac{1}{n^3}} \cdot \frac{n^3}{3n^3 - n} &= \lim \frac{1 + \frac{2}{n^2}}{3 - \frac{1}{n^2}} \\
&= \frac{\lim \left(1 + \frac{2}{n^2}\right)}{\lim \left(3 - \frac{1}{n^2}\right)} && \text{by (3)} \\
&= \frac{\lim 1 + \lim \frac{2}{n^2}}{\lim 3 + \lim \left(-\frac{1}{n^2}\right)} && \text{by (1)} \\
&= \frac{\lim 1 + 2 \lim \frac{1}{n} \lim \frac{1}{n}}{\lim 3 - \lim \frac{1}{n} \lim \frac{1}{n}} && \text{by (2) and (4)} \\
&= \frac{1}{3} && \text{by constant seq. and 1/n}
\end{aligned}$$

□

(2) $\lim \dfrac{n+1}{2n^2} = \lim \dfrac{\frac{1}{n^2}(n+1)}{\frac{1}{n^2} \cdot 2n^2} = \lim \dfrac{\frac{1}{n} + \frac{2}{n^2}}{2} = 0.$

(3) $\lim \frac{(-1)^n n}{n+1}$ does not exist, the sequence diverges

Proof. Assume the contrary: $\exists L \in \mathbb{C}$ s.t. $\lim \frac{(-1)^n n}{n+1} = L$, or equivalently $\lim \frac{(-1)^n}{1+\frac{1}{n}} = L$. Consider two sequences $\{s_n\} = \left\{\frac{(-1)^n}{1+\frac{1}{n}}\right\}$ and $\{t_n\} = \left\{1 + \frac{1}{n}\right\}$. By assumption, $s_n \to L$, and using the limit theorem (1), $t_n \to 1$. Thus, by part (2) of the limit theorem we have

$$\lim (-1)^n = \lim s_n t_n = L \cdot 1 = L,$$

which is a contradiction since $\{(-1)^n\}$ diverges. □

Real sequences

Theorem 10 (Comparison Theorem)**.** *Let* $\{s_n\}$ *and* $\{t_n\}$ *be* **real** *sequences s.t.* $s_n \to s$ *and* $t_n \to t$*. If* $\exists N \in \mathbb{N}$ *s.t.* $s_n \leq t_n \ \forall n \geq N$*, then* $s \leq t$*.*

Remark 11. *Limits do not preserve strict inequalities. I.e. it is* **NOT TRUE** *that* $s_n < t_n$ *implies* $s < t$*. Consider for example* $0 < 1/n$*, but* $\lim 0 = \lim 1/n = 0$*. However, since strict inequalities imply non-strict inequalities it is* **TRUE** *that* $s_n < t_n$ *implies* $s \leq t$*.*

Theorem 12 (Squeeze Theorem)**.** *Let* $\{a_n\}$*,* $\{b_n\}$*, and* $\{c_n\}$ *be* **real** *sequences s.t.* $a_n \to L$ *and* $c_n \to L$*. Suppose* $\exists N \in \mathbb{N}$ *s.t.* $a_n \leq b_n \leq c_n \ \forall n \geq N$*. Then* $\{b_n\}$ *converges and*

$$\lim_{n\to\infty} = L.$$

Proof. Let $\varepsilon > 0$, $\exists N_0 = \max\{N_1, N_2\}$ s.t. $|a_n - L| < \varepsilon$ and $|c_n - L| < \varepsilon \ \forall n \geq N_0$. We have

$$\begin{aligned} |a_n - L| < \varepsilon \iff -\varepsilon < a_n - L < \varepsilon \ \ \forall n \geq N_0 \\ |c_n - L| < \varepsilon \iff -\varepsilon < c_n - L < \varepsilon \ \ \forall n \geq N_0 \\ a_n \leq b_n \leq c_n \ \Rightarrow \ a_n - L \leq b_n - L \leq c_n - L \ \ \forall n \geq N. \end{aligned}$$

Combining we obtain for all $n \geq \max\{N_0, N\}$

$$-\varepsilon < a_n - L \leq b_n - L \leq c_n - L < \varepsilon \ \Rightarrow \ |b_n - L| < \varepsilon.$$

□

Examples.

(1) $\lim \dfrac{\sin n}{n} = 0$, since $-1 \leq \sin n \leq 1 \ \forall n \in \mathbb{N} \Rightarrow -\dfrac{1}{n} \leq \dfrac{\sin n}{n} \leq \dfrac{1}{n}$ and $-\dfrac{1}{n}, \dfrac{1}{n} \to 0$.

(2) $\lim \dfrac{n}{n!} = 0$, since $\forall n \geq 2$

$$0 \leq \frac{n}{n!} = \frac{1}{(n-1)!} \leq \frac{1}{n-1} \to 0.$$

Monotone Convergence Theorem.

Definition 13. *A sequence of real numbers* $\{s_n\}$ *is called*
increasing *(non-decreasing) if* $s_n \leq s_{n+1} \ \forall n \in \mathbb{N}$
decreasing *(non-increasing) if* $s_n \leq s_{n+1} \ \forall n \in \mathbb{N}$*.*

Examples.

(1) $\{1/n\}$, $\{-n\}$ are decreasing
(2) $\{n\}$, n^3 are increasing
(3) $\{c\}$ is both decreasing and increasing
(4) $(-1)^n n$, $\sin n/n$ are neither

Theorem 14 (MCT)**.** *Let $\{s_n\}$ be a sequence of real numbers. If $\exists N \in \mathbb{N}$ s.t. $\{s_n\}_{n\geq N}$ is increasing and bounded above, then $\{s_n\}$ converges and*

$$\lim s_n = \sup\{s_n : n \geq N\}.$$

If $\exists N \in \mathbb{N}$ s.t. $\{s_n\}_{n\geq N}$ is decreasing and bounded below, then $\{s_n\}$ converges and

$$\lim s_n = \inf\{s_n : n \geq N\}.$$

Example. *Define $\{s_n\}$ recursively as follows:*

$$s_1 = 1,\ s_{n+1} = \sqrt{1 + s_n}\ \ \forall n \geq 1.$$

We will first prove that $\{s_n\}$ converges in three steps.

(1) $\{s_n\} \nearrow$, i.e. $s_n \leq s_{n+1}\ \forall n \in \mathbb{N}$.

Proof. We will show this by induction. First, $s_2 = \sqrt{2} > 1 = s_1 \Rightarrow$ base case. Now assume $s_n \leq s_{n+1}$ for some $n \geq 1$. Then

$$s_{n+2} = \sqrt{1 + s_{n+1}} \leq \sqrt{1 + s_n} = s_{n+1},$$

which concludes the inductive step. □

(2) $0 < s_n \leq 2\ \forall n \geq 1$.

Proof. $0 < s_n$ is an **exercise**. We will show $s_n \leq 2$ by induction. First, $s_1 = 1 \leq 2$ $\Rightarrow$ base case. Now assume $s_n \leq 2$ for some $n \geq 1$. Then

$$s_{n+1} = \sqrt{1 + s_n} \leq \sqrt{1 + 2} = \sqrt{3} \leq 2,$$

which concludes the inductive step. □

(3) Using steps (1) and (2) we conclude by MCT $s_n \to s$.

(4) To find s we will use the following result without proof: if $t_n \to t$ and $t_n \geq 0\ \forall n$, then $\sqrt{t_n} \to \sqrt{t}$. We have shown that $s_n \to s$, and thus $1 + s_n \to 1 + s$. By a HW question we also have $s_{n+1} \to s$. Combining this we obtain

$$s = \lim s_n = \lim s_{n+1} = \lim \sqrt{1 + s_n} = \sqrt{1 + s},$$

which shows that the limit s must satisfy the quadratic equation

$$s^2 - s - 1 = 0.$$

Selecting the positive root gives

$$s = \frac{1 + \sqrt{5}}{2}.$$

Divergence to infinity.

Definition 15. *We say that a real sequence* $\{s_n\}$ **diverges to** ∞ *if*

$$\forall M > 0 \;\; \exists N \in \mathbb{N} \;\; s.t. \;\; s_n \geq M \;\; \forall n \geq N.$$

Write $\lim s_n = \infty$. *Similarly,* $\{s_n\}$ **diverges to** $-\infty$ *if*

$$\forall M > 0 \;\; \exists N \in \mathbb{N} \;\; s.t. \;\; s_n \leq -M \;\; \forall n \geq N.$$

Theorem 16. *Let* $\{s_n\}, \{t_n\}$ *be real sequences and suppose* $\exists N \in \mathbb{N}$ *s.t.* $s_n \leq t_n$ $\forall n \geq N$. *Then*

(1) If $\lim s_n = \infty$ *then* $\lim t_n = \infty$
(2) If $\lim t_n = -\infty$ *then* $\lim s_n = -\infty$.

Example. $\lim \frac{n^2-3}{n} = \infty$ *since*

$$\frac{n^2-3}{n} = n - \frac{3}{n} \geq n - 1 \; \forall n \geq 3,$$

and $\lim(n-1) = \infty$ *(***exercise***).*

Cauchy sequences

Definition 17. *A sequence of complex (or real) numbers,* $\{s_n\}$, *is called a* **Cauchy sequence** *if*

$$\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\; s.t. \;\; |s_n - s_m| < \varepsilon \;\; \forall n, m \geq N.$$

Theorem 18. *Let* $\{s_n\}$ *be a sequence of complex or real numbers.*

(1) $\{s_n\}$ *converges* $\Rightarrow$ $\{s_n\}$ *is Cauchy.*
(2) $\{s_n\}$ *is Cauchy* $\Rightarrow$ $\{s_n\}$ *converges.*

Proof. We will only prove part (1). Suppose $s_n \to s$ as $n \to \infty$. Then $\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ s.t. $|s_n - s| < \varepsilon/2 \; \forall n \geq N$. If $n, m \geq N$ we have using the triangle inequality

$$|s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s_m - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Example. *Consider* $\{(-1)^n\}$. *For any* $N \in \mathbb{N}$ *let* $n = 2N, m = 2N+1$. *We have* $n, m \geq N$ *but*

$$|(-1)^n - (-1)^m| = |(-1)^{2N} - (-1)^{2N+1}| = 2,$$

which cannot be smaller than 1 *no matter how large* N *is. This shows* $\{(-1)^n\}$ *is not Cauchy and therefore does not converge.*

SERIES

Definition 1. *Let $\{a_k\}_{k\in\mathbb{N}}$ be a sequence of complex numbers. The* **$\mathbf{n^{th}}$ partial sum** *of $\{a_k\}$ is $s_n := \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$. The* **infinite series** *with k^{th} term a_k is the sequence of partial sums $\{s_n\}_{n\in\mathbb{N}}$. Notation: $\sum_{k=1}^{\infty} a_k$.*

Definition 2. *The series $\sum_{k=1}^{\infty} a_k$* **converges** *to $s \in \mathbb{C}$, if the sequence of partial sums $\{s_n\}$ converges to s. Write $\sum_{k=1}^{\infty} a_k = s$. The series $\sum_{k=1}^{\infty} a_k$* **diverges** *if $\{s_n\}$ diverges.*

Example. *Consider the series*

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}, \quad a_k = \frac{1}{k(k+1)}.$$

To show the series converges, consider the sequence of partial sums

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}.$$

To figure out a formula for s_n first expand a_k in partial fractions

$$a_k = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1},$$

which implies

$$s_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}.$$

This is called a **telescoping sum**. *It is an* **exercise** *to prove this formula by induction. With that we have*

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 + \frac{1}{n+1}\right) = 1,$$

so the series converges to 1: $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$.

Arithmetic. If the series $\sum a_k$ and $\sum b_k$ converge, $\sum a_k = a$ and $\sum b_k = b$, then we can define

$$\sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} (a_k + b_k) = a + b.$$

However, if one of the series (or both) diverges, we cannot conclude anything about $\sum_{k=1}^{\infty}(a_k + b_k)$. Recall, that the same is true about sums of sequences.

Geometric series

Definition 3. *A* **geometric series** *is a series of the form*

$$\sum_{k=1}^{\infty} r^{k-1}$$

for some $r \in \mathbb{C}$.

Remark 4. *Note that*

$$\sum_{k=1}^{\infty} r^{k-1} = 1 + r + r^2 + \cdots = \sum_{k=0}^{\infty} r^k.$$

It will often be convenient to start the series at $k = 0$.

Theorem 5. *Let* $r \in \mathbb{C}$. *Then*

(1) If $|r| < 1$, *then* $\displaystyle\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$

(2) If $|r| \geq 1$, *then* $\displaystyle\sum_{k=0}^{\infty} r^k$ *diverges.*

Proof. Recall that we showed by induction that

$$(1-r)\sum_{k=0}^{n} r^k = 1 - r^{n+1} \quad \forall n \geq 0.$$

Now assume first $|r| < 0$, which implies $1 - r \neq 0$ so by above

$$s_n = \sum_{k=0}^{n} r^k = 1 = \frac{1 - r^{n+1}}{1-r}.$$

Since $|r| < 1$ we showed that $\lim_{n\to\infty} r^n = 0$, and thus using limit theorems

$$\lim_{n\to\infty} s_n = \frac{1}{1-r} \lim_{n\to\infty} (1 - r^{n+1}) = \frac{1}{1-r}.$$

This proves part (1).
To prove (2) consider first the case $r = 1$. In this case $s_n = \sum_{k=0}^{n} 1 = n + 1 \to \infty$ as $n \to \infty$, so the series diverges.
Finally, assume that $r \neq 1$ and $|r| \geq 1$. Then $1 - r \neq 0$ and as before we obtain

$$s_n = \frac{1 - r^{n+1}}{1-r}.$$

We show that s_n diverges by contradiction. Assume $s_n \to s$ for some $s \in \mathbb{C}$. Then expressing r^n via s_n from the above equality we obtain

$$r^n = \frac{1}{r}(1-(1-r)s_n) \to \frac{1}{r}(1-(1-r)s) \text{ using limit theorems.}$$

This is a contradiction since we know that the sequence $\{r^n\}$ diverges for $|r| \geq 1$, $r \neq 1$. □

Examples.

(1) $\displaystyle\sum_{k=0}^{\infty}\left(\frac{i}{2}\right)^k = \frac{1}{1-i/2} = \frac{2}{2-i} = \frac{2(2+i)}{5}.$

(2) $\displaystyle\sum_{k=0}^{\infty} 4^k$ diverges since $4 > 1$.

(3) $\displaystyle\sum_{k=5}^{\infty}\left(\frac{2}{3}\right)^k$. Note that the summation starts with 5 so we need to be a little bit careful. We have

$$\sum_{k=5}^{\infty}\left(\frac{2}{3}\right)^k = \left(\frac{2}{3}\right)^5 + \left(\frac{2}{3}\right)^6 + \cdots = \left(\frac{2}{3}\right)^5\left(1+\frac{2}{3}+\ldots\right)$$
$$= \left(\frac{2}{3}\right)^5 \sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k = \left(\frac{2}{3}\right)^5 \frac{1}{1-2/3} = 3\left(\frac{2}{3}\right)^5.$$

Harmonic series

The **harmonic series** is $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$.

Theorem 6. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ *diverges to* ∞.

Proof. Let $h_n = \sum_{k=1}^{n}\frac{1}{k}$. We claim $h_n \to \infty$ as $n \to \infty$, i.e.

$$\forall M > 0 \;\; \exists N \in \mathbb{N} \text{ s.t. } h_n \geq M \;\; \forall n \geq N.$$

Fix $M > 0$, and let m be any integer $\geq 2M$. Choose $N = 2^m$, we claim that $h_n \geq m/2 \geq M \; \forall n \geq N$. First notice that $h_{n+1} = h_n + \frac{1}{n+1} > h_n$, therefore, it is enough to show $h_N \geq m/2$. We will prove by induction that $h_{2^n} \geq n/2 \; \forall n \geq 1$, which will show that $h_N = h_{2^m} \geq m/2 \geq M$. First, if $n = 1$ we have

$$h_{2^n} = h_2 = 1 + \frac{1}{2} \geq \frac{1}{2} = \frac{n}{2},$$

which shows the base case. Now assume $h_{2^n} \geq n/2$ for some $n \geq 1$. Then

$$h_{2^{n+1}} = h_{2^n} + \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} \geq h_{2^n} + 2^n \frac{1}{2^{n+1}} \geq \frac{n}{2} + \frac{1}{2} = \frac{n+1}{2}.$$

This concludes the induction proof that $h_{2^n} \geq n/2 \; \forall n \in \mathbb{N}$, which shows in particular

$$h_N = h_{2^m} \geq \frac{m}{2} \geq M.$$

□

Remark 7. *Note that the harmonic series diverges to infinity, even though the k^{th} term $1/k \to 0$ as $k \to \infty$. The condition $a_k \to 0$ is therefore not sufficient for convergence of the series. However, the next theorem shows that it is necessary.*

Theorem 8. *If the series $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.*

Proof. The sequence $\{s_n\} = \{\sum_{k=1}^{n} a_k\}$ converges, and therefore is Cauchy. In particular,

$$\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\text{ s.t. }\; |s_n - s_{n+1}| < \varepsilon \;\; \forall n \geq N.$$

Since $s_{n+1} - s_n = a_{n+1}$ this shows $\lim_{n\to\infty} a_{n+1} = 0$, and thus $\lim a_n = 0$. □

Examples. Using Theorem 8 it is easy to see that all of the following series diverge

$$\sum_{k=1}^{\infty} (-1)^k, \;\; \sum_{k=1}^{\infty} (-1)^k \frac{k}{k+1}, \;\; \sum_{k=1}^{\infty} \frac{k^2+1}{2k^2+k-3}.$$

Indeed, for the first two series we have previously shown that $\{a_k\}$ diverges, so it certainly does not converge to 0. For the third example it is easy to show $a_k \to 1/2 \neq 0$.

Absolute convergence

Definition 9. *The series $\sum_{k=1}^{\infty} a_k$ is called* **absolutely convergent** *if the series $\sum_{k=1}^{\infty} |a_k|$ converges. If $\sum_{k=1}^{\infty} |a_k|$ diverges but $\sum_{k=1}^{\infty} a_k$ converges, then the series $\sum_{k=1}^{\infty} a_k$ is called* **conditionally convergent**.

Theorem 10. *Let $\sum_{k=1}^{\infty} a_k$ be absolutely convergent. Then $\sum_{k=1}^{\infty} a_k$ converges.*

Remark 11. *The converse to the theorem is not true, in other words there exist conditionally convergent series. For example, $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$ converges by the alternating series test. However, $\sum_{k=1}^{\infty} \left|(-1)^k \frac{1}{k}\right| = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges since it is the harmonic series. Therefore, $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$ converges conditionally.*

LIMITS OF FUNCTIONS

Let $F = \mathbb{R}$ or $\mathbb{C}$.

Definition 1. *Let $A \subseteq F$. Then $x \in F$ is a* **limit point** *of A if every ball centered at x contains a point of A not equal to x, i.e.*

$$\forall \varepsilon > 0 \quad \exists y \in B(x, \varepsilon) \cap A, \quad s.t. \quad y \neq x.$$

Examples.

(1) $A = (0, 1) \subseteq \mathbb{R}$, limit points $= [0, 1]$
(2) $A = (0, 1) \cup \{7\} \subseteq \mathbb{R}$, limit points $= [0, 1]$
(3) $B(0, 1) \subseteq \mathbb{C}$, limit points $= \overline{B(0,1)} = \{z \in \mathbb{C} : |z| \leq 1\}$

Definition 2. *Let $A \subseteq F$, $f : A \to F$, and $a \in F$ a limit point of A. The* **limit of f as $x \to a$ is L** *($\lim_{x \to a} f(x) = L$) if*

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad s.t. \quad 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

Equivalently,

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad s.t. \quad f\left((B(a, \delta) \backslash \{a\}) \cap A\right) \subseteq B(L, \varepsilon).$$

Examples.

(1) $\lim_{x \to 2}(5x + 3) = 13$

Proof. Let $\varepsilon > 0$, choose $\delta = \varepsilon/5$. If $0 < |x - 2| < \delta$, then

$$\begin{aligned} |f(x) - 13| &= |5x + 3 - 13| \\ &= |5x - 10| \\ &= 5|x - 2| \\ &< 5\delta = \varepsilon \end{aligned}$$

□

(2) $\lim_{x \to 4} x^2 = 16$

Proof. Let $\varepsilon > 0$, choose $\delta = \min\{1, \varepsilon/9\}$. If $0 < |x-4| < \delta$, then

$$\begin{aligned} |x^2 - 16| &= |x+4||x-4| \\ &= |x-4+8||x-4| \\ &\leq (|x-4|+8)|x-4| && \text{triangle inequality} \\ &< (\delta+8)\delta && \text{since } |x-4| < \delta \\ &\leq 9\delta && \text{since } \delta \leq 1 \\ &\leq \varepsilon && \text{since } \delta \leq \varepsilon/9. \end{aligned}$$

□

(3) $\lim_{x\to 1} \dfrac{x^2-1}{x-1} = 2$

Proof. The function $(x^2-1)/(x-1)$ is defined on $\mathbb{R}\backslash\{1\}$. Since 1 is a limit of $\mathbb{R}\backslash\{1\}$, the limit makes sense. Now given $\varepsilon > 0$, let $\delta = \varepsilon$. Then if $0 < |x-1| < \delta$ (which in particular implies $x \neq 1$), we have

$$\frac{x^2-1}{x-1} = x+1 \implies \left|\frac{x^2-1}{x-1} - 2\right| = |x+1-2| = |x-1| < \delta = \varepsilon.$$

□

(4) $\lim_{x\to 2i}(2x+i)^2 = -25$ (**exercise**)

(5) $\lim_{x\to 4}\sqrt{4} = 2$

Proof. Let $\varepsilon > 0$, choose $\delta = \varepsilon$, then if $0 < |x-4| < \delta$ we have

$$|\sqrt{x}-2| = \frac{|\sqrt{x}-2||\sqrt{x}+2|}{|\sqrt{x}-2|} \leq \frac{|x-4|}{2} < \frac{\delta}{2} < \varepsilon.$$

□

Limit theorems.

Theorem 3. *If a limit exists, it is unique*

Proof. Let $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} f(x) = L_2$. Then $\forall \varepsilon > 0$

$$\begin{aligned} &\exists \delta_1 \text{ s.t. } |f(x) - L_1| < \varepsilon/2 \text{ if } 0 < |x-a| < \delta_1 \\ &\exists \delta_2 \text{ s.t. } |f(x) - L_2| < \varepsilon/2 \text{ if } 0 < |x-a| < \delta_2 \end{aligned}$$

Then if $0 < |x-a| < \min\{\delta_1, \delta_2\}$

$$0 \leq |L_1 - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since this is true $\forall \varepsilon > 0$ we conclude $|L_1 - L_2| = 0 \Rightarrow L_1 = L_2$. □

Theorem 4. *Let $F = \mathbb{R}$ or $\mathbb{C}$, and $a, c \in F$. Let $f, g : A \to F$ and a be a limit point of A. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then*

(1) $\lim_{x\to a} c = c$
(2) $\lim_{x\to a} x = a$
(3) $\lim_{x\to a} (cf(x)) = cL$
(4) $\lim_{x\to a}(f(x) + g(x)) = L + M$
(5) $\lim_{x\to a}(f(x)g(x)) = LM$
(6) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$, *provided $M \neq 0$ and a is a limit point of the domain of $\frac{f}{g}$.*

Example. $\lim\limits_{x\to 1} \dfrac{x^2+5}{x^3-3x+1} = \dfrac{\lim(x^2+5)}{\lim(x^3-3x+1)} = \dfrac{1^2+5}{1^3-3\cdot 1+1} = -6.$

Continuity

Let $A, B \subseteq F$.

Definition 5. *A function $f : A \to B$ is* **continuous at** *$a \in A$ if*

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; s.t. \;\; |x-a| < \delta \implies |f(x) - f(a)| < \varepsilon.$$

Equivalently,

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; s.t. \;\; f\left((B(a,\delta)) \cap A\right) \subseteq B(f(a), \varepsilon).$$

f is **continuous** *if it is continuous at all $a \in A$.*

Remark 6. *Definitions of a limit and continuity have certain similarities, but here are some important differences.*

(1) f has to be defined at a
(2) a has to be in A, but it does not have to be a limit point of A
(3) the condition on x is $|x-a| < \delta$, not $0 < |x-a| < \delta$

Proposition 7. *Let $f : A \to B$, $a \in A$. Then*

(1) If a is NOT a limit point of A, then f is continuous at a
(2) If a is a limit point of A, then f is continuous at a iff $\lim_{x\to a} f(x) = f(a)$.

Proof. (1) Suppose that a is not a limit point of A. Then $\exists \delta > 0$ s.t. $B(a,\delta) \cap A = \{a\}$. Then, if $|x-a| < \delta$ we must have $x = a$ which implies $|f(x) - f(a)| = 0 < \varepsilon$ $\forall \varepsilon > 0$.

(2) Suppose now that a is a limit point of A. We need to prove both directions.
$\Rightarrow$: Let f be continuous at a, then

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \text{s.t.} \;\; |x-a| < \delta \implies |f(x) - f(a)| < \varepsilon.$$

Since $0 < |x-a| < \delta \Rightarrow |x-a| < \delta$ we conclude that

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \text{s.t.} \;\; 0 < |x-a| < \delta \implies |f(x) - f(a)| < \varepsilon, \;\; \text{so } \lim_{x\to a} f(x) = f(a).$$

$\Leftarrow$: Now assume $\lim_{x\to a} f(x) = f(a)$, i.e.

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \text{s.t.} \;\; 0 < |x-a| < \delta \implies |f(x) - f(a)| < \varepsilon.$$

However, if $x = a$ we have $|f(x) - f(a)| = 0 < \varepsilon$. Combining the two we get

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \text{s.t.} \;\; |x-a| < \delta \implies |f(x) - f(a)| < \varepsilon,$$

i.e. f is continuous at a,

□

Example. *Consider the function*

$$f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1, \\ 0, & x = 1. \end{cases}$$

We have shown $\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-1}{x-1} = 2$*, but* $f(1) = 0 \neq 2$*. Thus* $f(x)$ *is discontinuous at* $x = 1$*.*

Derivatives

Let $A, B \subseteq F$.

Definition 8. *Let* $f : A \to F$*,* $A \subseteq F$*, and* $a \in A$ *a limit point of* A*.* f *is* **differentiable at** $a \in A$ *if*

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \;\; \left(\text{or } \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \right) \;\; \text{exists.}$$

In this case this limit is called the **derivative of** f **at** a*. Notation:* $f'(a)$*,* $\frac{df}{dx}(a)$*,* $\frac{df}{dx}\big|_{x=a}$*.*

Example. *Find* $f'(i)$ *for the function* f *defined below*

$$f : \mathbb{C} \to \mathbb{C}$$
$$z \mapsto z^2$$

By definition

$$f'(i) = \lim_{h\to 0} \frac{f(i+h) - f(i)}{h} = \lim_{h\to 0} \frac{(i+h)^2 - i^2}{h} = \lim_{h\to 0} \frac{(i+h+i)h}{h} = 2i.$$

Basic properties

(1) if $f(z) = c \;\forall z \in A$, then $f'(z) = 0 \;\forall z \in A$.
(2) $(z^n)' = nz^{n-1}$
(3) $(f+g)' = f' + g'$
(4) $(fg)' = f'g + fg'$
(5) $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$
(6) $(f \circ g)'(a) = f'(g(a))g'(a)$

POWER SERIES

Definition 1. *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers. The series $f(z) := \sum_{n=0}^{\infty} a_n z^n$ is a* **power series** *with n^{th} coefficient a_n.*

Remark 2. *(1) For each $z \in \mathbb{C}$ we get a different series*
(2) The n^{th} term is $a_n z^n$, the n^{th} coefficient — a_n
(3) Suppose that for all $z \in D \subseteq \mathbb{C}$ the series $\sum_{n=0}^{\infty} a_n z^n$ converges to $f(z)$. Then f defines a function from D to $\mathbb{C}$, $z \mapsto f(z)$

Example. *Consider $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$. We will investigate absolute convergence, i.e. convergence of $\sum_{n=1}^{\infty} \frac{|z|^n}{n^2}$. Using the ratio test we have*

$$\frac{|z|^{n+1}}{(n+1)^2} \cdot \frac{n^2}{|z|^n} = |z| \left(\frac{n}{n+1}\right)^2 \to |z| \quad as \quad n \to \infty$$

Thus the series converges absolutely provided $|z| < 1$, and it does not converge absolutely if $|z| > 1$. If $|z| = 1$ we have $\sum_{n=1}^{\infty} \frac{|z|^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges by the p-series test. Thus, we have absolute convergence for $|z| \leq 1$. The series therefore defines a function $f : D \to \mathbb{C}$ on the closed unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$.

Theorem 3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then one of the following holds*

(1) $f(z)$ converges only when $z = 0$
(2) $f(z)$ converges for all $z \in \mathbb{C}$
(3) There exists a real number $R > 0$ s.t. $f(z)$ converges absolutely if $|z| < R$ and diverges for $|z| > R$.

Moreover, in case (2) convergence is absolute.

Definition 4. *The number R from Theorem 3 is called the* **radius of convergence** *of the power series $\sum_{n=0}^{\infty} a_n z^n$. We say $R = 0$ in case (1) and $R = \infty$ in case (2).*

Examples.

(1) $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!}$, R – ? Using the absolute convergence test and the ratio test we have

$$\frac{|z|^{n+1}}{(n+1)!} \cdot \frac{n!}{|z|^n} = \frac{|z|}{n} \to 0 < 1 \text{ as } n \to \infty, \ \forall z \in \mathbb{C}.$$

Thus, $f(z)$ converges absolutely $\forall z \in \mathbb{C} \Rightarrow R = \infty$.

(2) $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{5^n}$, R – ? Again, by the absolute convergence test and the ratio test we have

$$\frac{|z|^{n+1}}{5^{n+1}} \cdot \frac{5^n}{|z|^n} = \frac{|z|}{5} \to \frac{|z|}{5} \text{ as } n \to \infty, \text{ and } \frac{|z|}{5} < 1 \text{ if } |z| < 5.$$

Thus, $f(z)$ converges absolutely for all z satisfying $|z| < 5 \Rightarrow R = 5$.

(3) $f(z) = \sum_{n=1}^{\infty} \frac{(3n)!}{n!(2n)!} z^n$, R – ? As above we have

$$\frac{(3(n+1))!}{(n+1)!(2(n+1))!} \frac{n!(2n)!}{(3n)!}|z| = \frac{(3n+3)(3n+2)(3n+1)}{(n+1)(2n+2)(2n+1)}|z| \to \frac{27}{4}|z| \text{ as } n \to \infty,$$

and $\frac{27}{4}|z| < 1$ if $|z| < \frac{4}{27}$.

Thus, $R = 4/27$.

To prove Theorem 3 we need the following lemma

Lemma 5. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and suppose $f(w)$ converges for some $w \in \mathbb{C}\backslash\{0\}$. Then $f(z)$ converges absolutely for all $z \in \mathbb{C}$ s.t. $|z| < |w|$.*

Proof. Assume $w \in \mathbb{C}\backslash\{0\}$, $f(w)$ converges, and $|z| < |w|$. We will show that $f(z)$ converges absolutely. Since $f(w) = \sum_{n=0}^{\infty} a_n w^n$ converges, we must have $\lim_{n\to\infty} a_n w^n = 0 \Rightarrow$ the sequence $\{a_n w^n\}$ is bounded, i.e. $\exists M > 0$ s.t. $|a_n w^n| \leq M \ \forall n \in \mathbb{N}$. We can then write

$$|a_n z^n| = |a_n w^n| \left|\frac{z}{w}\right|^n \leq M \left|\frac{z}{w}\right|^n,$$

and note that the series $\sum M|z/w|^n$ converges since $|z/w| < 1$. By the comparison test this implies that the series $\sum a_n z^n$ converges absolutely. □

Proof of Theorem 3. Suppose cases (1) and (2) do not hold. Define the following subset of $\mathbb{R}$

$$S := \{r \in \mathbb{R} : \exists w \in \mathbb{C} \text{ s.t. } |w| = r \text{ and } f(w) = \sum a_n w^n \text{ converges}\}.$$

Note: 1) $0 \in S$ so $S \neq \emptyset$; 2) S is bounded by Lemma 5 since we assumed that case (2) does not hold. By the completeness of $\mathbb{R}$ this implies that there exists $\sup S$, and we denote $R := \sup S$. We now show that R satisfies the conditions in (3). First, let $|z| < R$ which implies that $|z|$ is not an upper bound for $S \Rightarrow \exists w \in \mathbb{C}$ s.t. $|w| > |z|$ and $f(w)$ converges $\Rightarrow$ by Lemma 5 that $f(z)$ converges absolutely.
Now let $|z| > R$, this implies $|z| \notin S \Rightarrow f(z)$ diverges by the definition of S. The proof that in case (2) the series converges absolutely is left as an **exercise**. □

Differentiation of power series. Recall that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defines a function on $B(0, R)$ where R is the radius of convergence. We now would like to investigate some properties of this function. In particular, the question is: is $f(z)$ differentiable on $B(0, R)$?
First note that each partial sum, that we now denote $s_n(z) := \sum_{k=0}^{n} a_k z^k$ is a polynomial in z, and is therefore differentiable for all $z \in \mathbb{C}$. Moreover, we can calculate

$$\frac{d}{dz} s_n(z) = \sum_{k=1}^{n} k a_k z^{k-1}.$$

Now, for each $z \in B(0, R)$ we have

$$f(z) = \lim_{n\to\infty} s_n(z) \;\Rightarrow\; \frac{d}{dz} f(z) = \frac{d}{dz} \lim_{n\to\infty} s_n(z).$$

If we could interchange the order of taking a limit and differentiation (which is also taking a limit) we would get

$$\frac{d}{dz} f(z) = \lim_{n\to\infty} \frac{d}{dz} s_n(z) = \lim_{n\to\infty} \sum_{k=1}^{n} k a_k z^{k-1} = \sum_{k=1}^{\infty} k a_k z^{k-1}.$$

So the derivative of $f(z)$ is also a power series. Using the ratio test to determine the radius of convergence we get

$$\frac{|a_{k+1}|(k+1)|z|^k}{|a_k| k |z|^{k-1}} = \frac{|a_{k+1}|}{|a_k|} \frac{k+1}{k} |z|$$
$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} \frac{k+1}{k} |z| = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} |z| = \frac{|z|}{R},$$

which shows that $f'(z)$ would have the same radius of convergence as $f(z)$. However, interchanging the limit and differentiation is a non-trivial task. We state the following theorem without proof

Theorem 6. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence R. Then f is differentiable on $B(0, R)$ and $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$. $f'(z)$ has radius of convergence R.*

Corollary 7. *Under the conditions of Theorem 6 the function $f(z)$ has derivatives of all orders on $B(0,R)$ that can be obtained by term-by-term differentiation of the power series defining $f(z)$.*

Proof. Note that $f'(z)$ can be written as a power series

$$f'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1} = \sum_{l=0}^{\infty} (l+1) a_{l+1} z^l.$$

Therefore, we can apply Theorem 6 to $f'(z)$ to obtain the result for the second derivative $f''(z)$. The rest follows by induction. □

Complex exponential

Let $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ with the convention $0! = 1$. We showed that the radius of convergence of this power series $R = \infty$, so $f(z)$ defines a function $f : \mathbb{C} \to \mathbb{C}$. By Theorem 6 we have

$$f'(z) = \sum_{n=1}^{\infty} \frac{n z^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = f(z), \quad \forall z \in \mathbb{C}.$$

Also note that $f(0) = 0$, and recall that for $x \in \mathbb{R}$ we have $(e^x)' = e^x$ and $e^0 = 1$. This motivates the following definition

Definition 8. *Define the* **exponential function** $exp(z) : \mathbb{C} \to \mathbb{C}$ *by*

$$exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Proposition 9. $\forall x \in \mathbb{R}$ $exp(x) = e^x$.

Proof. Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = \frac{exp(x)}{e^x}$. Note that $h(0) = 1$ and differentiating gives for all $x \in \mathbb{R}$

$$h'(x) = \frac{exp'(x)e^x - exp(x)(e^x)'}{(e^x)^2} = \frac{exp(x)e^x - exp(x)e^x}{(e^x)^2} = 0.$$

Since $h'(x) = 0$ $\forall x \in \mathbb{R}$ we conclude that $h(x) = const.$ Moreover, $h(0) = 1 \Rightarrow h(x) = 1$ $\forall x \in \mathbb{R} \Rightarrow exp(x) = e^x$ $\forall x \in \mathbb{R}$. □

Taylor series

Consider $f : F \to F$ and assume that f is continuous at $a \in F$, then near the point a the function f can be approximated by a constant function $y = f(a)$. Now assume in addition that f is differentiable at a. By the definition of derivative, if x is close to a, the derivative $f'(a)$ is approximated by $\frac{f(x)-f(a)}{x-a}$. We can therefore write

$$f(x) = f(x) - f(a) + f(a) = \frac{f(x) - f(a)}{x - a}(x - a) + f(a) \approx f'(a)(x - a) + f(a),$$

and thus f is approximated by a linear function $y = f'(a)(x - a) + f(a)$. Can we approximate by higher degree polynomials? In oder to do that we need higher order derivatives. **Notation**: if f' is differentiable we denote $(f')' = f'' = f^{(2)}$. We can define n^{th} derivative of f by induction, saying f is n times differentiable if its $n-1$st derivative is differentiable, and write $f^{(n)} = (f^{(n-1)})'$.

Definition 10. *Let the function f have n derivatives at the point a of the domain. The* **Taylor polynomial of f centered at a of order n** *is*

$$T_n(x) = f(a) + f'(a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k.$$

Example. *Consider $f(x) = x^4 + 3x^3 - 2x + 5$, and let $a = 0$. It is easy to calculate*

$$\begin{aligned} & & f(0) &= 5 \\ f'(x) &= 4x^3 + 9x^2 - 2 & f'(0) &= -2 \\ f''(x) &= 12x^2 + 18x & f''(0) &= 0 \\ f^{(3)}(x) &= 24x + 18 & f^{(3)}(0) &= 18 \\ f^{(4)}(x) &= 24 & f^{(4)}(0) &= 24 \\ f^{(5)}(x) &= 0 & \forall n \geq 5, &\quad \forall x \in \mathbb{R}. \end{aligned}$$

Proposition 11. *If f is a polynomial of degree d, then $\forall a \in F$ and any $n \geq d$, the n^{th} order Taylor polynomial of f centered at a, equals f.*

Definition 12. *Let the function f have derivatives of all orders at the point a of the domain. The* **Taylor series of f centered at** *a is*

$$T(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x - a)^k. \tag{1}$$

Notation: $f(x) \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x - a)^k$

Example. *Let $f(z) = \frac{1}{1-z}$, $z \in \mathbb{C}$, $z \neq 1$, and let $a = 0$. We can calculate*

$$\begin{aligned} & & f(0) &= 1 \\ f'(z) &= \frac{1}{(1-z)^2} & f'(0) &= 1 \\ f''(x) &= \frac{2}{(1-z)^3} & f''(0) &= 2 \\ f^{(3)}(z) &= \frac{6}{(1-z)^4} & f^{(3)}(0) &= 6. \end{aligned}$$

We can easily show by induction that $f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}$ $\forall n \in \mathbb{N}$. Indeed, the base case $n = 1$ is shown above. Now assume $f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}$ for some $n \geq 1$, then

$$f^{(n+1)}(z) = (f^{(n)}(z))' = \left(\frac{n!}{(1-z)^{n+1}}\right)' = \frac{(n+1)!}{(1-z)^{n+2}},$$

which concludes the inductive step and the general formula is proven. We therefore obtain $f^{(n)}(0) = n!$ and substituting into (1) we conclude

$$\frac{1}{1-z} \sim \sum_{n=0}^{\infty} z^n.$$

Finally, recall that the geometric series $\sum z^n$ converges for all $z \in \mathbb{C}$ s.t. $|z| < 1$ and the limit is $1/(1-z)$. We thus obtain

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad \forall z \in B(0,1).$$

The next two theorems give sufficient conditions for the Taylor series of a function to converge. Note that in the case of a complex function we seemingly require much less than in the case of a real function. See Remark 15.

Theorem 13. *Let $A \subseteq \mathbb{C}$, $f : A \to \mathbb{C}$, and let $T(z)$ be the Taylor series of $f(z)$ centered at $a \in A$. Suppose that the function f is differentiable on $B(a, R) \subseteq A$ for some $R > 0$. Then $T(z) = f(z)$ $\forall z \in B(a, R)$.*

Theorem 14. *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and let $T(x)$ be the Taylor series of $f(x)$ centered at $a \in A$. Suppose that the function f has derivatives of all orders on $B(a, R) \subseteq A$ for some $R > 0$. If f and all its derivatives are bounded on $B(a, R)$ (i.e. $\exists M > 0$ s.t. $|f^{(n)}(x)| \leq M$ $\forall x \in B(a, R)$, $\forall n \geq 0$), then $T(x) = f(x)$ $\forall x \in B(a, R)$.*

Remark 15. *(1) If $f(z)$ is differentiable at $z \in \mathbb{C}$ it actually has derivatives of all orders at z.*
(2) Recall that derivatives are defined using limits. If we are working in $F = \mathbb{R}$ then to define a limit at $a \in F$ we only need to consider $x \in B(a, \delta) \subseteq \mathbb{R}$, i.e. $x \in (a-\delta, a+\delta)$. While working in $F = \mathbb{C}$ means considering all $z \in B(a, \delta) \subseteq \mathbb{C}$ which is a much bigger set.
(3) Roughly speaking, it is much harder to be differentiable in the complex field, than being differentiable in the real field.

Example. *Define*

$$f(x) = \begin{cases} 0, & x = 0 \\ e^{-\frac{1}{x^2}}, & x \neq 0 \end{cases}$$

It is an **exercise** *to check that $\lim_{x\to 0} f(x) = 0$, and therefore f is continuous at 0. Next we calculate $\forall x \neq 0$*

$$f'(x) = \frac{2}{x^3} e^{-\frac{1}{x^2}}, \quad f''(x) = \left(\frac{4}{x^6} - \frac{6}{x^4}\right) e^{-\frac{1}{x^2}},$$

and it can be shown by induction that for $x \neq 0$

$$f^{(n)}(x) = P_{3n}\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}$$

where P_{3n} is a polynomial of degree n. Moreover,

$$\lim_{x\to 0} \frac{1}{|x|^k} e^{-\frac{1}{x^2}} = 0, \quad \forall k \in \mathbb{N}.$$

We then have

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{1}{x} e^{-\frac{1}{x^2}} = 0$$

and thus we can show by induction

$$f^{(n)}(0) = 0 \quad \forall n \in \mathbb{N}.$$

This implies that every term in the Taylor series of f centered at 0, is 0, and thus $T(x) \equiv 0$. Therefore, $f(x) \neq T(x)$ for any $x \neq 0$. Though f has derivatives of all orders in any ball around the origin, there is no uniform bound on those derivatives so Theorem 14 does not apply.

Trigonometric functions. Recall that we defined the complex exponential function as

$$exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Suppose now that z is purely imaginary, i.e. $z = ix$ for some $x \in \mathbb{R}$. We then have

$$\begin{aligned} exp(ix) = \sum_{0}^{\infty} \frac{(ix)^n}{n!} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \dots \\ &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \dots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} + i\sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{2k-1}}{(2k-1)!}. \end{aligned}$$

Exercise 16. *Calculating the Taylor series for* $\sin(x)$ *and* $\cos(x)$ *centered at* 0 *obtain*

$$\cos x \sim \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}; \quad \sin x \sim \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{2k-1}}{(2k-1)!}.$$

Using Theorem 14 we can show that $\forall x \in \mathbb{R}$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}; \quad \sin x = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{2k-1}}{(2k-1)!}.$$

We thus arrive at a famous formula (writing e^{ix} in place of $exp(ix)$)

$$\boxed{e^{ix} = \cos x + i \sin x \quad \forall x \in \mathbb{R}.}$$

We can also extend the definitions of sin and cos to the complex plane (note that the series below converge absolutely for all $z \in \mathbb{C}$)

$$\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}; \quad \sin x = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k-1}}{(2k-1)!}.$$