REAL ANALYSIS LECTURE NOTES

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SET THEORY

Lecture 1

Set — a collection of objects = elements. It is a basic concept and we will accept it without rigorous definition.

Definition 1. *A (binary)* ***relation*** *on a set S is a subset $R \subseteq S \times S$. If $(a, b) \in R$ write $a \sim b$.*
$\sim$ is an ***equivalence relation*** *if $\forall a, b, c \in S$:*

(1) $a \sim a$ — reflexivity
(2) $a \sim b \Longrightarrow b \sim a$ — symmetry
(3) $a \sim b,\ b \sim c \Longrightarrow a \sim c$ — transitivity

Examples of equivalence relations.

(1) $S = \mathbb{Z}$, $R = \{(a, b) \in \mathbb{Z} \colon a = b\}$ or $a \sim b$ if $a = b$
(2) $S = \mathbb{Z}$, $a \sim b$ if $a - b$ is divisible by 3
(3) $S = \mathbb{S}^2$, a sphere in $\mathbb{R}^3$, $a \sim b$ if $a = b$ or a and b are antipodal
(4) $S = L^2(\mathbb{R})$: all functions whose square is integrable on $\mathbb{R}$. $f \sim g$ if $\int_{\mathbb{R}} |f - g|^2 = 0$

Exercise 2. *Give examples of equivalence relations on $\mathbb{Q}$ and $\mathbb{R}$*

Non-examples.

(1) $S = \mathbb{R}^2$, $(a, b) \sim (c, d)$ if $ac = -bd$: symmetric, but not reflexive and not transitive
(2) $S = \mathbb{Z}$, $a \sim b$ if $a = 2b$: all three properties fail

Exercise 3. *Give examples of relations for which only one of the properties from Definition 1 fails.*

Partition.

Definition 4. *$\forall a \in S$, the* ***equivalence class*** *of a is the subset $S_a := \{b \in S : b \sim a\}$.*

Claim: $\forall a, b \in S$ either $S_a = S_b$ or $S_a \cap S_b = \emptyset$

Proof. Exercise □

Definition 5. ***Partition*** *of S is the collection of all distinct equivalence classes.*

Conversely, let $S = \bigcup_{\alpha \in I} T_\alpha$, T_α pairwise disjoint. Define $\sim$: $a \sim b$ if $a, b \in T_\alpha$ for some $\alpha \in I$. Then $\sim$ is an equivalence relation.

Proof. Exercise. □

Examples.

(1) $\mathbb{R}^3 = \bigcup_{\alpha \in \mathbb{R}_+} T_\alpha$, $T_\alpha = \{x \in \mathbb{R}^3 : |x| = \alpha\}$
(2) $\mathbb{R}_+ = \bigcup_{k \in \mathbb{N} \cup \{0\}} T_k$, $T_k = \{a \in \mathbb{R}_+ : \lfloor a \rfloor = k\}$
(3) $\mathbb{R} = \bigcup_{\alpha \in \mathbb{R}} T_\alpha$, $T_\alpha = \{a \in \mathbb{R} : a - \lfloor a \rfloor = \alpha\}$
(4) * **Vitali set.** Define $\sim$ on $\mathbb{R}^n$ by $x \sim y$ if $x - y \in \mathbb{Q}^n$. Then $\sim$ is an equivalence relation on $\mathbb{R}^n$ (exercise). Choose one element from each equivalence class to form the set $\mathcal{V}$ (using the Axiom of Choice). Let $q_1, q_2, \ldots$ be a list of the elements of $\mathbb{Q}^n$ (since $\mathbb{Q}^n$ is countable), and define $V_{q_i} := q_i + \mathcal{V}$. Then V_{q_i} are pairwise disjoint and $\mathbb{R}^n = \bigcup_{i=1}^\infty V_{q_i}$. We will prove and use this later.

Lecture 2

Cardinality.

Definition 6. *Sets S and T have the same* ***cardinality*** *if there exists a bijection $f \colon S \to T$.*

Recall: $f \colon S \to T$ is **surjective** (onto) if $\forall y \in T\ \exists x \in S$ s.t. $f(x) = y$. f is **injective** (1-1) if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$. f is **bijective** if it's surjective and injective.

Let $\mathcal{F}$ be a collection of sets, $S, T \in \mathcal{F}$.

Proposition 7. *Let $S \sim T$ if S and T have the same cardinality. Then $\sim$ is an equivalence relation on $\mathcal{F}$.*

Proof. (1) Symmetry: $S \sim S$, f – identity map
(2) Reflexivity: $S \sim T \Rightarrow \exists f : S \to T$ a bijection $\Rightarrow f^{-1} : T \to S$ also a bijection $\Rightarrow T \sim S$
(3) Transitivity: $S \sim T$, $T \sim Q \Rightarrow$ bijections $f : S \to T$ and $g : T \to Q \Rightarrow f \circ g : S \to Q$ also a bijection $\Rightarrow S \sim Q$.

□

Definition 8. *A set is* ***countable*** *if it has the same cardinality as $\mathbb{N} = \{1, 2, 3, \ldots\}$. A set is* ***finite*** *if it has finitely many elements.*

Properties.

Proposition 9. *Every infinite set contains a countable subset.*

Proof. Let S be an infinite set. Choose $s_1 \in S$ an arbitrary element of S. Since S is infinite we can choose $s_2 \in S$ s.t. $s_2 \neq s_1$. Next, choose $s_3 \in S \backslash \{s_1, s_2\}$, continue. $\square$

Proposition 10. *A subset of a countable set is either countable or finite.*

Proof. Let $B \subseteq A$, A countable. Write $A = \{a_1, a_2, \ldots, a_n, \ldots\}$, then $B = \{a_{n_1}, a_{n_2}, \ldots\}$ where $n_1, n_2, \ldots \in \mathbb{N}$. If $\sup\{n_1, n_2, \ldots\} < \infty$ then B is finite. Otherwise $B \sim \mathbb{N}$ since we have a bijection $B \to \mathbb{N}$ given by $a_{n_k} \mapsto k$. $\square$

Proposition 11. *The following sets are countable: (1) a countable union of countable sets; (2) a finite union of countable sets; (3) a countable union of finite sets. A finite union of finite sets is finite.*

Proof. We will give a proof of (1), the rest is an **exercise**. Let $S_1, S_2, \ldots$ be countable sets, and let $S_i = \{s_{i1}, s_{i2}, \ldots\}$. Make an array

$$\begin{array}{cccc} s_{11} & s_{12} & s_{13} & \cdots \\ s_{21} & s_{22} & s_{23} & \cdots \\ s_{31} & s_{32} & s_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{array}$$

List the elements as follows: read off the diagonals and throw out repeated elements, $s_{11}, s_{12}, s_{21}, s_{31}, s_{22}, s_{13}, \ldots$ $\square$

Example 12. ***Hilbert's Grand Hotel Paradox***. *Imagine a hotel with a countable number of rooms, each of which is occupied. If a new guest arrives can they be accommodated?* ***Yes:*** $a_n \to a_{n+1}$. *What if finitely many new guests arrive? What if countably many new guests arrive? (**Exercise**)*

Corollary 13. $\mathbb{Z}$ *is countable.*

Proof. $\mathbb{Z} = \{\ldots, -2, -1\} \cup \{0\} \cup \{1, 2, \ldots\}$. $\square$

Corollary 14. $\mathbb{Q}$ *is countable.*

Proof. $\forall n \in \mathbb{N}$ define $\mathbb{Q}_n := \{a/n \colon a \in \mathbb{Z}\}$. Then $\mathbb{Q} = \bigcup_{n\in\mathbb{N}} \mathbb{Q}_n$ and each Q_n is countable (**exercise**). $\square$

Exercise 15. *Prove the above corollaries directly by giving an explicit bijection into* $\mathbb{N}$*, without using Proposition 11.*

Cantor's theorem

Theorem 16 (Cantor, 1892)**.** $\mathbb{R}$ *is uncountable.*

Proof. (Cantor's diagonal process). First note that it is sufficient to show that a subset of $\mathbb{R}$ is uncountable (**exercise**). Define $T \subset \mathbb{R}$ as

$$T := \{a \in [0, 1) \colon \text{decimal expansion of } a \text{ contains only 0's and 1's}\}$$

Claim: T is uncountable.
Proof of Claim: assume the opposite: $\exists$ a bijection $f : \mathbb{N} \to T$, i.e. $\forall x \in T\ \exists! n \in \mathbb{N}$ s.t. $f(n) = x$. Write $f(n) = 0.x_{n1}x_{n2}\ldots$ Construct $y \in T$: $y = 0.y_1y_2\ldots$ where $y_n = 0$ if $x_{nn} = 1$ and $y_n = 1$ if $x_{nn} = 0$. Then $y \neq f(n)$ for any $n \in \mathbb{N}$ but $y \in T$. Therefore, f is not a bijection, contradiction. $\square$

Exercise 17. *Let* $S = [0, 1)$ *and suppose we have a bijection* $f : \mathbb{N} \to S$*, again write* $f(n) = 0.x_{n1}x_{n2}\ldots$ *Construct* $y \in S$*,* $y = 0.y_1y_2\ldots$ *where* $y_n \neq x_{nn}$ $\forall n \in \mathbb{N}$*. Does that show* S *is uncountable? If not, what is the problem? How can we fix it?*

Examples of uncountable sets.

(1) Any non-empty interval $(a, b) \subset \mathbb{R}$
(2) Set of points on a plane
(3) $C([a, b])$ — the set of continuous functions on $[a, b]$

Exercise 18. *Show that* $[0, 1)$ *has the same cardinality as* $\mathbb{R}$*.*

Lecture 3

Cardinal numbers.

Definition 19. S, T *sets, write*

(1) $|S| = |T|$ *if* S *and* T *have the same cardinality*
(2) $|S| \leq |T|$ *if there is an injection* $S \hookrightarrow T$*.* ***Example:*** $|\mathbb{N}| \leq \mathbb{R}$
(3) $|S| < |T|$ *if* $|S| \leq |T|$ *and there is no injection* $T \hookrightarrow S$*.* ***Example:*** $|\mathbb{N}| < \mathbb{R}$*, proof:* ***exercise****.*

Properties.

(1) If $|S| \leq |T|$ and $|T| \leq |S|$ then $|S| = |T|$ (Cantor-Bernstein theorem)
(2) If $|S| \leq |T|$ and $|T| \leq |U|$ then $|S| \leq |U|$
(3) $\forall S, T$ either $|S| \leq |T|$ or $|T| \leq |S|$ (assuming the Axiom of Choice, see below)

Now recall that by Proposition 7 all sets can be partitioned into equivalence classes based on their cardinalities. We can thus "define" cardinal numbers to be these equivalence classes. Properties 1.-3. show that the relation $\leq$ given in part 2. of Definition 19 is a *total order* on the set of cardinal numbers.

We can also think of natural numbers as cardinal numbers: e.g. number 2 is the equivalence class of sets that have exactly 2 elements. Thus, we can write for example $|\{-1, 2\}| = 2$.

Exercise 20. *Show that the "smallest" infinite cardinal number is the cardinality of* $\mathbb{N}$.

Aleph: $|\mathbb{N}| = \aleph_0$. $\aleph_1$ stands for the smallest cardinality which is strictly larger than $\aleph_0$.
Question: Is it true that $|\mathbb{R}| = \aleph_1$? More precisely, is there a set S such that $|\mathbb{N}| < |S| < |\mathbb{R}|$?

Continuum Hypothesis. (Cantor 1878): there is no set S such that $|\mathbb{N}| < |S| < |\mathbb{R}|$.

This was one of Hilbert's problems presented in 1900, and was resolved by Gödel (1938) and Cohen (1963). To gain more insight and answer the question, we first need to talk about Power Set and Axiom of Choice.

Power Set.

Definition 21. *Let S be any set, the* ***power set*** *of S, denoted 2^S is the set of all subsets of S.*

We will use an accepted notation: $|2^S| = 2^{|S|}$.

Exercise 22. *Let S be finite and $|S| = m$ (i.e. S has m elements). Show that $|2^S| = 2^m$, i.e. the power set of S has 2^m elements.*

Theorem 23. *For every set S we have $|S| < |2^S|$.*

Proof. First, we have $|S| \leq |2^S|$ using the natural injection

$$S \hookrightarrow 2^S$$
$$x \mapsto \{x\}.$$

Second, we need to show that there is no surjection $f : S \twoheadrightarrow 2^S$. Assume the opposite: $f : S \twoheadrightarrow 2^S$, and define

$$B := \{s \in S \colon s \notin f(s)\},$$

i.e. B is the set of all elements of s that are not in their images under f. Since f is surjective, $\exists b \in S$ s.t. $f(b) = B$. Does $b \in B$? If yes, then $b \in B = f(b)$ so by definition of B, $b \notin B$. This is a contradiction, so we must have $b \notin B$. But then $b \notin B = f(b)$ so by definition of B we must have $b \in B$. This is again a contradiction, so there cannot be a surjection from S to 2^S. $\square$

Note that the above proof is basically a reiteration of Cantor's diagonal argument.

Lecture 4

Bertrand Russell, 1901: let A be the set of all sets (the universal set). If A already contains all possible sets, how can 2^A be larger than A?

Russel's paradox: Let R be the set of all sets that are not elements of themselves, i.e. $R := \{A : A \notin A\}$. Does $R \in R$? Similar argument as before shows that both assumptions lead to a contradiction (**exercise**).

Example 24. *Let* $A = \{\ldots\{\{1\}\}\ldots\}$*, then* $A \in A$.

This contradiction led to the development of Zermelo-Fraenkel set theory (ZF), which in particular does not allow for the universal sets and sets that can be elements of themselves.

Axiom of Choice (Zermelo 1904)**.** *Let* I *be a collection of indices, and* $\{S_\alpha\}_{\alpha\in I}$ *a collection of non-empty subsets. Then there exists a family* $\{s_\alpha\}_{\alpha\in I}$ *s.t.* $s_\alpha \in S_\alpha$ $\forall\alpha \in I$*. In other words, for any collection of sets it is possible to create a set that contains exactly one element from each set in the family.*

ZFC stands for Zermelo-Fraenkel theory with the Axiom of Choice.

Back to **continuum hypothesis**. Recall the notation $\aleph_0 = |\mathbb{N}|$, $\aleph_1 > \aleph_0$, and also denote $c = |\mathbb{R}|$. The question is, does $\aleph_1 = c$?
If we try to search for a candidate for a set S s.t. $|\mathbb{N}| < |S| < |\mathbb{R}|$, the first set one might consider is $2^{\mathbb{N}}$.

Theorem 25. $|2^{\mathbb{N}}| = |\mathbb{R}|$*, in other words* $2^{\aleph_0} = c$.

Proof. We will show two inequalities: $c \le 2^{\aleph_0}$ and $2^{\aleph_0} \le c$.
1) $c \le 2^{\aleph_0}$: we need to construct an injection $\mathbb{R} \hookrightarrow 2^{\mathbb{N}}$. Since $|\mathbb{Q}| = |\mathbb{N}|$ it is sufficient instead to construct an injection from $\mathbb{R}$ to $2^{\mathbb{Q}}$. Let

$$\begin{aligned} f : \mathbb{R} &\to 2^{\mathbb{Q}} \\ x &\mapsto \{q \in \mathbb{Q} : q \le x\}, \end{aligned}$$

then f is $1-1$ (**exercise**).
2) $2^{\aleph_0} \le c$: Recall that $|\mathbb{R}| = |[0,1)|$ and define $f : 2^{\mathbb{N}} \to [0,1)$ as follows. For every $M \in 2^{\mathbb{N}}$ let

$$x_M = 0.x_{1,M}x_{2,M}\ldots, \quad \text{where } x_{i,M} = \begin{cases} 0, & \text{if } i \notin M, \\ 1 & \text{if } i \in M \end{cases}$$

Then if $f(M) = x_M$, it is $1-1$ (**exercise**). □

The above theorem shows that the set $2^{\mathbb{N}}$ is "too large" to confirm the continuum hypothesis. What is the answer then?

Gödel, 1938: CH is consistent with ZFC
Cohen, 1963: the negation of CH is consistent with ZFC

Thus, in the framework of ZFC, the continuum hypothesis is neither provable nor disprovable.

METRIC SPACES

Lecture 1

Definition 1. *A **metric space** is a set M and a function $d : M \times M \to \mathbb{R}$, called* **metric** *on M, such that $\forall x, y, z \in M$ the following three properties hold*

(1) $d(x, y) \geq 0$ *and* $d(x, y) = 0$ *iff* $x = y$ — positivity
(2) $d(x, y) = d(y, x)$ — symmetry
(3) $d(x, z) \leq d(x, y) + d(y, z)$ — triangle inequality

Examples.

(1) $M = \mathbb{R}^n$, $d(x, y) = |x - y| = \left(\sum_{i=1}^n (x_i - y_i)^2\right)^{1/2}$ — Euclidean metric
(2) $M = \mathbb{R}^n$, $d(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$
(3) $M = C([a, b])$, $d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|$
(4) $M = L^2([a, b])$, $d(f, g) = \left(\int_a^b |f(x) - g(x)|^2\right)^{1/2}$
(5) M any set, $d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$ — discrete metric

Proof. Positivity and symmetry are obvious. To prove triangle inequality, consider three points $x, y, z \in M$. There are four cases
(a) $x \neq y \neq z$, then $d(x, z) = 1 < 2 = d(x, y) + d(y, z)$
(b) $x = y \neq z$, then $d(x, z) = 1 = 0 + 1 = d(x, y) + d(y, z)$
(c) $x \neq y = z$, then $d(x, z) = 1 = 1 + 0 = d(x, y) + d(y, z)$
(d) $x = y = z$, then $d(x, z) = 0 = d(x, y) + d(y, z)$
This concludes the triangle inequality. □

Open sets.

Definition 2. *Let (M, d) be a metric space. The* **open ball** *centered at $x \in M$ with radius $r > 0$ is the set*

$$B(x, r) = \{y \in M : d(x, y) < r\}.$$

The **closed ball** *is the set $\overline{B}(x, r) = \{y \in M : d(x, y) \leq r\}$.*

Definition 3. *A subset $U \subseteq M$ is* **open** *if it contains an open ball around each of its points, i.e. $\forall x \in U$ $\exists r > 0$ s.t. $B(x, r) \subseteq U$.*

Proposition 4. *An open ball is open (as a set).*

Proof. Let $y \in B(x,r)$.
Claim: $B(y,R) \subseteq B(x,r)$ where $R := r - d(x,y) > 0$.
Proof of Claim: First, since $y \in B(x,r)$, by definition we get $d(x,y) < r$ and thus $R := r - d(x,y) > 0$. Now let $z \in B(y,R)$, then $d(z,y) < r - d(x,y) \implies d(x,y) + d(z,y) < r$. By triangle inequality, $d(x,z) \leq d(x,y) + d(y,z) < r$, and therefore $z \in B(x,r)$. $\square$

Proposition 5. *Let (M,d) be a metric space*

(1) $\emptyset$ is open
(2) M is open
(3) $\{U_\alpha\}_{\alpha \in I}$ a family of open sets, then $\bigcup_{\alpha \in I} U_\alpha$ is open (in other words, arbitrary unions of open sets are open)
(4) Let $U_1, U_2, \ldots, U_k$ be open sets, then $\bigcap_{i=1}^k U_i$ is open (in other words, finite intersections of open sets are open)

Proof. We will prove (4), the rest is an **exercise**. Let $z \in \bigcap_{i=1}^k U_i$, then $z \in U_i$ for all $i = 1, \ldots, k$. Since U_i is open, for each i there exists $r_i > 0$ s.t. $B(z, r_i) \subseteq U_i$. Define $r := \min_{1 \leq i \leq k}\{r_i\}$, then $B(z,r) \subseteq U_i \ \forall i = 1, \ldots, k \implies B(z,r) \subseteq \bigcap_{i=1}^k U_i$. $\square$

Example 6. *Let $M = \mathbb{R}$ with Euclidean distance, and let $U_n = (-1/n, 1/n),\ n \in \mathbb{N}$. Note that each U_n is an open ball centered at 0, and is therefore open. However, $\bigcap_{n \in \mathbb{N}} U_n = \{0\}$ which is not open. This shows that* arbitrary intersections of open sets are not necessarily open.

Definition 7. *A* **neighborhood** *of a point $x \in M$ is any set containing an open set containing x.*

Remark 8. *A neighborhood is often taken to be an open set.*

Example 9. $M = \mathbb{R}$, $d(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$. *What is $B(x,r)$?*
Recall $B(x,r) = y \in \mathbb{R} : d(x,y) < r$. Since the distance between x and y is either 0 or 1 we need to consider two cases. First, if $r \leq 1$, the condition $d(x,y) < r$ is only satisfied when $d(x,y) = 0$, i.e. $x = y$. Thus if $r \leq 1$, $B(x,r) = \{x\}$. Now, if $r > 1$ we have $d(x,y) < r$ satisfied for any y, and therefore $B(x,y) = \mathbb{R}$.

Lecture 2

Closed sets. Let (M, d) be a metric space.

Definition 10. *Let $S \subseteq M$. A point $x \in M$ is*

(1) A **contact point of** *S if every neighborhood of x contains a point of S*
(2) A **limit point of** *S if every neighborhood of x contains infinitely many points of S*

A point $s \in S$ is

(3) An **isolated point of** *S if there exists a neighborhood of s containing no other points of S besides s*
(4) An **interior point of** *S if there exists a neighborhood of s contained in S*

Exercise 11. *Can a contact point be not a limit point?*
Can every neighborhood of a point contain exactly two points of a given set?

Examples. Let $M = \mathbb{R}$, $d = |\cdot|$

(1) $S = (0, 1)$. The set of contact points = the set of limit points = $[0, 1]$
(2) $S = \mathbb{N}$. The set of contact points = S. No limit points. Every point of S is isolated
(3) $S = \mathbb{Q}$. The set of contact points = the set of limit points = $\mathbb{R}$

Definition 12. *The* **closure** *of $S \subseteq M$, denoted $\overline{S}$, is the set of all contact points of S. S is* **closed** *if $\overline{S} = S$.*

Proposition 13. *(1) $S \subseteq \overline{S}$*
(2) $S \subseteq T \implies \overline{S} \subseteq \overline{T}$
(3) $\overline{\overline{S}} = \overline{S}$
(4) $\overline{S \cup T} = \overline{S} \cup \overline{T}$
(5) $\overline{\emptyset} = \emptyset$

Remark 14. *It is not true in general that $\overline{S \cap T} = \overline{S} \cap \overline{T}$. See example 15 below.*

Proof. We will prove (2) and (4), the rest is an **exercise**.
(2) Let $x \in \overline{S} \implies x$ is a contact point of $S \implies$ every nbhd of x contains a point $y \in S$. $S \subseteq T \implies y \in T \implies$ every nbhd of x contains a point of $T \implies x$ is a contact point of $T \implies x \in \overline{T}$.
(4) We have $S \subseteq S \cup T$ and $T \subseteq S \cup T$ and therefore by (2) $\overline{S} \subseteq \overline{S \cup T}$ and $\overline{T} \subseteq \overline{S \cup T}$ $\implies \overline{S} \cup \overline{T} \subseteq \overline{S \cup T}$
OTOH: Suppose $x \notin \overline{S} \cup \overline{T} \implies \exists U, V$, neighborhoods of x, s.t. $U \cap S = \emptyset$ and $V \cap T = \emptyset$. Now let $W := U \cap V$, then W is a nbhd of x (**exercise**) and $W \cap (S \cup T) = \emptyset$ $\implies x \notin \overline{S \cup T}$. This shows $\overline{S \cup T} \subseteq \overline{S} \cup \overline{T}$. $\square$

Example 15. *Consider the Euclidean* $\mathbb{R}^2$*, and let* $S = B((0,0),1)$ *and* $T = B((2,0),1)$*. We have* $S \cap T = \emptyset$*, and thus by (5) Proposition 13 we get* $\overline{S \cap T} = \emptyset$*. On the other hand,* $\overline{S} \cap \overline{T} = \{(1,0)\}$ *— an isolated point.*

Examples.

(1) $M = \mathbb{R}$, $d = |\cdot|$, $\overline{[a,b]} = [a,b]$
(2) Closed ball in (Euclidean) $\mathbb{R}^n$ is closed
(3) $M = \mathbb{R}$ with discrete metric.
$B(x,1) = \{y \in \mathbb{R} : d(x,y) < 1\} = \{x\} \Longrightarrow \overline{B(x,1)} = \{x\} = B(x,1)$.
OTOH: $\overline{B}(x,1) = \{y \in \mathbb{R} : d(x,y) \leq 1\} = \mathbb{R} \Longrightarrow \overline{\overline{B}(x,1)} = \mathbb{R} = \overline{B}(x,1)$.

Proposition 16. *S is closed iff S contains all its limit points*

Proof. $\Longrightarrow$: **exercise**
$\Longleftarrow$: S contains all its contact points, moreover, isolated points are by definition contained in S. By Homework problem contact points are either limit points or isolated points, therefore, S contains all its limit points $\Longrightarrow S = \overline{S} \Longrightarrow S$ is closed. □

Proposition 17. *S is closed iff S^c is open*

Proof. $\Longrightarrow$: S is closed $\Longrightarrow$ S contains all its contact points $\Longrightarrow$ S^c does not contain any contact points of S $\Longrightarrow$ every point of S^c has a nbhd that doesn't contain any point of S, thus only contains points of S^c $\Longrightarrow$ S^c is open.
$\Longleftarrow$: **exercise**. □

Corollary 18. *Single point sets are closed*

Proof. Homework □

Corollary 19. *(1) $\emptyset$ is closed*
(2) M is closed
(3) $\{U_\alpha\}_{\alpha \in I}$ a family of closed sets, then $\bigcap_{\alpha \in I} U_\alpha$ is closed (in other words, arbitrary intersections of closed sets are closed)
(4) Let $U_1, U_2, \ldots, U_k$ be closed sets, then $\bigcup_{i=1}^k U_i$ is closed (in other words, finite unions of closed sets are closed)

Proof. $\left(\bigcap_{\alpha \in I} U_\alpha\right)^c = \bigcup_{\alpha \in I} U_\alpha^c$, apply Proposition 5. □

The Cantor set.

$C = \bigcap_{i \geq 0} F_i$

(1) C is closed by (3) Proposition 19 since each F_i is closed by (4)
(2) Clearly, $0, 1, 1/3, 2/3, 1/9, 2/9, 7/9, 8/9, \ldots \in C$ (these are the endpoints of the removed intervals)

What other points are in the Cantor set?
Let $x \in [0,1]$, write x in ternary (base-3) expansion

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, \quad a_n \in \{0,1,2\}$$

Then $x \in C \iff a_n \neq 1 \ \forall n$, i.e. x is not in the 'middle interval' at any scale.

Remark 20. *Here we use the notation convention*

$$\frac{1}{3^N} = \sum_{n=N+1}^{\infty} \frac{2}{3^n}$$

Theorem 21. *C is uncountable, moreover, $|C| = |\mathbb{R}|$*

Proof. Define $f : C \to [0,1]$ as follows:
if $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ where $a_n \in \{0,2\}$, let $b_n = \begin{cases} 0, & \text{if } a_n = 0 \\ 1, & \text{if } a_n = 2 \end{cases}$, and let $f(x) = \sum_{n=1}^{\infty} \frac{b_n}{2^n}$.
Then f is surjective, which shows $|C| \geq |[0,1]|$ (**exercise**).
$|C| \leq |[0,1]|$ is obvious by using the natural injection $C \hookrightarrow [0,1]$. $\square$

Lecture 3

Limits and continuity.

Definition 22. *Let (M, d) be a metric space. A sequence $\{x_n\}$ in M* **converges** *to $x \in M$ if $\forall \varepsilon$ $\exists N > 0$ s.t. $\forall n \geq N$ $d(x, x_n) < \varepsilon$.*

Remark 23. *$x_n \to x$ in $M \iff d(x_n, x) \to 0$ in $\mathbb{R}$.*

Theorem 24. *Let $x \in M$, $S \subseteq M$.*

(1) x is a contact point of S iff there exists a sequence $\{x_n\}$ in S s.t. $x_n \to x$
(2) x is a limit point of S iff there exists a sequence $\{x_n\}$ of distinct points *in S s.t. $x_n \to x$*

Proof. We will prove (1), the proof of (2) is similar.
$\Longleftarrow$: **exercise**.
$\Longrightarrow$: if x is a contact point of S, then $\forall n \in \mathbb{N}$ $B(x, 1/n)$ contains a point of S, call it x_n. Obtain a sequence $\{x_n\}$ in S s.t. $d(x, x_n) < 1/n \Longrightarrow x_n \to x$. □

Continuity.

Definition 25. *Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is* **continuous at** *$x_0 \in X$ if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $d_X(x_0, x) < \delta \Longrightarrow d_Y(f(x_0), f(x)) < \varepsilon$. Equivalently, $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $f(B_X(x_0, \delta)) \subseteq B_Y(f(x_0), \varepsilon)$. f is* **continuous** *if it is continuous at every $x \in X$.*

Theorem 26. *$f : X \to Y$ is continuous at $x_0 \in X$ iff for every nbhd V of $f(x_0)$ there exists a nbhd U of x_0 s.t. $f(U) \subseteq V$.*

Proof. HW □

Theorem 27. *f is continuous at $x_0 \iff$ for each sequence $x_n \to x_0$ we have $f(x_n) \to f(x_0)$, i.e. $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$.*

Proof. $\Longrightarrow$: let f be continuous at x_0, and let $x_n \to x_0$. Let $\varepsilon > 0$, by continuity of f, $\exists \delta > 0$ s.t. $d_X(x_0, x) < \delta \Longrightarrow d_Y(f(x_0), f(x)) < \varepsilon$. Since $x_n \to x_0$, for that δ $\exists N \in \mathbb{N}$ s.t. $d_X(x_0, x_n) < \delta$ $\forall n \geq N$. Combining, we get that $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $d_Y(f(x_0), f(x_n)) < \varepsilon$ $\forall n \geq N \Longrightarrow f(x_n) \to f(x_0)$.
$\Longleftarrow$: Assume that f is not continuous at x_0. Then $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$ $\exists x_\delta$ s.t. $d_X(x_0, x_\delta) < \delta$ but $d_Y(f(x_0), f(x)) \geq \varepsilon$. Consider a sequence $\delta_n = 1/n$, $n \in \mathbb{N}$. Then $\forall n$ $\exists x_n$ s.t. $d_X(x_0, x_n) < 1/n$ but $d_Y(f(x_0), f(x)) \geq \varepsilon$. Thus, $x_n \to x_0$ but $f(x_n) \nrightarrow f(x_0)$. □

Theorem 28. *Let $f : X \to Y$. The following are equivalent (TFAE):*

(1) f is continuous

(2) $S \subseteq Y$ open $\Longrightarrow f^{-1}(S) \subseteq X$ is open
(3) $S \subseteq Y$ closed $\Longrightarrow f^{-1}(S) \subseteq X$ is closed

Proof. (1)$\Longrightarrow$(2): Let $x \in f^{-1}(S)$, i.e. $f(x) \in S$. Since S is open, $\exists \varepsilon > 0$ s.t. $B_Y(f(x), \varepsilon) \subseteq S$. By continuity of f, $\exists \delta > 0$ s.t. $f(B_X(x, \delta)) \subseteq B_Y(f(x), \varepsilon) \subseteq S \Longrightarrow B_X(x, \delta) \subseteq f^{-1}(S) \Longrightarrow f^{-1}(S)$ is open.
(2)$\Longrightarrow$(1): Let $x \in X$, and let V be an open nbhd of $f(x)$. By assumption, $f^{-1}(V)$ is open. Moreover, $x \in f^{-1}(V)$, so $U := f^{-1}(V)$ is a nbhd of x. It is straightforward to check that $f(U) = f\left(f^{-1}(V)\right) \subseteq V$, so by Theorem 26 we conclude that f is continuous at x.
(2) $\Longleftrightarrow$ (3): **exercise**. $\square$

Remark 29. *If $f : X \to Y$ is continuous, and $y \in Y$, then $f^{-1}(\{y\})$ is a closed subset of X.*

Definition 30. *A subset $X \subseteq M$ is* **dense** *if $\overline{X} = M$.*

Example 31. *$\mathbb{Q}$ is dense in $\mathbb{R}$ (with $|\cdot|$)*

Lecture 4

Completeness.

Definition 32. *A sequence $\{x_n\}$ in a metric space (M, d) is a* **Cauchy sequence** *if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t. $d(x_m, x_n) < \varepsilon\ \forall m, n \geq N$*

Theorem 33. *Every convergent sequence is a Cauchy sequence*

Proof. Let $x_n \to x$, then $\forall \varepsilon > 0\ \exists N$ s.t. $d(x_n, x) < \varepsilon/2\ \forall n \geq N \Longrightarrow$ if $m, n \geq N$, then $d(x_n, x_m) \leq$(by triangle inequality)$d(x_n, x) + d(x_m, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon$ $\square$

What about the converse?

Example 34. *$M = \mathbb{Q}$, $d = |\cdot|$. Let x_n be the n-th decimal approximation of $\sqrt{2}$. Then $\{x_n\}$ is a Cauchy sequence in M, but $\nexists x \in \mathbb{Q}$ s.t. $x_n \to x$.*
Alternatively, you can define

$$x_0 = 1 + \frac{1}{2},\ x_1 = 1 + \frac{1}{1 + 1 + \frac{1}{2}}, \ldots, x_n = 1 + \frac{1}{1 + x_{n-1}}$$

This is called a continued fraction expansion. *It's an* **exercise** *to check that x_n is Cauchy and $x_n \to \sqrt{2}$ in $\mathbb{R}$ (thus, x_n doesn't have a limit in $\mathbb{Q}$).*

Theorem 35. *Every Cauchy sequence is bounded, i.e. $\exists B(a, R)$ s.t. $\{x_n\} \subseteq B(a, R)$.*

Proof. $\{x_n\}$ is Cauchy $\Longrightarrow \exists N \in \mathbb{N}$ s.t. $d(x_n, x_m) < 1\ \forall m, n \geq N$.
Let $r := \max_{1 \leq n \leq N} d(x_n, x_N)$ and $R := \max\{r, 1\}$.

Then $\forall n \geq 1$ we have $d(x_n, x_N) \leq \begin{cases} r, & \text{if } n \leq N, \\ 1, & \text{if } n \geq N \end{cases} \implies d(x_n, x_N) \leq R \; \forall n \geq 1 \implies$ $\{x_n\} \subseteq B(x_N, R+1)$ □

Theorem 36. *If $\{x_n\}$ is a Cauchy sequence, and it has a subsequence $\{x_{n_i}\}$ converging to a point x, then $\{x_n\} \to x$.*

Proof. HW □

Definition 37. *A metric space M is* **complete** *if every Cauchy sequence in M converges.*

Examples.

(1) $\mathbb{Q}$ with Euclidean metric is not complete (see Example 34)
(2) $\mathbb{R}$ with Euclidean metric is complete

Proof. Recall the least upper bound property of $\mathbb{R}$ (completeness of $\mathbb{R}$ as a field): for any non-empty subset $T \subseteq \mathbb{R}$ that is bounded above, $\sup(T)$ exists in $\mathbb{R}$. Now, let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}$, by Theorem 35 $\{s_n\}$ is bounded. This implies that in particular the set $\{x_n, x_{n+1}, \ldots\}$ is bounded for each $n \in \mathbb{N}$. By the least upper bound property $u_n := \sup_{i \geq n} x_i$ is well defined. Moreover, the sequence $\{u_n\}$ is non-increasing and bounded below (**exercise**) and therefore converges by Monotone Convergence Theorem. Denote $u = \lim u_n$. **Claim:** $x_n \to u$.
To prove that, fix $\varepsilon > 0$, and let N_1 be s.t. $|x_n - x_m| < \varepsilon/2 \; \forall m, n \geq N_1$, where we are using the fact that $\{x_n\}$ is a Cauchy sequence. Fixing $n \geq N_1$ we thus have $x_m < x_n + \varepsilon/2 \; \forall m \geq n$. Since $u_n = \sup_{m \geq n}\{x_m\}$, this gives $x_m \leq u_n < x_n + \varepsilon/2$ $\forall m \geq n$, and in particular $x_n \leq u_n < x_n + \varepsilon/2$. We therefore obtain $|x_n - u_n| < \varepsilon$ $\forall n \geq N_1$. Now let N_2 be s.t. $|u_n - u| < \varepsilon/2 \; \forall n \geq N_2$. Choosing $N = \max\{N_1, N_2\}$ and using the triangle inequality we obtain

$$|x_n - u| \leq |x_n - u_n| + |u_n - u| < \varepsilon \quad \forall n \geq N.$$

□

(3) $\mathbb{R}^n$ with Euclidean metric is complete

Proof. Follows from completeness of $\mathbb{R}$, **exercise**. □

(4) $(0, 1)$ with Euclidean metric is not complete, $[0, 1]$ is.

Theorem 38. *Let M be a complete metric space, $S \subseteq M$. Then S is complete $\iff$ S is closed. (The implication S is complete $\implies$ S is closed is true even if M is not complete).*

Proof. HW □

(5) $M = C([a, b])$ with $d(f, g) = \sup_{t \in [a,b]} |f(t) - g(t)|$ is complete.

Proof. See Theorem 40 in the next section. □

Lecture 5

Uniform convergence. Let $\{f_n\}$ be a sequence of functions, $f_n : X \to Y$, where (X, d_X) and (Y, d_Y) are metric spaces.

Definition 39. $f_n \to f$ **pointwise** *if* $\forall x \in X, \forall \varepsilon > 0\ \exists N = N(\varepsilon, x)$ *s.t.* $d_Y(f_n(x), f(x)) < \varepsilon\ \forall n \geq N$.
$f_n \to f$ **uniformly** *if* $\forall \varepsilon > 0\ \exists N = N(\varepsilon)$ *s.t.* $d_Y(f_n(x), f(x)) < \varepsilon\ \forall n \geq N, \forall x \in X$. *Equivalently,* $\forall \varepsilon > 0\ \exists N = N(\varepsilon)$ *s.t.* $\sup_{x\in X} d_Y(f_n(x), f(x)) < \varepsilon\ \forall n \geq N$.

Consider $M = C([a, b])$ with $d(f, g) = \sup_{t\in[a,b]} |f(t) - g(t)|$. Let $\{f_n\}$ be a sequence in M. Note that $f : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ is endowed with the Euclidean metric. Suppose now that $f_n \to f$ uniformly, i.e.

$$\forall \varepsilon > 0\ \exists N \text{ s.t. } \sup_{t\in[a,b]} |f_n(t) - f(t)| < \varepsilon\ \forall n \geq N,$$

which is equivalent to

$$\forall \varepsilon > 0\ \exists N \text{ s.t. } d_M(f_n, f) < \varepsilon\ \forall n \geq N.$$

Therefore, the uniform convergence of a sequence of functions from $C([a, b])$ is equivalent to the convergence of the sequence in the metric space (M, d). We have the following result

Theorem 40. *The space $C([a, b])$ with the supremum metric is complete.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $C([a, b])$. In proving completeness, there are usually three steps: 1. find a candidate for f, the limit of f_n; 2. prove that $f_n \to f$ in the right metric; 3. prove that f belongs to the right space (in this case $C([a, b])$)
1. Since $\{f_n\}$ is Cauchy, we have

$$\forall \varepsilon > 0\ \exists N\colon\ \sup_{t\in[a,b]} |f_n(t) - f_m(t)| < \varepsilon \quad \forall m, n \geq N,$$

which implies $|f_n(t) - f_m(t)| < \varepsilon\ \forall t \in [a, b] \implies$ for each fixed $t \in [a, b]$, $\{f_n(t)\}$ is a Cauchy sequence in $(\mathbb{R}, |\cdot|) \implies$ (by completness of $\mathbb{R}$)$\implies f_n(t) \to f(t)$ in $\mathbb{R}$ $\forall t \in [a, b]$ (pointwise convergence). This f is our candidate.
2. Let $\varepsilon > 0$, since $\{f_n\}$ is Cauchy $\exists N$ s.t. $d(f_m, f_n) < \varepsilon/6\ \forall m, n \geq N$. Moreover, $\forall t \in [a, b]\ \exists m = m(t, \varepsilon)$ which we can take to be $\geq N$ s.t. $|f_m(t) - f(t)| < \varepsilon/6$ by pointwise convergence. Thus $\forall t \in [a, b], \forall n \geq N$

$$|f_n(t) - f(t)| \leq |f_n(t) - f_m(t)| + |f_m(t) - f(t)| < \varepsilon/3$$

$\implies \sup_{t\in[a,b]} |f_n(t) - f(t)| < \varepsilon/3$, so $f_n \to f$ uniformly.
3. Let $t \in [a, b]$, $\varepsilon > 0$, by part 2. $\exists N$ s.t. $d(f_n, f) < \varepsilon/3\ \forall n \geq N$. Since f_N is continuous, $\exists \delta = \delta(\varepsilon, N, t)$ s.t. $|f_N(t) - f_N(x)| < \varepsilon/3\ \forall x \in B(t, \delta)$. This gives

$$|f(t) - f(x)| \leq |f(t) - f_N(t)| + |f_N(t) - f_N(x)| + |f_N(x) - f(x)| < \varepsilon, \quad \forall x \in B(t, \delta).$$

□

Corollary 41. *Let $\{f_n\} \subseteq C([a,b])$, and suppose $f_n \to f$ uniformly. Then f is continuous on $[a,b]$.*

Proof. Follows from part 3. of the proof of Theorem 40. □

Example 42. *Let $f_n(x) = \begin{cases} 1 - nx, & \text{if } 0 \le x < 1/n \\ 0, & \text{if } 1/n \le x \le 1 \end{cases}$. It is easy to check that $f_n \in C([0,1])$ $\forall n$. Moreover, $\forall x \in (0,1]$ $\exists N > 1/x$ s.t. $f_n(x) = 0$ $\forall n \ge N$, which implies $f_n(x) \to 0$ $\forall x \in (0,1]$. We also have $f_n(0) = 1$ $\forall n$, so*

$$f_n(x) \to f(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } 0 < x \le 1 \end{cases} \quad \textit{pointwise,}$$

and $f(x)$ is clearly discontinuous at $x = 0$, so $f \notin C([0,1])$.
Moreover, $\sup_{x\in[0,1]} |f_n(x) - f(x)| = 1$ $\forall n$, so f_n does not converge to f uniformly.

Lecture 6

Let (M,d) be a metric space.

Definition 43. *The* **diameter** *of $S \subseteq M$ is $diam(S) := \sup_{t,s\in S} d(t,s)$.*

Note that S is bounded iff $diam(S) < \infty$ — **exercise**.

Theorem 44. *TFAE:*

(1) M is complete
(2) For every nested sequence $S_1 \supseteq S_2 \supseteq S_3 \supseteq \dots$ of closed non-empty sets S_i with $diam(S_i) \to 0$, we have $\bigcap_i S_i \neq \emptyset$

Proof. (1)$\Longrightarrow$(2): let M be complete, and $S_1 \supseteq S_2 \supseteq S_3 \supseteq \dots$ a sequence as above. $\forall i$ choose $s_i \in S_i$ (axiom of choice). **Claim:** $\{s_i\}$ is Cauchy.
Proof of Claim: let $\varepsilon > 0$, choose N s.t. $diam(S_n) < \varepsilon$ $\forall n \ge N$. Then $\forall n,m \ge N$ we have $s_n, s_m \in S_N \Longrightarrow d(s_n,s_m) \le diam(S_N) < \varepsilon$.
Since M is complete, $s_n \to s \in M$. We are left to show $s \in \bigcap_i S_i$. Fix $i \in \mathbb{N}$ and consider the sequence $\{s_{i+n}\}_{n\ge1}$. It is easy to see that $\lim_{n\to\infty} s_{i+n} = s$, and since S_i is closed we have $s \in S_i$. This is true for all i, and thus $s \in \bigcap_i S_i$.
(2)$\Longrightarrow$(1): let $\{x_n\}$ be a Cauchy sequence in M. For each $i \in \mathbb{N}$ let $T_i := \{x_i, x_{i+1}, \dots\}$, and define $S_i := \overline{T_i}$. Then $\{S_i\}$ satisfies all the conditions in (2) (**exercise**) and therefore $\exists x \in \bigcap_i S_i$. **Claim:** $\lim_{n\to\infty} x_n = x$.
Proof of Claim: let $\varepsilon > 0$, choose N s.t. $diam(S_n) < \varepsilon$ $\forall n \ge N$. Since $x \in S_n$ $\forall n$ and $x_n \in S_n$, $d(x,x_n) < diam(S_n) < \varepsilon$ $\forall n \ge N$. □

Contraction mappings.

Definition 45. *Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is a* **contraction mapping** *if*

$$\exists \alpha < 1 \ \ s.t. \ \ d_Y(f(s), f(t)) \leq \alpha d_X(s,t) \ \ \forall s,t \in X.$$

Proposition 46. *f is a contraction mapping $\Longrightarrow$ f is continuous.*

Proof. HW. □

Banach Fixed Point Theorem. *Let $f : M \to M$ be a contraction mapping, and M is complete. Then f has a unique* **fixed point**, *i.e. $x \in M$ s.t. $f(x) = x$.*

Proof. 1. Uniqueness: Let $f(x) = x$ and $f(x) = y$, then

$$d(x,y) = d(f(x), f(y)) \leq \alpha d(x,y) \Longrightarrow (1-\alpha)d(x,y) \leq 0 \Longrightarrow d(x,y) = 0 \Longrightarrow x = y.$$

2. Existence: We will construct the point x as a limit of successive approximations. Let $x_0 \in M$, define $x_{n+1} = f(x_n)$, $\forall n \geq 0$, i.e. $x_n = f^{(n)}(x_0)$. **Claim:** $\{x_n\}$ converges. Proof of claim: Let $m \leq n$, then

$$\begin{aligned} d(x_n, x_m) &= d(f^{(n)}(x_0), f^{(m)}(x_0)) \leq \alpha^m d(x_0, f^{(n-m)}(x_0)) = \alpha^m d(x_0, x_{n-m}) \\ &\leq \alpha^m \left(d(x_0,x_1) + d(x_1,x_2) + \ldots + d(x_{n-m-1}, x_{n_m})\right) \\ &\leq \alpha^m \left(1 + \alpha + \alpha^2 + \ldots + \alpha^{n-m-1}\right) d(x_0,x_1) \leq \frac{\alpha^m}{1-\alpha} d(x_0,x_1) \end{aligned}$$

Thus, given $\varepsilon > 0$ can choose N (**exercise**) s.t. $\forall m,n \geq N$ $d(x_n, x_m) < \varepsilon \Longrightarrow$ $\{x_n\}$ is Cauchy $\Longrightarrow$(by completeness of M) $x_n \to x \in M$. Since f is continuous, $\lim_{n\to\infty} f(x_n) = f(x)$. OTOH, $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x \Longrightarrow f(x) = x$. □

Examples.

(1) Let $f \in C^1([a,b])$ with $|f'(x)| \leq K < 1$ $\forall x \in [a,b]$. By the Mean Value Theorem $\forall x,y \in [a,b]$ $\exists \xi \in [a,b]$ s.t.

$$|f(x) - f(y)| = |f'(\xi)||x-y| \leq K|x-y|$$

so f is a contraction. Moreover, $[a,b]$ with Euclidean distance is complete. Therefore, by Banach fixed pt thm $\exists$ a unique $x \in [a,b]$ s.t. $f(x) = x$.
OK, let $f(x) = x/2 - 1 \in C([0,1])$. Clearly, f is differentiable and $f'(x) = 1/2 < 1$. But $f(x) = x$ gives $x/2 - 1 = x \Longrightarrow x = -2 \notin [0,1]$. What is the problem?

(2) Let $f : [0,\infty) \to [0,\infty)$ be defined by $f(x) = \sqrt{1+x^2}$. Then $f'(x) = \frac{x}{\sqrt{1+x^2}} < 1$, so $|f(x) - f(y)| < |x-y|$. However, $f(x) = x$ implies $1 + x^2 = x^2$ which has no solutions. This example shows that we need $\alpha < 1$ that works for all x.

(3) Let $f : \mathbb{R}\backslash\{2\} \to \mathbb{R}\backslash\{2\}$ given by $f(x) = x/2 + 1$. $f'(x) = 1/2$, therefore it is a contraction. There is no fixed point in $\mathbb{R}\backslash\{2\}$, which shows the necessity of the completeness assumption.

Lecture 7

Picard's theorem. Suppose we want to solve the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$. The right hand side can be a nonlinear function of y so how do we solve it? Idea: solve $y' = f(x, y_0)$, get y_1. Plug this into the RHS, solve $y' = f(x, y_1)$, get y_2. Continue forever, and hope that this process converges to a solution.

Picard's Theorem. *Let $f(x, y)$ be a continuous function on a closed rectangle $R \subseteq \mathbb{R}^2$ whose interior contains (x_0, y_0). Suppose f is* Lipschitz in y uniformly in x, *i.e.* $\exists M \geq 0$ *s.t*

$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2|, \quad \forall x \text{ s.t. } (x, y_1), (x, y_2) \in R.$$

Then there exists an interval $|x - x_0| \leq \delta$ in which the differential equation $y' = f(x, y)$ has a unique solution $y = \varphi(x)$ satisfying the initial condition $\varphi(x_0) = y_0$.

Proof. 1. Choose $\delta > 0$ and $b > 0$ s.t. $[x_0 - \delta, x_0 + \delta] \times [y_0 - b, y_0 + b] \subseteq R$ and $M\delta < 1$. Denote $I_\delta = [x_0 - \delta, x_0 + \delta]$.
2. Let $D \subseteq C(I_\delta)$ be the set $D := \{f \in C(I_\delta) : f(I_\delta) \subseteq [y_0 - b, y_0 + b]\}$. Then D is closed with respect to the uniform metric (**exercise**) $\Longrightarrow$ D is complete.
3. Define $\lambda : D \to D$ by $\lambda[\varphi](t) = y_0 + \int_{x_0}^t f(s, \varphi(s))ds$. Then λ is continuous (**exercise**) and

$$\lambda[\varphi](t) - y_0 = \int_{x_0}^t f(s, \varphi(s))ds \leq \int_{x_0}^t |f(s, \varphi(s))|ds \leq K|t - x_0| \leq K\delta \leq b$$

In the above inequalities we used $|f(x, y)| \leq K$ on R since f is continuous and R is closed and bounded (Extreme Value Theorem). The last inequality follows by choosing δ sufficiently small. We therefore get $\lambda(D) \subseteq D$.
4. The last step is to show λ is a contraction. Using the fact that f is Lipschitz, we get

$$\begin{aligned} |\lambda[\varphi](t) - \lambda[\psi](t)| &\leq \int_{x_0}^t |f(s, \varphi(s)) - f(s, \psi(s))|ds \\ &\leq M \int_{x_0}^t |\varphi(s) - \psi(s)|ds \\ &\leq M\delta \sup_{s \in I_\delta} |\varphi(s) - \psi(s)| = M\delta d(\varphi, \psi) \end{aligned}$$

Taking $\sup_{t \in I_\delta}$ in the above inequality we therefore obtain

$$d(\lambda[\varphi] - \lambda[\psi]) \leq M\delta d(\varphi, \psi) < d(\varphi, \psi)$$

by making δ even smaller if necessary to satisfy $M\delta < 1$.
From 1.-4. we conclude that $\exists$ a unique fixed point $\varphi \in D$ s.t. $\lambda[\varphi] = \varphi$, i.e. $\varphi(t) = y_0 + \int_{x_0}^t f(s, \varphi(s))ds$. It is an **exercise** to check that $\varphi(t)$ is differentiable, and taking the derivative we get $\varphi'(t) = f(t, \varphi(t))$. $\varphi(x_0) = y_0$ trivially. ∎

Example 47. *Consider the following IVP*

$$y' = y^{2/3},\ y(0) = 0$$

Solve:

$$\frac{dy}{y^{2/3}} = dx \implies 3y^{1/3} = x + C \implies y = (x+C)/27$$

$y(0) = 0 \implies y = x^3/27$. *However,* $y = \begin{cases} 0 & \text{if } x \in (a,b) \\ (x-b)^3/27 & \text{if } x \geq b \\ (x-a)^3/27 & \text{if } x \leq a \end{cases}$ *is a solution for any* $a < 0 < b$*, as well as* $y \equiv 0$*, therefore, we have infinitely many solutions.* **Exercise:** *check that* $f(x,y) = y^{2/3}$ *is not Lipschitz in* y*.*

Lecture 8

Completion.

Definition 48. *Let* M *be a metric space. Complete metric space* M^* *is called a* **completion** *of* M *if* M *is a subspace of* M^* *(meaning* $M \subseteq M^*$ *and* $d_{M^*}\big|_M = d_M$*) and* $\overline{M} = M^*$*.*

Here $\overline{M}$ is the closure of M in M^*.

Theorem 49. *Every metric space* M *has a completion. This completion is unique up to an isometry that has all the points in* M *as its fixed points.*

Proof. **Uniqueness**. Let M^* and M^{**} be two completions of M. We need to show there exists $\varphi : M^* \to M^{**}$ s.t.

(1) $\varphi(x) = x\ \forall x \in M$ (fixed points)
(2) $d_{M^{**}}(\varphi(x^*), \varphi(y^*)) = d_{M^*}(x^*, y^*)\ \forall x^*, y^* \in M^*$ (isometry)

We will define φ as follows. Let $x^* \in M^*$, by definition of completion x^* is a contact point of M, therefore, $\exists \{x_n\} \subseteq M$ s.t. $x_n \to x^* \in M^*$. Since $\{x_n\}$ converges in M^* it is Cauchy in M^*, and thus in M. We also have $\{x_n\} \subseteq M^{**}$ and M^{**} is complete, thus $x_n \to x^{**} \in M^{**}$. **Exercise:** x^{**} does not depend on the choice of the sequence $\{x_n\}$ converging to x^*. Define $\varphi : M^* \to M^{**}$ by $\varphi(x^*) = x^{**}$. First, if $x \in M$ and $x_n \to x \in M$, then $x^{**} = x$. So $\varphi(x) = x$. Second, if $x_n \to x^* \in M^*$, $x_n \to x^{**} \in M^{**}$, and $y_n \to y^* \in M^*$, $y_n \to y^{**} \in M^{**}$, then

$$d_{M^{**}}(x^{**}, y^{**}) = \lim_{n\to\infty} d_{M^{**}}(x_n, y_n) = \lim_{n\to\infty} d_M(x_n, y_n) = \lim_{n\to\infty} d_{M^*}(x_n, y_n) = d_{M^*}(x^*, y^*)$$

by the result from HW.
Existence. Given any metric space M consider the set of Cauchy sequences in M. Define the equivalence relation on it by $\{x_n\} \sim \{y_n\}$ if $d_M(x_n, y_n) \to 0$ as $n \to \infty$. It is an **exercise** to check that it is indeed an equivalence relation (reflexive, symmetric,

transitive). We can therefore partition the set of Cauchy sequences in M into disjoint equivalence classes, call this set M^*. Define the distance on M^* as follows

$$d_{M^*}(x^*, y^*) = \lim_{n\to\infty} d_M(x_n, y_n), \tag{1}$$

where $\{x_n\}$ and $\{y_n\}$ are some representatives of x^* and y^* respectively. We need to show that it is in fact a distance, namely, that the limit exists and doesn't depend on the choice of representatives; and the three properties defining a metric hold. First, by one of the HW questions

$$|d_M(x_n, y_n) - d_M(x_m, y_m)| \leq d_M(x_n, x_m) + d_M(y_n, y_m)$$

and the right hand side can be made arbitrarily small by choosing n, m large enough, since both $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in M. This shows that $\{d(x_n, y_n)\}$ is a Cauchy sequence in $\mathbb{R}$ and therefore converges. Now, if $\{x_n\}$ and $\{x_n'\}$ are two representatives of x^*, and $\{y_n\}$ and $\{y_n'\}$ are two representatives of y^*, we have

$$|d_M(x_n, y_n) - d_M(x_n', y_n')| \leq d_M(x_n, x_n') + d_M(y_n, y_n') \to 0, \quad \text{as } n \to \infty$$

since $\{x_n\} \sim \{x_n'\}$ and $\{y_n\} \sim \{y_n'\}$. This shows that the limit doesn't depend on the choice of the representative of the equivalence class. Symmetry and non-negativity follow immediately from the definition. The triangle inequality follows from the triangle inequality for d_M and definition (1).
Finally, we need to show that M^* is a completion of M. For each $x \in M$ let $x^* \in M$ be the equivalence class of Cauchy sequences converging to x. If $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$, then

$$d_M(x, y) = \lim_{n\to\infty} d_M(x_n, y_n) = d_{M^*}(x^*, y^*),$$

so the map $M \to M^*$ given by $x \mapsto x^*$ is an isometry, and we can consider M as a subspace of M^*. Write d for d_{M^*}. We are left to show that

(1) $\overline{M} = M^*$
(2) M^* is complete

To see (1) we only need to show $M^* \subseteq \overline{M}$ (**exercise**). Let $x^* \in M^*$, fix $\varepsilon > 0$, and choose some representative of x^*, i.e. a Cauchy sequence $\{x_n\} \subseteq M$. Then $\exists N$ s.t. $d(x_n, x_m) < \varepsilon \ \forall n, m \geq N$. Therefore,

$$d_{M^*}(x_m, x^*) = \lim_{n\to\infty} d_M(x_m, x_n) < \varepsilon \ \forall n \geq N$$

so $x_m \to x^*$ as $m \to \infty$, which shows x^* is a contact point of M. Finally, to show completeness of M^* we note that by definition of M^*, every Cauchy sequence $\{x_n\}$ in M converges to a point $x^* \in M^*$. Moreover, M is a dense subset of M^*. By one of the homework problems it shows that M^* is complete. $\square$

Example 50. *Construct $\mathbb{R}$ as the completion of $\mathbb{Q}$ with Euclidean distance. Problem: extend the arithmetic.*

TOPOLOGICAL SPACES

Lecture 1

Definition 1. *A **topology** on a set X is a collection of subsets τ of X such that*

(1) $\emptyset \in \tau$
(2) $X \in \tau$
(3) τ is closed under arbitrary unions
(4) τ is closed under finite intersections

Elements of τ are called **open sets***; (X, τ) —* **topological space**.

Definition 2. *A subset of a topological space is* **closed** *if its complement is open.*

Definition 3. *A* **neighborhood** *of $x \in X$ is a set containing an open set containing x.*

Examples.

(1) Any metric space
(2) Let X be any set, let $\tau := \{\emptyset, X\}$. This is the **trivial or indiscrete** topology, the coarsest possible topology.
(3) Let X be any set, let $\tau := 2^X$. This is the **discrete** topology, the finest possible topology.
(4) $X = \{a, b\}$, $\tau = \{\emptyset, X, \{b\}\}$. Closed sets: $X, \emptyset, \{a\}$.

Definition 4. *A collection of subsets $\mathcal{B}$ is a* **base (basis)** *for topology τ if every open set in X is a union of sets in $\mathcal{B}$. A collection of sets $\mathcal{S}$ is a* **subbase (subbasis)** *if the collection of finite intersections of elements of $\mathcal{S}$ is a basis for τ.*

Example 5. *(M, d) metric space, open balls = base*

Definition 6. *Let (X, τ) be a topological space, $Y \subseteq X$. The* **subspace topology** *ζ in Y is defined by*

$$G \in \zeta \iff \exists T \in \tau \colon G = T \cap Y.$$

Product topology.

Definition 7. *X, Y — topological spaces, the* **product topology** *on $X \times Y$ is the topology whose base consists of sets of the form $U \times V$ with U open in X and V open in Y.*
Equivalently, the product topology is the one generated by the subbase of the form $\mathcal{S} = \{p_1^{-1}(U) : U \text{ open in } X\} \cup \{p_2^{-1}(V) : V \text{ open in } Y\}$. This can be generalized to $\prod_{i \in I} X_i$ by defining the subbase as $\mathcal{S} = \bigcup_{i \in I} S_i$, $S_i = \{p_i^{-1}(U_i) : U_i \text{ open in } X_i\}$.

Here p_k is the **canonical projection** $p_k : \prod_{i\in I} X_i \to X_k$ defined by $p_k((x_i)_{i\in I}) = x_k$.

Example 8. *Let X, Y be two topological spaces. Consider the set F of all functions from X to Y. We can define* **topology of pointwise convergence** *on it as follows. Consider $f : X \to Y$ as a point in $Y^X = \prod_{x\in X} Y_x$ whose x-coordinate is $f(x)$. Here Y_x is just a copy of Y, but we take one for each $x \in X$. Then $Y^X = \prod_{x\in X} Y_x$ has the product topology, i.e. given a point $x \in X$ and an open set $U \subseteq Y$ the sets of the form*

$$S(x, U) = \{f : f \in Y^X \text{ and } f(x) \in U\}$$

form a subbase for a topology on F. We have the following result

Definition 9. *Let (X, τ) be a topological space. A sequence $x_n \to x$ in X if for every nbhd U of x $\exists N \in \mathbb{N}$ s.t. $x_n \in U$ $\forall n \geq N$.*

Theorem 10. *A sequence of functions $\{f_n\}$ converges to f in the topology of pointwise convergence iff $\forall x \in X$ $f_n(x) \to f(x)$ in Y, i.e. $f_n \to f$ pointwise.*

Proof. $\Longrightarrow$: Suppose $f_n \to f$ in the topology of pointwise convergence, τ. This means that for any nbhd V of f $\exists N \in \mathbb{N}$ s.t. $f_n \in V$ for all $n \geq N$. For any $x \in X$ and an open set U containing $f(x)$, $S(x, U)$ is a nbhd of f in τ. Therefore, $f_n \in S(x, U)$ for all $n \geq N \Longrightarrow f_n(x) \in U$ $\forall n \geq N$.
$\Leftarrow$: Suppose $f_n(x) \to f(x)$ $\forall x \in X$. It is sufficient to show that for any arbitrary element of the subbase, $S(x, U)$, containing f, we have $f_n \in S(x, U)$ for all n sufficiently large (depending on x and U) (**exercise**). But since $f_n(x) \to f(x)$ $\forall x \in X$ and for any open U containing $f(x)$ $\exists N$ s.t. $f_n(x) \in U$ $\forall n \geq N \Longrightarrow f_n \in S(x, U)$ $\forall n \geq N$. ☐

Lecture 2

Hausdorff spaces. Axioms of separation.

Definition 11. *A topological space X is $\mathbf{T_1}$ if $\forall x, y \in X$, $x \neq y$, $\exists$ nbhd U_x of x and U_y of y s.t. $x \notin U_y$ and $y \notin U_x$. We say that X satisfies the* **first axiom of separation**.

Examples.

(1) Any discrete space is T_1
(2) Metric spaces are T_1
(3) Any space with trivial topology that has at least two points is not T_1
(4) $X = \{a, b\}$, $\tau = \{\emptyset, X, \{b\}\}$, is not T_1

Theorem 12. *Every finite subset of a T_1 space is closed.*

Proof. First note that it is sufficient to show that a single point set is closed (**exercise**). Let X be a T_1 space and $x \in X$. Then $\forall y \in X$ $\exists U_y$ open s.t. $x \notin U_y$. We have $X \backslash \{x\} = \bigcup_{y \in X, y \neq x} U_y$ which is open as a union of open sets. Since $X \backslash \{x\} = \{x\}^c$ is open, $\{x\}$ is closed. $\square$

Definition 13. *X is* **Hausdorff** *or* $\mathbf{T_2}$ *if $\forall x, y \in X$, $x \neq y$, $\exists$ nbhd U_x of x and U_y of y s.t. $U_x \cap U_y = \emptyset$.*

Proposition 14. *Every Hausdorff space is T_1. The reverse is not necessarily true, see Example (4) below.*

Examples.

(1) Any metric space
(2) A discrete topological space
(3) Any space with trivial topology that has at least two points is not T_1
(4) Let X be any set. The **finite complement topology** is defined as follows: $\tau = \{\emptyset, X$, all sets obtained from X by removing a finite number of elements$\}$. Then X is T_1 but not Hausdorff (HW)

Proposition 15. *If X is Hausdorff, then a sequence converges to at most one point.*

Proof. Suppose $x_n \to x$ and $y \neq x$. Then $\exists U_x, U_y$ — nbhds of x and y respectively s.t. $U_x \cap U_y = \emptyset$. $x_n \to x$ implies $\exists N$ s.t. $x_n \in U_x$ $\forall n \geq N \implies \forall n \geq N$ $x_n \notin U_y$. Therefore, $x_n \nrightarrow y$. $\square$

Proposition 16. *Let X be finite T_1 topological space. Then X is discrete, and therefore Hausdorff.*

Proof. Let $X = \{x_1, \ldots, x_n\}$, and fix $x_l \in X$. By T_1 for each $1 \leq k \leq n$, $k \neq l$, $\exists$ a nbhd U_k of x_l s.t. $x_k \notin U_k$. Define $U = \cap_{k \neq l} U_k$, then U is open and $x_l \in U$. By construction, U does not contain any point of X other than x_l and thus $U = \{x_l\}$. $\square$

Continuity.

Definition 17. *Let X and Y be two topological spaces. $f : X \to Y$ is* **continuous** *at $x \in X$ if $\forall$ nbhd V of $f(x)$ $\exists$ a nbhd U of x s.t. $f(U) \subseteq V$.*

Theorem 18. *$f : X \to Y$ is continuous iff $f^{-1}(V)$ is open for every open set $V \subseteq Y$.*

Proof. $\implies$: let $V \subseteq Y$ be an open set and $x \in f^{-1}(V) \implies f(x) \in V$, so V is a nbhd of $f(x) \implies \exists U$ a nbhd of x s.t. $f(U) \subseteq V$ (by continuity) $\implies U \subseteq f^{-1}(V)$
$\impliedby$: let $f(x) \in V$ and V open $\implies x \in f^{-1}(V)$. Since $f^{-1}(V)$ is open we can take $U := f^{-1}(V)$ and it is the required nbhd of x. $\square$

Proposition 19. *If $x_n \to x$ in the topological space X and $f : X \to Y$ is continuous, then $f(x_n) \to f(x)$. The reverse implication is not necessarily true for general topological spaces X and Y.*

Lecture 3

Compactness

Definition 20. *An* **open cover** *of $Y \subseteq X$, a subset of a topological space, is a collection of open sets whose union contains Y.*

Example 21. $\mathbb{R}_+ = (0, \infty)$, *open cover* $\{(0, 2n), (n, 3n) : n \in \mathbb{N}\}$

Definition 22. *$Y \subseteq X$ is* **compact** *if every open cover of Y has a finite subcover.*

Examples.

(1) $\mathbb{R}$ is not compact (with Euclidean topology)
(2) $(0, 1) \subseteq \mathbb{R}$ is not compact: $\{(1/n, 1) : n \in \mathbb{N}\}$
(3) $[0, 1]$ is compact. More generally, $[a, b] \subseteq \mathbb{R}$ is compact.

Proof. Let $\mathscr{C}$ be an open cover of $[a, b]$. Define

$$S := \{x \in [a, b] : [a, x] \text{ has a finite cover}\}$$

First, $a \in S \implies S \neq \emptyset$. Moreover, $S \subseteq [a, b]$, thus by the least upper bound property of $\mathbb{R}$ $\exists c := \sup S$, and $c \in [a, b]$ since b is an upper bound for S. Since $\mathscr{C}$ is an open cover of $[a, b]$, $\exists U_c \in \mathscr{C}$ s.t. $c \in U_c$. By the openness of U_c $\exists \varepsilon > 0$ s.t. $(c - \varepsilon, c + \varepsilon) \subseteq U_c$. Let $x \in S \cap (c - \varepsilon, c]$, **exercise:** why such an x exists? Then by the definition of S, $[a, x]$ has a finite subcover, call it $\mathscr{C}'$. It then follows that $\mathscr{C}' \cup U_c$ is a finite cover of $[a, c + \varepsilon/2]$, i.e. a finite cover of $[a, x]$ for $x = c + \varepsilon/2$. Since c was the supremum of such $x \in [a, b]$ we must have $c + \varepsilon/2 > b$ $\forall \varepsilon > 0$ $\implies c = b$. ☐

(4) $\{0\} \cup \{1/n : n \in \mathbb{N}\}$ (HW)

Theorem 23 (Connection between closed and compact)**.**

(1) A closed subset Y of a compact space X is compact
(2) A compact subset K of a Hausdorff space X is closed

Proof. (1) Let $\{U_\alpha\}$ be an open cover of Y. Then $Y^c \cup \bigcup_\alpha U_\alpha = X$. X is compact $\implies \exists \alpha_1, \ldots, \alpha_k$ s.t. $Y^c \cup \bigcup_{i=1}^k U_{\alpha_i} = X \implies Y \subseteq \bigcup_{i=1}^k U_{\alpha_i}$.

(2) Fix $x \in X \backslash K$, and let $y \in K$. Since X is Hausdorff $\exists U_y \ni y$ and $V_y \ni x$ open s.t. $U_y \cap V_y = \emptyset$. $\{U_y\}_{y \in K}$ is an open cover of K, K is compact $\implies \exists y_1, \ldots y_k \in K$ s.t. $K \subseteq \bigcup_{i=1}^k U_{y_i}$. Thus $V := \bigcap_{i=1}^k V_{y_i}$ is an open nbhd of x s.t. $V \cap \left(\bigcap_{i=1}^k V_{y_i}\right) = \emptyset$. Thus $V \subseteq X \backslash K \implies X \backslash K$ is open $\implies K$ is closed.

☐

Corollary 24. *A compact subset of a metric space is closed.*

Example 25. $[0, \infty) \subseteq \mathbb{R}$ *is closed but not compact*

Theorem 26. *If $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact.*

Proof. HW $\square$

Compactness in metric spaces.

Theorem 27. *Let (M, d) be a metric space, and $K \subseteq M$ a compact subset. Then K is closed and bounded.*

Proof. M is a metric space $\Longrightarrow$ M is Hausdorff $\Longrightarrow$ K is closed. To show boundedness note that $\bigcup_{x\in K} B(x, 1)$ is an open cover of K. Since K is compact, $\exists\{x_1, \dots, x_n\} \subseteq K$ s.t. $\bigcup_{i=1}^n B(x_i, 1) \supseteq K$. Let $R := \max_{i=2,\dots,n} d(x_1, x_i)$, then $B(x_1, R + 1) \supseteq \bigcup_{i=1}^n B(x_i, 1)$ (**exercise**) $\Longrightarrow B(x_1, R+1) \supseteq K \Longrightarrow K$ is bounded. $\square$

Example 28. *Let $S = [0, 1] \cap \mathbb{Q} \subseteq \mathbb{Q}$, where $\mathbb{Q}$ is equipped with Euclidean metric. Then S is closed and bounded in $\mathbb{Q}$ (*__exercise__*), but it is not compact. To see this fix $r \in [0, 1]\backslash\mathbb{Q}$ and let $U_n := ((-1/n, r - 1/n) \cup (r + 1/n, 1 + 1/n)) \cap \mathbb{Q}$. Then $S \subseteq \bigcup_{n\in\mathbb{N}} U_n$, U_n is open in $\mathbb{Q}$, but there is no finite subcover.*

Example 29. *Let $M = l^2$ with the usual distance. Consider the unit sphere*

$$S = \{x \in l^2 : \sum_{n=1}^{\infty} x_n^2 = 1\}.$$

*This set is clearly bounded, and it is also closed (*__exercise__*). Since M is complete, S is also complete. Now, $\bigcup_{x\in S} B(x, 1/2)$ is an open cover of S, and if $y, z \in B(x, 1/2)$ we have $d(y, z) < 1$. Consider $\{e_n\} \subseteq S$ where $e_n = (0, 0, \dots, 1, 0, \dots)$, i.e. e_n has 1 in the nth spot and 0 everywhere else. Then $d(e_m, e_n) = \sqrt{2} > 1$ $\forall m \neq n$ so e_m and e_n cannot belong to the same ball $B(x, 1/2)$. Since $\{e_n\}$ is an infinite subset and we need at least one ball to cover each element, we cannot select a finite subcover from $\bigcup_{x\in S} B(x, 1/2)$ to cover $\{e_n\}$, and thus S. This implies that S is not compact.*

Total boundedness.

Definition 30. *Let (M, d) be a metric space, and $\varepsilon > 0$. A set $A \subseteq M$ is called an* **ε-net** *for a set $S \subseteq M$, if $\forall x \in S$ $\exists a \in A$ s.t. $d(x, a) < \varepsilon$.*

Example 31. *Let $(M, d) = (\mathbb{R}^n, |\cdot|)$, and $A = \mathbb{Z}^n$ (all the points with integer coordinates). Let $x \in \mathbb{R}^n$, then $\forall k = 1, \dots, n$ $\exists a_k \in \mathbb{Z}$ s.t. $|x_k - a_k| \le 1 \Longrightarrow d(x, a) = \left(\sum_{k=1}^n (x_k - a_k)^2\right)^{1/2} \le \sqrt{n}$, where we defined $a := (a_1, \dots, a_n) \in A$. Therefore, A is $\sqrt{n}/2$-net for M. Let*

Definition 32. *$S \subseteq M$ is* **totally bounded** *if $\forall \varepsilon > 0$ S has a finite ε-net.*

Properties.

(1) every subset of a totally bounded set is bounded
(2) totally bounded $\Longrightarrow$ bounded
(3) bounded $\not\Longleftarrow$ totally bounded

Proof. (1) trivial
(2) **exercise**
(3) Recall Example 29, in particular $d(e_m, e_n) = \sqrt{2}$. It follows that there is no ε-net for S for $\varepsilon < \sqrt{2}/2$. Thus S is not totally bounded.

□

Theorem 33. *(M, d) is compact $\iff$ it is complete and totally bounded*

Proof. We will prove this theorem later, after we discuss sequential compactness. □

Corollary 34. *Let (M, d) be a complete metric space. Then $S \subseteq M$ is compact iff it is closed and totally bounded.*

Example 35. *Cantor set is compact (as a subset of $(\mathbb{R}, |\cdot|)$). First recall that C is closed, and therefore complete since $\mathbb{R}$ is complete. Now we claim that C is totally bounded. For every $\varepsilon > 0$ choose $n \in \mathbb{N}$ s.t. $\frac{1}{3^n} < \varepsilon$ and let $A = \{$endpoints of $F_n\}$. Then $\forall x \in C$, $x \in F_n$ since $C = \bigcap_{k\in\mathbb{N}} F_k \Longrightarrow \exists a \in A$ s.t. $d(a, x) \leq \frac{1}{3^n} < \varepsilon$. Therefore, A is a finite ε-net.*

Sequential compactness.

Definition 36. *$S \subseteq M$ is* **sequentially compact** *if every sequence contains a convergent subsequence.*

Theorem 37. *(M, d) is compact $\iff$ it is sequentially compact*

Proof. $\Longrightarrow$: Let M be compact, and $\{x_n\} \subseteq M$ any sequence. Define $S = \{x_n : n \in \mathbb{N}\}$, i.e. S is the set of all the points in the sequence. If S is finite then $\exists$ a constant subsequence of $\{x_n\}$ which clearly converges. Thus assume S is infinite. We have $\overline{S}$ is a closed subset of a compact space M, which implies $\overline{S}$ is compact (by Theorem 23). **Claim:** $\exists x \in \overline{S}$ s.t. $\forall \varepsilon > 0$ $B(x, \varepsilon)$ contains infinitely many points of S (i.e. x is a limit point of S). Proof of claim: assume the contrary: $\forall x \in \overline{S}$ $\exists \varepsilon_x$ s.t. $B(x, \varepsilon_x) \cap \overline{S} = \{x\}$. Then $U := \bigcup_{x\in\overline{S}} B(x, \varepsilon_x)$ is an open cover of $\overline{S}$ with no finite subcover $\Longrightarrow$ $\overline{S}$ is not compact, which is a contradiction.
Thus, x is a limit point of $S \Longrightarrow \exists\{x_{n_k}\}$ s.t. $x_{n_k} \to x$.
$\Longleftarrow$: Let M be sequentially compact. We will prove compactness in five steps.
Step 1: M is **totally bounded**.
Fix $\varepsilon > 0$, $x_1 \in M$. If $B(x_1, \varepsilon) = M$ then done. If not, let $x_2 \in M \backslash B(x_1, \varepsilon)$. If $B(x_1, \varepsilon) \cup B(x_2, \varepsilon) = M$ then done, otherwise continue. If for some $n \in \mathbb{N}$ we

get $\cup_{k=1}^{n} B(x_k, \varepsilon) = M$ then we constructed a finite ε-net. Otherwise we obtain a sequence $\{x_n\}$ with $d(x_n, x_m) \geq \varepsilon \; \forall m \neq n$. Such a sequence cannot have a convergent subsequence (**exercise**) which is a contradiction to sequential compactness. Thus we must have a finite ε-net, so M is totally bounded.

Step 2: M is **separable**, i.e. has a countable dense subset.
For every $n \in \mathbb{N}$ let S_n be a finite $1/n$-net, and let $S := \bigcup_{n\in\mathbb{N}} S_n$. Then S is countable, and $\forall x \in M \; \forall \varepsilon > 0 \; \exists n \in \mathbb{N}, s \in S$ s.t. $1/n < \varepsilon$ and $d(x, s) < 1/n < \varepsilon \Longrightarrow s \in B(x, \varepsilon)$. This shows that x is a contact point of S, and since this is true $\forall x \in M$ we obtain $\overline{S} = M$

Step 3: M is **second countable**, i.e. has a countable basis for its topology. Let $S = \{x_1, x_2, \ldots\}$ be a countable dense subset.
Let $U \subseteq M$ be any open set, and $x \in U$. Since U is open and S is dense, $\exists \varepsilon > 0, x_m \in S$ s.t. $x_m \in B(x, \varepsilon/3) \subseteq B(x, \varepsilon) \subseteq U \Longrightarrow \exists n \in \mathbb{N}$ s.t. $x \in B(x_m, 1/n) \subseteq U$. Thus the open balls $B(x_m, 1/n)$ form a countable base for M

Step 4: Every open cover has a countable subcover.
Let $\mathscr{C} = \{U_\alpha\}$ be an open cover of M, and $\mathscr{S}$ a countable base constructed in the previous step. $\forall x \in M \; \exists \alpha$ s.t. $x \in U_\alpha$ and $\exists B_x \in \mathscr{S}$ s.t. $x \in B_x \subseteq U_\alpha$. The collection of all such B_x is countable or finite (since $\mathscr{S}$ is countable) and covers M. Choosing one $U_\alpha \in \mathscr{C}$ s.t. $B_x \subseteq U_\alpha$ for each of B_x gives a countable subcover.

Step 5: Every countable open cover has a finite subcover.
Let $\mathscr{C} = \{U_n\}_{n\in\mathbb{N}}$ be a countable open cover. Let $F_n := (\cup_{k=1}^{n} U_k)^c = \cap_{k=1}^{n} U_k^c$, then $F_1 \supseteq F_2 \supseteq \ldots$. Suppose that $\forall n \in \mathbb{N} \; F_n \neq \emptyset$ and choose $x_n \in F_n$. By sequential compactness $\{x_n\}$ has a subsequence converging to some $x \in M$. For each n one can choose a subsequence of that subsequence $\subseteq F_n$ converging to x (**exercise**). Since F_n is closed, $x \in F_n \; \forall n \Longrightarrow x \in \cap_n F_n \Longrightarrow x \notin \cup_n F_n^c \Longrightarrow x \notin \cup_n U_n$ which contradicts to $\cup_n U_n$ being an open cover. Thus we must have $F_n = \emptyset$ for some n which implies $\{U_1, \ldots, U_n\}$ is a finite subcovering. $\square$

We can now prove Theorem 33.

Proof of Theorem 33. $\Longrightarrow$: Let M be compact. Then $\forall \varepsilon > 0$ the open cover $\bigcup_{x\in M} B(x, \varepsilon)$ contains a finite subcover $\bigcup_{i=1}^{n} B(x_i, \varepsilon) = M$. It follows that $\forall x \in M \; \exists i \in \{1, \ldots, n\}$ s.t. $x \in B(x_i, \varepsilon) \Longrightarrow \{x_1, \ldots, x_n\}$ is a finite ε-net. Thus, M is totally bounded. Next, since M is compact it is also sequentially compact ($\Longrightarrow$ part of Theorem 37). Thus for every Cauchy sequence there exists a convergent subsequence $\Longrightarrow$ the whole sequence converges (proved in HW). This shows M is complete.
$\Longleftarrow$: Suppose M is totally bounded and complete. We will prove sequential compactness. Let $\{x_n\}$ be any sequence in M. First cover M by finitely many balls of radius 1 (by total boundedness). At least one of these balls contains an infinite subsequence $\{x_n^{(1)}\}$ of $\{x_n\}$, call it B_1. Now cover M by finitely many balls of radius

$1/2$, choose B_2 and $\{x_n^{(2)}\} \subseteq \{x_n^{(1)}\}$ as above. Thus for each $k \in \mathbb{N}$ we obtain a subsequence of $\{x_n\}$ contained in B_k, a ball of radius $1/2^k$. For each k inductively choose $x_{n_k} \in \{x_n^{(k)}\} \subseteq B_k$ s.t. $n_k > n_{k-1}$. Then $\{x_{n_k}\}$ is a Cauchy sequence (**exercise**) and thus it converges by completeness of M.

□

Compactness in $\mathbb{R}^n$.

Theorem 38. *A subset $S \subseteq \mathbb{R}^n$ is compact $\iff$ it is closed and bounded.*

Proof. $\Longrightarrow$: Theorem 27
$\Longleftarrow$: S is bounded $\Longrightarrow \exists$ a cube $[-R, R]^n \supseteq S$. We claim that $[-R, R]^n$ is totally bounded. Let $\varepsilon > 0$, consider the set

$$A := \left\{ \frac{\varepsilon}{2\sqrt{n}} (k_1, k_2, \ldots, k_n) : k_i \in \mathbb{Z}, -\frac{2\varepsilon}{R} \leq k_i \leq \frac{2\varepsilon}{R} \quad \forall i = 1, \ldots, n \right\}$$

Then A is finite and it is an ε-net for $[-R, R]^n$ (**exercise**). Thus $[-R, R]^n$ is totally bounded and S is totally bounded as a subset. Since S is also closed, and $\mathbb{R}^n$ is complete, it is compact by Theorem 33. □

Example 39 (Bolzano-Weierstrass Theorem)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. Let $\{x_n\} \subseteq \mathbb{R}$ bounded, i.e. $\exists M > 0$ s.t. $|x_n| \leq M \; \forall n \in \mathbb{N} \Longrightarrow \{x_n\} \subseteq [-M, M]$. $[-M, M]$ is compact $\Longrightarrow$ sequentially compact $\Longrightarrow$ every sequence has a convergent subsequence. □

Continuity and compactness.

Theorem 40. *If $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact.*

Proof. HW □

Proposition 41. *A metric space (M, d) is compact $\iff$ there exists a continuous function f mapping the Cantor set onto M*

Example 42 (Cantor function)**.** *Let $x \in C$, then $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$, $a_i \in \{0, 2\}$. Define*

$$f(x) := \sum_{i=1}^{\infty} \frac{a_i/2}{2^i}, \quad a_i/2 \in \{0, 1\}$$

*Then $f : C \to [0, 1]$ is onto (***exercise***). To see that f is continuous, fix $y \in C$ and $\varepsilon > 0$. Let $N \in \mathbb{N}$ be s.t. $1/2^N < \varepsilon$ and choose $\delta = 1/3^{N+1}$. Then for any $x \in B(y, \delta)$*

we have $a_n = b_n\ \forall n \leq N$*, where* $a_n, b_n \in \{0,2\}$ *are the coefficients in the ternary expansions of* x *and* y *respectively. Therefore,*

$$|f(x) - f(y)| = \left|\sum_{n=1}^{\infty} \frac{a_n/2 - b_n/2}{2^n}\right| = \left|\sum_{n=N+1}^{\infty} \frac{a_n/2 - b_n/2}{2^n}\right| = \frac{1}{2^N}\left|\sum_{i=1}^{\infty} \frac{a_{i+N}/2 - b_{i+N}/2}{2^i}\right|$$

$$\leq \frac{1}{2^N}\sum_{i=1}^{\infty}\frac{1}{2^i} = \frac{1}{2^N} < \varepsilon.$$

The Cantor function f *can be extended to a continuous function* $g : [0,1] \to [0,1]$ *by making it constant on the middle-third intervals. This function is constant a.e., but grows from* 0 *to* 1 *on a finite interval. It's called* Devil's staircase*.*

MEASURE THEORY

LECTURE 1

LEBESGUE OUTER MEASURE

We will first construct a measure on $\mathbb{R}^n$.

Definition 1. *A* **rectangle** *in* $\mathbb{R}$ *is a set of the form* $(a,b),[a,b),(a,b],$ *or* $[a,b]$ *with* $a,b\in\mathbb{R}$, $a\le b$. *An* **interval** *is either a rectangle or a set of the same form with either* $a=\infty$ *or* $b=\infty$ *(or both). The* **length** *of a rectangle* I *with endpoints* $a\le b$ *is* $l(I)=b-a$. *The length of an infinite interval is set to be equal* ∞.

Definition 2. *A* **rectangle (interval)** *in* $\mathbb{R}^n$ *is a Cartesian product* $I=I_1\times\ldots\times I_n$, *where each* I_i *is a rectangle (interval) in* $\mathbb{R}$. *The* $\boldsymbol{n}$**-dimensional volume** *of* I *is* $l(I)=\prod_{i=1}^n l(I_i)$.

Convention: if $l(I_i)=0$ for some i, then $l(I)=0$. If not, and $l(I_i)=\infty$ for some i, then $l(I)=\infty$.

Example 3. $l([1,1]\times(0,\infty))=0$

Definition 4. *If* $X\subseteq\mathbb{R}^n$, *the* **outer measure** *of* X *is*

$$m^*(X)=\inf\left\{\sum_k l(I_k)\colon\ \cup_k I_k\supseteq X\right\},$$

where the infimum is taken over all sequences of rectangles $I_1,I_2,\ldots$ *in* $\mathbb{R}^n$ *that cover* X.

Examples.

(1) $I\subseteq\mathbb{R}^n$ a rectangle in $\mathbb{R}^n$, then $m^*(I)=l(I)$

Proof. See Example 5. $\square$

(2) $m^*(\emptyset)=0$

(3) $m^*(\mathbb{Q})=0$. More generally, if X is any countable subset of $\mathbb{R}^n$, then $m^*(X)=0$.

Proof. Let $\{x_1,x_2,\ldots\}\subset\mathbb{R}^n$, $\varepsilon>0$. Define I_k^ε to be the cube centered at x_k with side length $=\left(\varepsilon/2^k\right)^{1/n}$. Then $l(I_k^\varepsilon)=\varepsilon/2^k$, and

$$X\subseteq\cup_{k=1}^\infty I_k^\varepsilon,\quad \sum_{k=1}^\infty l(I_k^\varepsilon)=\sum_{k=1}^\infty\frac{\varepsilon}{2^k}=\varepsilon.$$

Therefore, $m^*(X)\le\varepsilon\ \forall\varepsilon>0\Longrightarrow m^*(X)=0.$ $\square$

(4) $C \subseteq \mathbb{R}$ the Cantor set, then $m^*(C) = 0$.

Proof. Recall $C = \cap_{n=0}^{\infty} F_n$, so $C \subseteq F_n$ $\forall n \geq 0$. Moreover, F_n is a union of 2^n intervals of length $1/3^n$ each, thus $m^*(F_n) = (2/3)^n \implies m^*(C) \leq (2/3)^n$ $\forall n \implies m^*(C) = 0$. □

Properties of the outer measure.

(1) $0 \leq m^*(X) \leq \infty$ $\forall X \subseteq \mathbb{R}^n$
(2) $X \subseteq Y \implies m^*(X) \leq m^*(Y)$
(3) m^* is **translation invariant**: $\forall X \subseteq \mathbb{R}^n$, $\forall p \in \mathbb{R}^n$

$$m^*(X + p) = m^*(X)$$

Proof. **Exercise**. Show that the n-dimensional volume, l, is translation invariant first. □

(4) The definition of m^* does not change if we restrict I_k's to be open, or closed, or half-open, etc.

Proof. Let $\{I_k\}$ be any cover of X, and let $\varepsilon > 0$. Define $\{I'_k\}$ to be an open cover of X, where $I'_k \supseteq I_k$ is open and $l(I'_k) \leq l(I_k) + \varepsilon/2^k$. Then

$$m^*(X) \leq \sum_{k=1}^{\infty} l(I'_k) \leq \sum_{k=1}^{\infty} l(I_k) + \varepsilon$$

Taking infimum over all covers we obtain

$$m^*(X) \leq \inf \left\{ \sum_k l(I'_k) \colon \cup_k I'_k \supseteq X,\ I'_k \text{ open} \right\} \leq m^*(X) + \varepsilon \quad \forall \varepsilon > 0.$$

On the other hand, if $\{I_k\}$ any cover of X, then $\{\overline{I_k}\}$ is a closed cover, and $l(\overline{I_k}) = l(I_k)$. □

(5) Given any $\delta > 0$ we may in addition assume $diam(I_k) < \delta$ $\forall I_k$ in the cover.

Proof. **Exercise**: Split each I_k in an arbitrary cover into sufficiently small subintervals □

(6) **Countable subadditivity:** If $X_1, X_2, \ldots \subseteq \mathbb{R}^n$, then

$$m^*(\cup_i X_i) \leq \sum_i m^*(X_i)$$

Proof. If $m^*(X_i) = \infty$ for some i, then the result is trivial. Assume $m^*(X_i) < \infty$ $\forall i$. Let $\varepsilon > 0$, $\forall i$ $\exists$ cover $\{I^i_k\}_{k \geq 1}$ of X_i s.t.

$$m^*(X_i) \leq \sum_{k=1}^{\infty} l(I^i_k) \leq m^*(X_i) + \frac{\varepsilon}{2^i}$$

(**exercise**: why?). Since $\cup_i \cup_k I_k^i$ is a cover of $\cup_i X_i$, we have

$$m^*\left(\cup_i X_i\right) \leq \sum_i \sum_k l(I_k^i) \leq \sum_i \left(m^*(X_i) + \frac{\varepsilon}{2^i}\right) = \sum_i m^*(X_i) + \varepsilon.$$

Since this is true $\forall \varepsilon > 0$ we obtain the result. $\square$

Lecture 2

Question 1. *Let $A, B \in \mathbb{R}^n$ s.t. $A \cap B = \emptyset$. Do we have $m^*(A \cup B) = m^*(A) + m^*(B)$?*

(7) Let $A, B \in \mathbb{R}^n$ and suppose $d(A, B) = \inf_{a \in A, b \in B} d(a, b) > 0$. Then

$$m^*(A \cup B) = m^*(A) + m^*(B).$$

Proof. First, by countable subadditivity we have $m^*(A \cup B) \leq m^*(A) + m^*(B)$, so we only need to show the reverse inequality. Fix $\varepsilon > 0$, let $\{I_k\}$ be a cover of $A \cup B$ s.t. $\sum_k l(I_k) \leq m^*(A \cup B) + \varepsilon$. By property (5) we may assume $diam(I_k) < d(A, B)$ $\forall k$, which implies no I_k contains points from both A and B. Write $\{I_k\} = \{I_k^A\} \cup \{I_k^B\} \cup \{I_k^\emptyset\}$, where $I_k^A \cap A \neq \emptyset$, $I_k^B \cap B \neq \emptyset$, and $I_k^\emptyset$ are all other intervals. Then $\{I_k^A\}$ covers A and $\{I_k^B\}$ covers B, and therefore

$$m^*(A \cup B) + \varepsilon \geq \sum_k l(I_k) = \sum_k l(I_k^A) + \sum_k l(I_k^B) + \sum_k l(I_k^\emptyset) \geq m^*(A) + m^*(B).$$

Since $\varepsilon > 0$ was arbitrary, the result follows. $\square$

Example 5. *$I \subseteq \mathbb{R}^n$ a rectangle in $\mathbb{R}^n$, then $m^*(I) = l(I)$*

Proof. First, since I is a cover of I we have $m^*(I) \leq l(I)$. Next, $\forall \varepsilon > 0$ we can choose an open cover $\{I_k\}$ of I s.t.

$$m^*(I) \leq \sum_k l(I_k) \leq m^*(I) + \frac{\varepsilon}{2}.$$

If I is closed, and thus compact, we can choose a finite subcover. If not, let $J \subseteq I$ be a closed interval s.t. $l(J) \geq l(I) + \varepsilon/2$. Since $\{I_k\}$ is a cover of J and J is compact, we can choose a finite subcover $I_1, \ldots, I_m$ of J. It is an **exercise** to show that $l(J) \leq \sum_{k=1}^m l(I_k)$. We therefore obtain

$$l(I) \leq l(J) + \frac{\varepsilon}{2} \leq \sum_{k=1}^m l(I_k) + \frac{\varepsilon}{2} \leq \sum_{k=1}^\infty l(I_k) + \frac{\varepsilon}{2} \leq m^*(I) + \varepsilon.$$

$\square$

General measure theory

σ-algebra.

Definition 6. *Let X be any set. A* σ-**algebra** *on X is a collection of subsets of X, Σ, s.t.*

(1) $\emptyset \in \Sigma$
(2) if $A \in \Sigma$, *then* $A^c \in \Sigma$
(3) Σ *is* σ-**finite**, *i.e. if* $\{A_i\}_{i=1}^{\infty} \subseteq \Sigma$ *then* $\bigcup_{i=1}^{\infty} A_i \in \Sigma$

Examples.

(1) $\Sigma = \{\emptyset, X\}$
(2) $\Sigma = 2^X$
(3) Let $A \subseteq X$, $\Sigma = \{\emptyset, X, A, A^c\}$

Theorem 7. *Let $\mathcal{A}$ be a collection of subsets of X. Then there exists a unique smallest σ-algebra, $\Sigma_{\mathcal{A}}$, on X that contains $\mathcal{A}$, i.e. for any σ-algebra Σ containing $\mathcal{A}$, $\Sigma_{\mathcal{A}} \subseteq \Sigma$.*

Proof. 1. 2^X is a σ-algebra and $\mathcal{A} \subseteq 2^X$, so there exists at least one.
2. Let $\Sigma_{\mathcal{A}} = \bigcap_{\Sigma \supseteq \mathcal{A}} \Sigma$, where the intersection is taken over all σ-algebras containing $\mathcal{A}$. Then $\Sigma_{\mathcal{A}}$ is a σ-algebra (**exercise**), $\Sigma_{\mathcal{A}} \supseteq \mathcal{A}$, and $\Sigma_{\mathcal{A}} \subseteq \Sigma$ for any σ-algebra Σ. $\square$

Definition 8. *The smallest σ-algebra in $\mathbb{R}^n$ containing all open sets is called the* **Borel σ-algebra**, *$\mathcal{B}$. Elements =* **Borel sets**.

Theorem 9.

$$\mathcal{B} = \Sigma_{\textit{open sets}} = \Sigma_{\textit{closed sets}} = \Sigma_{\textit{open rectangles}} = \Sigma_{\textit{closed rectangles}} = \Sigma_{\textit{rectangles}} = \Sigma_{\textit{intervals}}$$

Measure.

Definition 10. *Let X be a set, and Σ — a σ-algebra on X. A function $\mu \colon \Sigma \to \mathbb{R} \cup \{\infty\}$ is a* **measure** *on (X, Σ) if*

(1) $\mu(\emptyset) = 0$
(2) $0 \leq \mu(A) \leq \infty \ \forall A \in \Sigma$
(3) **Countable additivity (σ-additivity)***:* $\forall \{A_i\}_{i \in \mathbb{N}} \subseteq \Sigma$ *pairwise disjoint*

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

Proposition 11. *Any measure is countably subadditive*

Proof. Let $B_1, B_2, \ldots \in \Sigma$, not necessarily disjoint. We want to show that $\mu\left(\cup_{i=1}^{\infty} B_i\right) \leq \sum_{i=1}^{\infty} \mu(B_i)$. Let

$$A_1 := B_1, \quad A_2 := B_2 \backslash B_1, \quad A_3 := B_3 \backslash (B_1 \cup B_2), \ \ldots, \ A_n := B_n \backslash \cup_{i=1}^{n-1} B_i$$

Then A_i's are pairwise disjoint, $A_i \subseteq B_i$, $\cup_{i=1}^{\infty} B_i = \cup_{i=1}^{\infty} A_i$, and $A_i \in \Sigma$, $\forall i$ (**exercise**), so by σ-additivity we have

$$\mu\left(\cup_{i=1}^{\infty} B_i\right) = \mu\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) \leq \sum_{i=1}^{\infty} \mu(B_i).$$

$\square$

Exercise: If $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

Examples.

(1) X — any set, $\Sigma = 2^X$, $\mu \colon \Sigma \to \mathbb{R} \cup \{\infty\}$ given by

$$\mu(A) = \begin{cases} |A|, & \text{if } A \text{ is finite} \\ \infty, & \text{if } A \text{ is infinite} \end{cases}, \text{where } |A| \text{ is the cardinality of } A.$$

This is called a **counting measure**.

(2) X — any set, $\Sigma = \{\emptyset, X\}$, $\mu(X) = 1$, $\mu(\emptyset) = 0$.

(3) $X = [0,1]$, $\Sigma = \{\emptyset, X, [1/4, 1/2], [0, 1/4) \cup (1/2, 1]\}$ with $\mu = l$

(4) $X = \mathbb{R}^n$, $\Sigma \supseteq \mathcal{B}$, $\mu = m^*$.

To do: construct Σ, prove that $m^*\big|_{\Sigma}$ is a measure

Definition 12. *Let m^* be the outer measure on $\mathbb{R}^n$. A subset $X \subseteq \mathbb{R}^n$ satisfies the* **Carathéodory condition** *if $\forall A \in \mathbb{R}^n$*

$$m^*(A) = m^*(A \cap X) + m^*(A \cap X^c). \tag{1}$$

(X splits every set additively).

Lecture 3

Theorem 13. *Let $\mathcal{L} := \{X \in \mathbb{R}^n : X$ satisfies the Carathéodory condition $\}$. Then $\mathcal{L}$ is a σ-algebra on $\mathbb{R}^n$, and $m := m^*\big|_{\mathcal{L}}$ is a measure on $\mathcal{L}$.*

Definition 14. *$\mathcal{L}$ is called the* **Lebesgue σ-algebra** *on $\mathbb{R}^n$, and $m = m^*\big|_{\mathcal{L}}$ the* **Lebesgue measure** *on $\mathbb{R}^n$.*

Proof of Theorem 13. 1. It follows immediately from (1) that $\emptyset \in \mathcal{L}$ and if $E \in \mathcal{L}$ then $E^c \in \mathcal{L}$.
2. Claim: $E, F \in \mathcal{L} \Longrightarrow E \cap F \in \mathcal{L}$.
Proof of claim: Let $A \subseteq \mathbb{R}^n$ any subset. $E \in \mathcal{L} \Longrightarrow m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$. Using (1) with $A \cap E$ in place of A and $F \in \mathcal{L} \Longrightarrow m^*(A \cap E) = m^*(A \cap E \cap F) + m^*(A \cap E \cap F^c)$. Combining,

$$m^*(A) = m^*(A \cap E \cap F) + m^*(A \cap E \cap F^c) + m^*(A \cap E^c).$$

We now only need to show that the sum of the last two terms is equal to $m^*(A \cap (E \cap F)^c)$. Using (1) with $A \cap (E \cap F)^c$ in place of A and $E \in \mathcal{L}$ we obtain

$$m^*(A \cap (E \cap F)^c) = m^*(A \cap (E \cap F)^c \cap E) + m^*(A \cap (E \cap F)^c \cap E^c) = m^*(A \cap F^c \cap E) + m^*(A \cap E^c),$$

where the last equality is left as an **exercise**.
3. $E, F \in \mathcal{L} \Longrightarrow E \cup F \in \mathcal{L}$. Indeed, $E, F \in \mathcal{L} \Longrightarrow E^c, F^c \in \mathcal{L} \Longrightarrow E^c \cap F^c \in \mathcal{L} \Longrightarrow E \cup F = (E^c \cap F^c)^c \in \mathcal{L}$
4. $E_1, E_2, \ldots, E_n \in \mathcal{L} \Longrightarrow \cup_{k=1}^n E_k \in \mathcal{L}$. If in addition E_k's are pairwise disjoint, then

$$m^*(A \cap (\cup_{k=1}^n E_k)) = \sum_{k=1}^n m^*(A \cap E_k)$$

This is an **exercise**. Use part 3. and induction.
5. Let $E_1, E_2, \ldots \in \mathcal{L}$ and $E := \cup_{k=1}^\infty E_k$. Claim: $E \in \mathcal{L}$.
Proof of claim: First write $E = \cup_{k=1}^\infty E'_k$, where E'_k are pairwise disjoint, $E'_k \subseteq E_k$ and $E'_k \in \mathcal{L}$ $\forall k$ (see proof of Proposition 11). By 4. we have $F_n := \cup_{k=1}^n E'_k \in \mathcal{L}$ $\forall n \in \mathbb{N}$, and by (1) $\forall A \subseteq \mathbb{R}^n$

$$\begin{aligned} m^*(A) = m^*(A \cap F_n) + m^*(A \cap F_n^c) &= m^*(\cup_{k=1}^n (A \cap E'_k)) + m^*(A \cap F_n^c) \\ &= \sum_{k=1}^n m^*(A \cap E'_k) + m^*(A \cap F_n^c), \end{aligned}$$

where in the last equality we used the second part of 4. Since $F_n \subseteq E$, $A \cap F_n^c \supseteq A \cap E^c$ $\forall n$, and therefore

$$m^*(A) \geq \sum_{k=1}^n m^*(A \cap E'_k) + m^*(A \cap E^c) \tag{2}$$

OTOH by countable subadditivity of m^*

$$m^*(A \cap E) = m^*(\cup_{k=1}^\infty (A \cap E'_k)) \leq \sum_{k=1}^\infty m^*(A \cap E'_k). \tag{3}$$

Thus, $m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$. The reverse inequality follows immediately from the subadditivity of m^*, which concludes the claim.
6. From 1.-5. we obtain that $\mathcal{L}$ is a σ-algebra. We are only left to show that m is σ-additive. Let $A = E$ in (2), (3) to obtain $m^*(E) = \sum_{k=1}^\infty E'_k$. $\square$

Properties of Lebesgue measure.

(1) m is translation invariant
(2) $I \subseteq \mathbb{R}^n$ a rectangle, then $I \in \mathcal{L}$ and $m(I) = l(I)$. **Exercise**. (Hint: I splits any cover of A by rectangles into a cover of $A \cap I$ and a cover of $A \cap I^c$).
(3) Let μ be a measure on $\mathcal{L}$ s.t. $\mu(I) = m(I)$ for all rectangles I. Then $\mu = m$.
(4) Let $X \subseteq \mathbb{R}^n$ and $m^*(X) = 0$. Then $X \in \mathcal{L}$ and if $Y \subseteq X$, then $m^*(Y) = 0$, and thus $Y \in \mathcal{L}$. (HW)
(5) Every open set and every closed set in $\mathbb{R}^n$ is Lebesgue measurable, i.e. $\mathcal{B} \subseteq \mathcal{L}$.

Proof. $\mathcal{L}$ is closed under complements, so it is sufficient to consider open sets. Every open set in $\mathbb{R}^n$ is a countable union of open rectangles (HW), and rectangles are measurable by (2). Since $\mathcal{L}$ is closed under countable unions the result follows. □

Lecture 4

Approximation theorem.

Theorem 15 (Approximation theorem)**.** *The following are equivalent for $X \subseteq \mathbb{R}^n$*

(1) $X \in \mathcal{L}$
(2) $\forall \varepsilon > 0 \; \exists$ *open set* $U \supseteq X$ *s.t.* $m^*(U \backslash X) < \varepsilon$
(3) $\forall \varepsilon > 0 \; \exists$ *closed set* $F \subseteq X$ *s.t.* $m^*(X \backslash F) < \varepsilon$

If, in addition, $m^*(X) < \infty$, then (1)-(3) are also equivalent to

(4) $\forall \varepsilon > 0 \; \exists$ *compact set* $K \subseteq X$ *s.t.* $m^*(X \backslash K) < \varepsilon$

Lemma 16. *Let $A \subseteq \mathbb{R}^n$, $\varepsilon > 0$. There exists an open set $G \subseteq \mathbb{R}^n$ s.t. $A \subseteq G$ and $m^*(G) \leq m^*(A) + \varepsilon$. This implies*

$$m^*(A) = \inf\{m(G) : A \subseteq G, \; G \text{ open}\}.$$

Proof. If $m^*(A) = \infty$, then let $G = \mathbb{R}^n$. Assume $m^*(A) < \infty$, and let $\{I_k\}$ be an open cover of A s.t. $\sum_{k=1}^{\infty} l(I_k) \leq m^*(A) + \varepsilon$. Define $G := \cup_k I_k$, then G is open, $G \supseteq A$, and by σ-subadditivity of m^*

$$m(G) \leq \sum_k m^*(I_k) = \sum_k l(I_k) \leq m^*(A) + \varepsilon.$$

□

Proof of Theorem 15. $\Longrightarrow$: Assume $X \in \mathcal{L}$ and $m(X) < \infty$. By Lemma 16 $\exists G$ open, s.t. $X \subseteq G$ and $m(G) < m(X) + \varepsilon$. Since $X \in \mathcal{L}$ we have using the Carathéodory condition and the fact $X \subseteq G$,

$$m(G) = m(G \cap X) + m(G \cap X^c) = m(X) + m(G \backslash X).$$

Thus, $m(G\backslash X) = m(G) - m(X) < \varepsilon$.
If $m(X) = \infty$, define $X_1 := X \cap \{x \in \mathbb{R}^n : |x| \le 1\}$, $X_k := X \cap \{x \in \mathbb{R}^n : k-1 < |x| \le k\}$ $\forall k \ge 2$. For each $k \in \mathbb{N}$ let G_k be open, $X_k \subseteq G_k$ and $m(G_k\backslash X_k) < \varepsilon/2^k$. Such G_k exists since $X_k \in \mathcal{L}$ and $m(X_k) < \infty$ (**exercise**). Define $G := \cup_{k=1}^\infty G_k$, then G is open, $X \subseteq G$ and $G\backslash X \subseteq \cup_{k=1}^\infty G_k\backslash X_k$, which implies

$$m(G\backslash X) \le \sum_{k=1}^\infty m(G_k\backslash X_k) < \sum_{k=1}^\infty \frac{\varepsilon}{2^k} = \varepsilon.$$

$\Longleftarrow$: $\forall k \in \mathbb{N}$ $\exists G_k$ open s.t. $m^*(G_k\backslash X) < 1/k$. Let $H := \cap_{k=1}^\infty G_k$, then $H \in \mathcal{L}$ (**exercise**), and $H \subseteq G_k$ $\forall k \implies H\backslash X \subseteq G_k\backslash X \implies 0 \le m^*(H\backslash X) \le m^*(G_k\backslash X) < 1/k$ $\forall k \in \mathbb{N} \implies m^*(H\backslash X) = 0 \implies H\backslash X \in \mathcal{L}$ (HW) $\implies X = H\backslash(H\backslash X) \in \mathcal{L}$. □

The approximation theorem allows us to establish the following two results which will help us construct a non-measurable set, and a measurable set that is not Borel.

Proposition 17. *Let $K \subseteq \mathbb{R}^n$ be compact, and $m(K) > 0$. Then the difference set $K \ominus K := \{x - y : x, y \in K\}$ contains an open ball around the origin.*

Proof. $K \subseteq \mathbb{R}^n$ is compact $\implies K$ is closed $\implies K \in \mathcal{L}$. By Theorem 15 $\exists$ open set $U \supseteq K$ s.t. $m(U\backslash K) < m(K)$ (taking $\varepsilon = m(K)$). Since K is compact, U^c closed, and $K \cap U^c = \emptyset$, we have $\delta := d(K, U^c) > 0$ (**exercise**). Let $x \in B(0, \delta)$, i.e. $|x| < \delta$. Then $x + K \subseteq U$ since otherwise $\exists p \in K$ s.t. $x + p \notin U \implies x + p \in U^c$ $\implies d(K, U^c) \le d(p, x + p) = |x| < \delta$ which is a contradiction. We now claim that $(x + K) \cap K \ne \emptyset$. Proof of claim: assume the opposite, $(x + K) \cap K = \emptyset$. Then by additivity and translation invariance of m we have

$$m(K \cup (x + K)) = m(K) + m(x + K) = 2m(K).$$

OTOH since $K \cup (x + K) \subseteq U$, we get

$$m(K \cup (x + K)) \le m(U) = m(U\backslash K) + m(K) < 2m(K),$$

which is a contradiction. This concludes the claim, which shows $\exists p \in K \cap (x + K)$ $\implies p = x + q$ for some $q \in K \implies x \in K \ominus K$. □

Proposition 18. *Let $X \subseteq \mathbb{R}^n$ be Lebesgue measurable, and $m(X) > 0$. Then the difference set $X \ominus X := \{x - y : x, y \in X\}$ contains an open ball around the origin.*

Proof. If $m(X) < \infty$, then by 15 $\exists$ compact $K \subseteq X$ s.t. $m(X\backslash K) < m(X) \implies m(K) = m(X) - m(X\backslash K) > 0$ and we can apply the previous proposition to $K \subseteq X$. If $m(X) = \infty$, let X_k be as in the proof of Theorem 15. Then $X = \cup_{k=1}^\infty X_k \implies 0 < m(X) = \sum_{k=1}^\infty m(X_k) \implies m(X_k) > 0$ for some k. Since $m(X_k) < \infty$ we can find $K \subseteq X_k$ as before. □

Lecture 5

Vitali set. Define $\sim$ on $\mathbb{R}^n$ by $x \sim y$ if $x-y \in \mathbb{Q}^n$. Then $\sim$ is an equivalence relation on $\mathbb{R}^n$ (exercise). Choose one element from each equivalence class to form the set $\mathcal{V}$ (using the Axiom of Choice). Let $q_1, q_2, \dots$ be a list of the elements of $\mathbb{Q}^n$ (since $\mathbb{Q}^n$ is countable), and define $V_{q_i} := q_i + \mathcal{V}$.

Lemma 19. *V_{q_i}'s are pairwise disjoint and $\mathbb{R}^n = \bigcup_{i=1}^{\infty} V_{q_i}$.*

Proof. First, we show that $\mathbb{R}^n \subseteq \bigcup_{i=1}^{\infty} V_{q_i}$ (the other inclusion is immediate). Let $x \in \mathbb{R}^n$, then $\exists y \in \mathcal{V}$ s.t. $x \sim y$. By definition, this implies $\exists q_j \in \mathbb{Q}^n$ s.t. $x - y = q_j$ $\implies x = q_j + y \implies x \in V_{q_j}$.
To show V_{q_i}'s are disjoint, assume the contrary: $\exists x \in V_{q_i} \cap V_{q_j}$ for some $i \neq j$. Then $x = q_i + v = q_j + v'$ where $v, v' \in \mathcal{V} \implies v - v' = q_j - q_i \in \mathbb{Q}^n \backslash \{0\} \implies v \sim v'$ which is a contradiction since $\mathcal{V}$ contains only one element from each equivalence class. $\square$

Theorem 20. *$\mathcal{V}$ is not Lebesgue measurable.*

Suppose $\mathcal{V} \in \mathcal{L}$. Since m is translation invariant, we have $m(\mathcal{V}) = m(V_{q_i})$ $\forall i \in \mathbb{N}$. By Lemma 19 and σ-additivity of m we obtain

$$m(\mathbb{R}^n) = m\left(\bigcup_{i=1}^{\infty} V_{q_i}\right) = \sum_{i=1}^{\infty} m(V_{q_i}) = \sum_{i=1}^{\infty} m(\mathcal{V}).$$

If $m(\mathcal{V}) = 0$ this gives $m(\mathbb{R}^n) = 0$ which is a contradiction. If $m(\mathcal{V}) > 0$, then by Proposition 18 the difference set $\mathcal{V} \ominus \mathcal{V}$ contains an open ball around the origin. This implies $\exists q \in Q^n \backslash \{0\}$ s.t. $q \in \mathcal{V} \ominus \mathcal{V}$, i.e. $\exists v, v' \in \mathcal{V}$ s.t. $q = v - v' \implies v \sim v'$ (and $v \neq v'$), which is a contradiction again.

Non-Borel measurable sets.

Theorem 21. *If $X \subseteq \mathbb{R}^n$ and $m^*(X) > 0$, then X contains a non-measurable set.*

Proof. Define $X_i := X \cap V_{q_i}$, and assume X_i is measurable $\forall i$ (otherwise $X_i \subseteq X$ is the required non-measurable subset). By Lemma 19 X_i's are disjoint and $X = \cup_i X_i$, which by σ-additivity of m gives

$$0 < m^*(X) = m(\cup_i X_i) = \sum_i m(X_i).$$

$\square$

This implies $m(X_i) > 0$ for some $i \in \mathbb{N} \implies X_i \ominus X_i$ contains a ball around the origin $\implies \mathcal{V} \ominus \mathcal{V} = V_{q_i} \ominus V_{q_i} \supseteq X_i \ominus X_i$ contains a ball around the origin, and a contradiction follows as in the proof of Theorem 20.

Theorem 22. *There exists a Lebesgue measurable set that is not Borel.*

Proof. Let $f : C \to [0,1]$ be the Cantor function:

$$x = (0.x_1x_2\dots)_3 \mapsto (x_1/2, x_2/2, \dots)_2$$

We showed before that it is continuous and onto, and it can be extended to a continuous onto function $g : [0,1] \to [0,1]$ called the Devil's staircase. This is done by assigning a constant value on all the intervals missing from C. Now define $h : [0,1] \to [0,2]$ by $h(x) = x + g(x)$. It is an **exercise** to check that $h(x)$ is continuous and bijective. Recall that if h is a continuous bijective map between compact Hausdorff spaces (which $[0,1]$ and $[0,2]$ are), the inverse function h^{-1} is also continuous (HW). This implies that for any open subset U of the corresponding interval, $h(U)$ and $h^{-1}(U)$ are open. By question 4 from HW 7 we get that $\forall B \in \mathcal{B}\ h(B), h^{-1}(B) \in \mathcal{B}$, since the Borel σ-algebra is generated by open sets. Thus, $C \in \mathcal{B} \Longrightarrow h(C) \in \mathcal{B} \Longrightarrow h(C) \in \mathcal{L}$.
Exercise: $m(h(C)) = 1$. Hint: $m(C) = 0 \Longrightarrow m([0,1]\backslash C) = 1 \Longrightarrow$ (since g is constant on all the "middle third" intervals) $m(h([0,1]\backslash C)) = 1$. OTOH $m(h([0,1])) = m([0,2]) = 2$.
It follows from Theorem 21 that $\exists W \subseteq h(C)$ which is not measurable, and therefore not Borel. Define $\tilde{W} := h^{-1}(W) \subseteq C$. $m(C) = 0 \Longrightarrow m^*(\tilde{W}) = 0 \Longrightarrow \tilde{W} \in \mathcal{L}$. However, $\tilde{W}$ is not Borel, since otherwise $W = h(\tilde{W})$ would be Borel. □

Hausdorff measure

Idea: Let $I \subseteq \mathbb{R}$ be any interval of finite length, $|I| < \infty$. If we scale I by m, we can write $mI = I_1 \cup I_2 \cdots \cup I_m$ where I_k is just a translated copy of I. Doing the same thing with a square, $I \subseteq \mathbb{R}^2$, we get $mI = \cup_{k=1}^{m^2} I_k$; doing this with a cube we get m^3 copies of the original cube, etc. This suggests an intuitive way of determining dimension: if the set mE contains m^α copies of E, then E has dimension α.
Let us look at the Cantor set $C \subseteq [0,1]$. Scaling by 3 gives $3C \subseteq [0,3]$ which has 2 copies of the Cantor set: one in $[0,1]$ and one in $[2,3]$. This suggests the dimension of the Cantor set should be $\alpha = \ln 2/\ln 3$ since $2 = 3^\alpha$. Hausdorff measure makes it precise.

Definition 23. *Let $E \subseteq \mathbb{R}^n$, for $\alpha > 0$ define*

$$h^*_{\alpha,\delta}(E) := \inf\left\{\sum_{j=1}^{\infty}(diam(E_j))^\alpha \colon E \subseteq \cup_j E_j, diam(E_j) \leq \delta\right\}$$

$$h^*_\alpha(E) := \lim_{\delta\to 0} h^*_{\alpha,\delta}(E) = \sup_{\delta>0} h^*_{\alpha,\delta}(E).$$

Finally, define

$$\mathcal{H}^\alpha(E) = \gamma_\alpha h^*_\alpha(E), \quad \gamma_\alpha = \frac{\pi^{\alpha/2}}{2^\alpha\Gamma(\alpha/2)+1}$$

the α-dimensional **Hausdorff measure** *of E.*

Proposition 24. *Hausdorff measure restricted to Borel sets satisfies the three properties of a measure (non-negative, countably additive, and $\mathcal{H}^\alpha(\emptyset) = 0$). Moreover, $\forall A \in \mathcal{B}$*

$$\mathcal{H}^n(A) = m(A),$$

where m is the Lebesgue measure on $\mathbb{R}^n$.

Proposition 25. $\forall E \subseteq \mathbb{R}^n$, $0 \leq s < t < \infty$

(1) $\mathcal{H}^s(E) < \infty \Longrightarrow \mathcal{H}^t(E) = 0$
(2) $\mathcal{H}^t(E) > 0 \Longrightarrow \mathcal{H}^s(E) = \infty$

Definition 26. Hausdorff dimension *of $E \subseteq \mathbb{R}^n$ is*

$$\alpha := \inf\{s : \mathcal{H}^s(E) = 0\}$$

Cantor set.

Theorem 27. *$\mathcal{H}^\alpha(C) = \gamma_\alpha$, where $\alpha = \ln 2/\ln 3$, and γ_α as in Definition 23. In particular, the Hausdorff dimension of the Cantor set C is $\alpha = \ln 2/\ln 3$.*

Proof. Recall $C = \cap_{k\geq 0} F_k$. Each F_k is a cover of C by 2^k intervals of length $1/3^k \Longrightarrow h^*_{\alpha,\delta}(C) \leq 2^k(1/3^k)^\alpha$ for $\delta = 1/3^k$. If $3^\alpha = 2$, i.e. $\alpha = \ln 2/\ln 3$, get $h^*_{\alpha,\delta}(C) \leq 1 \Longrightarrow \mathcal{H}^\alpha(C) \leq \gamma_\alpha$.
To show the inverse, let $\{I_j\}$ be an open cover of C by rectangles. Since C is compact, there exists a finite subcover, $\{I_j\}_{j=1}^m$. $\{I_j\}$ finite $\Longrightarrow$ (WLOG) $\inf_j l(I_j) > 0 \Longrightarrow \exists k \in \mathbb{N}$ s.t. $\{I_j\}$ covers F_k, and each interval in F_k is contained in some I_j, $j = 1, \ldots, m$. By shrinking I_j's and taking their closure we can arrange: (1) each interval from F_k only intersects one of I_j's; and (2) all the endpoints of I_j's are also endpoints of the intervals from F_k. Thus, $\forall j = 1, \ldots,$ either (1) I_j coincides with an interval from F_k; or (2) $I_j \supseteq L_j$, $L_j \notin F_k$, $l(L_j) \geq l(I_j)/3$. In case (2) partition I_j into $J_j \cup L_j \cup J'_j$ s.t. $J_j \cup J'_j$ covers $I_j \cap F_k$. Therefore,

$$\begin{aligned}
l(I_j)^\alpha &= (l(J_j) + l(L_j) + l(I'_j))^\alpha \\
&\geq \left(\frac{3}{2}(l(J_j) + l(I'_j))\right)^\alpha \quad \text{since } l(L_k) \geq \frac{1}{3}(l(J_j) + l(L_j) + l(I'_j)) \\
&= 2\left(\frac{1}{2}(l(J_j) + l(I'_j))\right)^\alpha \quad \text{since } 3^\alpha = 2 \\
&\geq l(J_j)^\alpha + l(J'_j)^\alpha \quad \text{by concavity of } t^\alpha,\ 0 < \alpha < 1.
\end{aligned}$$

Iterating, obtain a cover by the intervals from F_k, so

$$\sum_{j=1}^m l(I_j)^\alpha \geq \frac{2^k}{(3^k)^\alpha} = 1.$$

□

INTEGRATION

Lecture 1

Recall Riemann integral: upper and lower Darboux sums, U and L. Consider $f := \chi_{\mathbb{Q}\cap[0,1]}$, and note that $m(\mathbb{Q}\cap[0,1]) = 0$. However, $U(f) = 1$ and $L(f) = 0$, so f is not Riemann integrable.

Measurable functions

Let (X, Σ, μ) be a measure space. It will be convenient to allow real-valued functions on X to take infinite values, that is $f(x)$ belongs to the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. We say that f is **finite valued** if $-\infty < f(x) < \infty$ $\forall x \in X$.

Definition 1. *A function $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is* **measurable** *if $f^{-1}(B) \in \Sigma$ for all Borel sets $B \subseteq \mathbb{R}$ and $f^{-1}(\infty), f^{-1}(-\infty) \in \Sigma$. The set of all measurable functions on X is denoted $\mathcal{M}(X)$, and $\mathcal{M}^+(X)$ is the set of all measurable non-negative functions.*

Proposition 2. *Let $f : X \to \overline{\mathbb{R}}$, TFAE*

(1) $f \in \mathcal{M}(X)$
(2) $A_\alpha := \{x \in X : f(x) > \alpha\} = f^{-1}((\alpha, \infty]) \in \Sigma \quad \forall \alpha \in \mathbb{R}$
(3) $B_\alpha := \{x \in X : f(x) \leq \alpha\} = f^{-1}([-\infty, \alpha]) \in \Sigma \quad \forall \alpha \in \mathbb{R}$
(4) $C_\alpha := \{x \in X : f(x) \geq \alpha\} = f^{-1}([\alpha, \infty]) \in \Sigma \quad \forall \alpha \in \mathbb{R}$
(5) $D_\alpha := \{x \in X : f(x) < \alpha\} = f^{-1}([-\infty, \alpha)) \in \Sigma \quad \forall \alpha \in \mathbb{R}$

Proof. First note that we can write $A_\alpha = f^{-1}((\alpha, \infty)) \cup f^{-1}(\infty)$ and similarly for all other sets. Then (1)$\Longrightarrow$ (2)-(5) since each interval of the real line is Borel, and $f^{-1}(\pm\infty) \in \Sigma$ by assumption. It is also easy to see that (2) $\Longleftrightarrow$ (3) and (4) $\Longleftrightarrow$ (5) since Σ is closed under complements. Moreover, (2)$\Longrightarrow$(4) since $C_\alpha = \cap_{n=1}^\infty A_{\alpha-1/n}$; and (4)$\Longrightarrow$(2) since $A_\alpha = \cup_{n=1}^\infty C_{\alpha+1/n}$. It remains to show (2)-(5)$\Longrightarrow$ (1). First, $f^{-1}(\infty) = \cap_{n\in\mathbb{N}} A_n \in \Sigma$, and $f^{-1}(-\infty) = \cap_{n\in\mathbb{N}} D_n \in \Sigma$. Now, to show $f^{-1}(B) \in \Sigma$ for every Borel set $B \subseteq \mathbb{R}$ it is sufficient to show $f^{-1}(U) \in \Sigma$ for every open $U \subseteq \mathbb{R}$ since the Borel σ-algebra is generated by open sets. Every open set is a countable union of open rectangles (HW), and every open rectangle $(a, b) = (a, \infty] \cap [-\infty, b)$. $\square$

Examples.

(1) $f = \mathit{const}$ is measurable

(2) Let $E \in \Sigma$, $\chi_E : X \to \mathbb{R}$ the characteristic function of E, i.e. $\chi_E(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E \end{cases}$.

Then $\chi_E \in \mathcal{M}(X)$

Proof. Let $\alpha \in \mathbb{R}$, denote $I = (\alpha, \infty)$, it is sufficient to show $\chi_E^{-1}(I) \in \Sigma$. There are 4 cases: (1) $0 \notin I, 1 \in I \implies \chi_E^{-1}(I) = E \in \Sigma$; (2) $0 \in I, 1 \notin I \implies \chi_E^{-1}(I) = E^c \in \Sigma$; (3) $0, 1 \in I \implies \chi_E^{-1}(I) = X \in \Sigma$; (4) $0, 1 \notin I \implies \chi_E^{-1}(I) = \emptyset \in \Sigma$. □

(3) $X = \mathbb{R}^n$, $\Sigma = \mathcal{L}$, $f : \mathbb{R}^n \to \mathbb{R}$ continuous $\implies f$ measurable.

Proof. $U \subseteq \mathbb{R}$ open $\implies f^{-1}(U) \subseteq \mathbb{R}^n$ open $\implies f^{-1}(U) \in \mathcal{L}$. □

Proposition 3. *Let $f, g \in \mathcal{M}(X)$, $c \in \mathbb{R}$. Then $cf, f^2, f+g, fg, |f| \in \mathcal{M}(X)$.*

Proof. cf — **exercise**. For f^2 write

$$\{f^2 > \alpha\} = \begin{cases} X, & \text{if } \alpha < 0, \\ \{f > \sqrt{\alpha}\} \cup \{f < -\sqrt{\alpha}\} & \text{if } \alpha \geq 0. \end{cases}$$

Similarly,

$$\{|f| > \alpha\} = \begin{cases} X, & \text{if } \alpha < 0, \\ \{f > \alpha\} \cup \{f < -\alpha\} & \text{if } \alpha \geq 0. \end{cases}$$

For $f + g$ write (**exercise**):

$$\{f + g > \alpha\} = \bigcup_{q \in \mathbb{Q}} \{f > q\} \cap \{g > \alpha - q\}.$$

Finally, note that we can write $fg = \dfrac{[(f+g)^2 - (f-g)^2]}{4}$ and use the previous results for sums, squares and multiplications by a constant. □

Definition 4. *The* **positive part** *of f is the function f^+ defined by $f^+(x) = \max\{f(x), 0\}$. The* **negative part** *of f is the function f^- defined by $f^-(x) = \min\{f(x), 0\}$.*

Corollary 5. *$f \in \mathcal{M}(X) \implies f^+, f^- \in \mathcal{M}(X)$.*

Proof. Note that

$$f = f^+ - f^-, \quad |f| = f^+ + f^-$$
$$f^+ = \frac{|f| + f}{2}, \quad f^- = \frac{|f| - f}{2}$$

and use Proposition 3. □

Lecture 2

lim, inf, sup, liminf, limsup.

Definition 6. *Let $\{f_n\}$ be a sequence of functions $X \to \overline{\mathbb{R}}$. Define functions $\inf_n f_n$, $\sup_n f_n$, $\liminf_n f_n$, $\limsup_n f_n$ to be the functions $X \to \overline{\mathbb{R}}$ that assign to each element $x \in X$ the following elements of $\overline{\mathbb{R}}$*

$$\inf_{n\in\mathbb{N}} f_n(x),\ \sup_{n\in\mathbb{N}} f_n(x),$$

$$\liminf_{n\in\mathbb{N}} f_n(x) = \lim_{n\to\infty}\left[\inf_{m\geq n} f_m(x)\right] = \sup_{n\in\mathbb{N}}\left[\inf_{m\geq n} f_m(x)\right]$$

$$\limsup_{n\in\mathbb{N}} f_n(x) = \lim_{n\to\infty}\left[\sup_{m\geq n} f_m(x)\right] = \inf_{n\in\mathbb{N}}\left[\sup_{m\geq n} f_m(x)\right]$$

Remark 7. $\inf, \sup, \liminf, \limsup$ *always exist in $\overline{\mathbb{R}}$. If $\liminf_n f_n = \limsup_n f_n$, then there exists a limit (maybe infinite) of f_n and $\lim_{n\to\infty} f_n = \liminf_n f_n = \limsup_n f_n$.*

Examples.

(1) $a_n = (-1)^n - \dfrac{1}{n^2}$

$$\sup_{n\in\mathbb{N}} a_n = 1,\ \inf_{n\in\mathbb{N}} a_n = -2,\ \limsup_n a_n = 1,\ \liminf_n a_n = -1,\ \textit{no limit}$$

(2) $f_n(x) = \begin{cases} 1, & \text{if } x \leq 0 \\ 1 - nx, & \text{if } 0 < x \leq 1/n \\ 0, & \text{if } x > 1/n \end{cases}$

$$\sup_{n\in\mathbb{N}} f_n(x) = f_1(x),\ \inf_{n\in\mathbb{N}} f_n(x) = \chi_{(-\infty,0]}(x),$$
$$\limsup_n f_n(x) = \lim_{n\to\infty} f_n(x) = \chi_{(-\infty,0]}(x) = \liminf_n f_n(x)$$

(3) $f_n(x) = \begin{cases} n - n^2|x|, & \text{if } |x| \leq 1/n \\ 0, & \text{if } |x| > 1/n \end{cases}$ **challenge exercise**

Proposition 8. *Let $\{f_n\} \subseteq \mathcal{M}(X)$. Then $\inf_n f_n$, $\sup_n f_n$, $\liminf_n f_n$, $\limsup_n f_n$ $\in \mathcal{M}(X)$. If there exists a pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ $\forall x \in X$, then $f \in \mathcal{M}(X)$.*

Proof.

$$\{x \in X : \inf_n f_n(x) \geq \alpha\} = \bigcap_{n\in\mathbb{N}} \{x \in X : f_n(x) \geq \alpha\}$$

$$\{x \in X : \sup_n f_n(x) > \alpha\} = \bigcup_{n\in\mathbb{N}} \{x \in X : f_n(x) > \alpha\},$$

so C_α for $\inf f_n$ and A_α for $\sup f_n$ are measurable, and thus the functions are measurable. Moreover, $\liminf_n f_n = \sup_n(\inf_{m\geq n} f_m)$ and $\limsup_n f_n = \inf_n(\sup_{m\geq n} f_m)$, therefore

$$\{x \in X : \liminf_n f_n(x) \geq \alpha\} = \bigcup_{n\in\mathbb{N}} \{x \in X : \inf_{m\geq n} f_m(x) \geq \alpha\}$$
$$\{x \in X : \limsup_n f_n(x) > \alpha\} = \bigcap_{n\in\mathbb{N}} \{x \in X : \sup_{m\geq n} f_m(x) > \alpha\},$$

which similarly shows that lim inf and lim sup are measurable. Finally, if $\lim_{n\to\infty} f_n(x)$ exists, we have $\lim_{n\to\infty} f_n(x) = \liminf_n f_n(x) = \limsup_n f_n(x)$, and thus it is measurable. □

Simple functions

Definition 9. *A* **simple function** *is a function of the form*

$$f(x) = \sum_{k=1}^{N} a_k \chi_{E_k}(x),$$

where $E_k \in \Sigma$ and $\mu(E_k) < \infty$ $\forall k$. If E_k's are disjoint and a_k's are distinct, we say that the function is in a **standard form**.

Remark 10. *Every simple function can be put into standard form (see proof of Proposition 14). Every simple function is measurable.*

Remark 11. *To construct the Riemann integral one needs to consider functions*

$$f(x) = \sum_{k=1}^{N} a_k \chi_{I_k}(x).$$

While these functions are simpler, the class of functions they approximate is much smaller.

Theorem 12 (Approximation by simple functions)**.** *Let $f \in \mathcal{M}^+(X)$. There exists an increasing sequence of non-negative simple functions $\{\varphi_k\}_{k=1}^\infty$ converging to f pointwise, i.e. $\varphi_n(x) \to f(x)$ $\forall x \in X$.*

Proof. For every $n \in \mathbb{N}$ define

$$\varphi_n(x) = \begin{cases} \frac{k}{2^n}, & \text{if } \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n} \text{ for } k = 0, 1, \ldots, n2^n - 1 \\ n, & \text{if } n \leq f(x) \end{cases}$$

For example,

$$\varphi_1(x) = \begin{cases} 0, & \text{if } 0 \leq f(x) < \frac{1}{2}, \\ \frac{1}{2} & \text{if } \frac{1}{2} \leq f(x) < 1, \\ 1, & \text{if } 1 \leq f(x). \end{cases}$$

Then we have
1. φ_n is a simple function since it takes the same value on each of the sets $f^{-1}\left([k/2^n,(k+1)/2^n)\right)$, $k=0,\dots,n2^n-1$, and $f^{-1}\left([n,\infty]\right)$ which are all measurable since f is measurable
2. $0\le\varphi_n(x)\le\varphi_{n+1}(x)\le f(x)\ \forall x\in X$ (**exercise**)
3. Fix $x\in X$. If $f(x)=\infty$, then $\varphi_n(x)=n\to\infty$. If $f(x)<\infty$, then let $\varepsilon>0$ and choose $N\in\mathbb{N}$ s.t. $\frac{1}{2^N}<\varepsilon$ and $N>f(x)$. Then $\forall n\ge N$ $\varphi_n(x)=\frac{k}{2^n}$ for some $k\le n2^n-1$ and $\frac{k}{2^n}<f(x)\le\frac{k+1}{2^n}$. Thus,

$$|\varphi_n(x)-f(x)|\le\frac{1}{2^n}\le\frac{1}{2^N}<\varepsilon.$$

□

Lecture 3

Lebesgue integral

$\mathcal{M}(X)=\{f:X\to\overline{\mathbb{R}},\ \text{measurable}\}$, $\mathcal{M}^+(X)=\{f\in\mathcal{M}(X):f(x)\ge 0\ \forall x\in X\}$.

Definition 13. *Let $f=\sum_{i=1}^k a_i\chi_{E_i}\in\mathcal{M}^+(X)$ be a non-negative simple function. Then the* **Lebesgue integral** *of f is defined to be*

$$\int f\,d\mu=\sum_{i=1}^k a_i\mu(E_i)$$

Proposition 14. *The definition above does not depend on the representation of a simple function.*

Proof. Let $f=\sum_{i=1}^k a_i\chi_{E_i}\in\mathcal{M}^+(X)$, we will first put it into standard form. For any subset $K\subseteq\{1,\dots,k\}$ define $H_K:=(\cap_{i\in K}E_i)\cap(\cap_{i\notin K}E_i^c)$. We have $H_{K_1}\cap H_{K_2}=\emptyset$ if $K_1\ne K_2$ and $E_i=\cup_{K\in i}H_k$, which implies $\chi_{E_i}=\sum_{K\ni i}H_k$ and

$$f=\sum_{i=1}^k a_i\sum_{K\ni i}\chi_{H_K}=\sum_{i=1}^k\sum_{K\ni i}a_i\chi_{H_K}=\sum_{K\subseteq\{1,\dots,k\}}\sum_{i\in K}a_i\chi_{H_K}=\sum_{K\subseteq\{1,\dots,k\}}b_K\chi_{H_K}$$

where we have defined $b_K:=\sum_{i\in K}a_i$. Now let $\{c_j\}_{j=1}^m$ be the set of all distinct values from $\{b_K\}_{K\subseteq\{1,\dots,k\}}$, and $F_j:=\cup_{K:b_K=c_j}H_K$. Then $f=\sum_{j=1}^m c_j\chi_{F_j}$ is a standard form, moreover

$$\int f\,d\mu=\sum_{i=1}^k a_i\mu(E_i)=\sum_{i=1}^k a_i\sum_{K\ni i}\mu(H_K)=\sum_{K\subseteq\{1,\dots,k\}}b_K\mu(H_K)=\sum_{j=1}^m c_j\mu(F_j)$$

□

Examples.

(1) $X = \mathbb{R}$, $\Sigma = \mathcal{L}$, $\mu = m$, $f = \chi_{[0,2]} + 2\chi_{[1,4]}$

$$\int f dm = 2 + 2 \cdot 3 = 8$$

In standard form: $f = \chi_{[0,1)} + 3\chi_{[1,2]} + 2\chi_{(2,4]}$

(2) $X = \{x_1, x_2, \ldots, x_n\}$, $\Sigma = 2^X$, $\mu(E) = \frac{|E|}{n}$, $f(x_i) = i$.

$$\int f d\mu = \int \sum_{i=1}^{n} i\chi_{x_i} d\mu = \sum_{i=1}^{n} i\mu(\{x_i\}) = \sum_{i=1}^{n} \frac{i}{n} = \frac{n+1}{2}.$$

Proposition 15 (Linearity)**.** *Let ϕ, ψ be simple functions in $\mathcal{M}^+(X)$ and $c \geq 0$.*

(1) $\int c\phi d\mu = c \int \phi d\mu$
(2) $\int (\phi + \psi) d\mu = \int \phi d\mu + \int \psi d\mu$

Proof. (1) **Exercise**

(2) It has been shown in Proposition 14 that the definition of the Lebesgue integral for simple functions does not depend on the representation. Thus, let $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$, and $\phi = \sum_{j=1}^{k} b_j \chi_{F_j}$. We have

$$(\phi + \psi)(x) = \sum_{i=1}^{n} a_i \chi_{E_i} + \sum_{j=1}^{k} b_j \chi_{F_j}$$

$$\implies \int (\phi + \psi) d\mu = \sum_{i=1}^{n} a_i \mu(E_i) + \sum_{j=1}^{k} b_j \mu(F_j)$$

$$= \int \phi d\mu + \int \psi d\mu$$

□

Lecture 4

Definition 16. *Let $f \in \mathcal{M}^+(X)$, the* **Lebesgue integral** *of f is*

$$\int f d\mu = \sup \int \phi d\mu,$$

where the supremum is taken over all nonnegative simple functions satisfying $\phi(x) \leq f(x)$ $\forall x \in X$. For each $E \in \Sigma$ we can define $\int_E f d\mu := \int f \chi_E d\mu$.

Proposition 17.

(1) $f, g \in \mathcal{M}^+(X)$, $f(x) \leq g(x)$ $\forall x \in X \implies \int f d\mu \leq \int g d\mu$
(2) $E, F \in \Sigma$, $E \subseteq F$, and $f \in \mathcal{M}^+(X) \implies \int_E f d\mu \leq \int_F f d\mu$

Proof. (1) $\int f d\mu = \sup_{\phi \le f} \int \phi d\mu \le \sup_{\phi \le g} \int \phi d\mu = \int g d\mu$
(2) Apply (1) to $f\chi_E \le f\chi_F$

□

Monotone convergence theorem.

Theorem 18 (MCT)**.** *Let $\{f_n\} \subseteq \mathcal{M}^+(X)$ be an increasing sequence of functions s.t. $\lim_{n\to\infty} f_n(x) = f(x)$ $\forall x \in X$. Then*

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

Proof. 1. $f_n(x) \le f(x)$ $\forall n \in \mathbb{N}$, $\forall x \in X \Longrightarrow$ (by Proposition 17) $\int f_n d\mu \le \int f d\mu$ $\forall n \in \mathbb{N} \Longrightarrow \lim_{n\to\infty} \int f_n d\mu \le \int f d\mu$.
2. Let ϕ be a simple function s.t. $0 \le \phi(x) \le f(x)$ $\forall x \in X$. Let $\alpha \in (0,1)$, define $A_n := \{x \in X : f_n(x) \ge \alpha\phi(x)\}$. Since $\{f_n\}$ is increasing we have $A_n \subseteq A_{n+1}$ $\forall n$, and moreover $X = \bigcup_n A_n$ (**exercise**). $f_n - \alpha\phi \in \mathcal{M} \Longrightarrow A_n \in \Sigma$, and thus

$$\int_{A_n} \alpha\phi d\mu \le \int_{A_n} f_n d\mu \le \int f_n d\mu \Longrightarrow \lim_{n\to\infty} \int_{A_n} \alpha\phi d\mu \le \lim_{n\to\infty} \int f_n d\mu \tag{1}$$

3. Claim: $\lim_{n\to\infty} \int_{A_n} \phi d\mu = \int \phi d\mu$. Proof of claim: let $\phi = \sum_{i=1}^k a_i \chi_{E_i}$ in the standard form. Then

$$\int_{A_n} \phi = \int \phi \chi_{A_n} = \int \sum_{i=1}^k a_i \chi_{E_i \cap A_n} = \sum_{i=1}^k a_i \mu(E_i \cap A_n)$$

Exercise: $E_i \cap A_n \subseteq E_i \cap A_{n+1}$ $\forall n$ and $E_i = \bigcup_n (E_i \cap A_n) \Longrightarrow \lim_{n\to\infty} \mu(E_i \cap A_n) = \mu(E_i)$ (see q. 6 on HW7). This gives

$$\lim_{n\to\infty} \int_{A_n} \phi d\mu = \sum_{i=1}^k a_i \mu(E_i) = \int \phi d\mu,$$

which proves the claim.
4. Taking lim in (1) we get using part (1) of Proposition 15, $\alpha \int \phi d\mu \le \lim_{n\to\infty} \int f_n d\mu$. Since this is true for any $\alpha \in (0,1)$ we obtain $\int \phi d\mu \le \lim_{n\to\infty} \int f_n d\mu$, and taking the supremum over all ϕ as in 2. we conclude $\int f d\mu \le \lim_{n\to\infty} \int f_n d\mu$ □

Examples.

(1) Let $f : [0,1] \to [0,1]$ be defined by $f(x) = x$, and $[0,1]$ is endowed with Lebesgue measure m. Define for each $n \in \mathbb{N}$ $f_n = \sum_{k=1}^{n-1} \frac{k}{n} \cdot \chi_{(k/n,(k+1)/n]}$. Then the sequence $\{f_n\}$ satisfies all the conditions of MCT, and therefore

$$\int f dm = \lim_{n\to\infty} \int f_n dm = \lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{k}{n^2} = \lim_{n\to\infty} \frac{n(n-1)}{2n^2} = \frac{1}{2}$$

(2) Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{1}{n}\chi_{[n,\infty)}(x)$, $n \in \mathbb{N}$. We have

$$\int f_n dm = \frac{1}{n} m([n,\infty)) = \infty, \ \forall n \in \mathbb{N}.$$

On the other hand, the sequence $\{f_n\}$ is nonnegative, and converges to 0 monotonically, so

$$\int \lim_{n\to 0} f_n dm = 0.$$

This shows that the MCT does not hold for decreasing sequences.

Lecture 5

Corollaries.

(1) Given $f \in \mathcal{M}^+(X)$ consider the sequence of simple functions $\{\varphi_k\}$ from Theorem 12. It is easy to check that this sequence satisfies all the conditions of MCT, and therefore

$$\int f d\mu = \lim_{n\to\infty} \int \phi_n d\mu$$

(2) If $E \in \Sigma$ and $\mu(E) = 0$, then $\forall f \in \mathcal{M}^+(X)$ $\int_E f d\mu = 0$

Proof. Let $\{\varphi_n\}$ be the sequence of simple functions from Theorem 12 approximating f. Then $0 \le \varphi_n \le n$ $\forall n$, and by the previous corollary

$$0 \le \int_E f d\mu = \lim_{n\to\infty} \int_E \varphi_n d\mu \le \lim_{n\to\infty} \int_E n d\mu = \lim_{n\to\infty} n\mu(E) = 0.$$

□

(3) Linearity: let $f, g \in \mathcal{M}^+(X)$, $c \ge 0$. Then $cf, f+g \in \mathcal{M}^+(X)$ and
  (a) $\int c\phi d\mu = c \int \phi d\mu$
  (b) $\int (\phi + \psi) d\mu = \int \phi d\mu + \int \psi d\mu$

Proof. Let $\{\phi_n\}, \{\psi_n\} \subseteq \mathcal{M}^+(X)$ be simple and s.t. $\phi_n \nearrow f$, $\psi_n \nearrow g$ pointwise. Then $c\phi_n \nearrow cf$, $\phi_n + \psi_n \nearrow f + g$, and using Corollary (1) and Proposition 15 we obtain

$$\int cf = \lim_{n\to\infty} \int c\phi_n = \lim_{n\to\infty} c \int \phi_n = c \lim_{n\to\infty} \int \phi_n = c \int f$$

Similarly for $f + g$. □

(4) Let $\{g_k\} \subseteq \mathcal{M}^+(X)$, then

$$\int \sum_{k=1}^{\infty} g_k = \sum_{k=1}^{\infty} \int g_k$$

Proof. Consider $f_n := \sum_{k=1}^{n} g_k$, apply MCT □

(5) Let $f \in \mathcal{M}^+(X)$, define $\lambda : \Sigma \to \overline{\mathbb{R}}$ by $\lambda(E) = \int_E f d\mu$. Then λ is a measure on (X, Σ)

Proof. HW+MCT □

Fatou's lemma

Theorem 19. *Let $\{f_n\} \subseteq \mathcal{M}^+(X)$. Then*

$$\int \liminf_n f_n d\mu \leq \liminf_n \int f_n d\mu$$

Proof. Recall, $\liminf_n f_n = \lim_{n\to\infty} \inf_{m\geq n} f_m = \sup_n \inf_{m\geq n} f_m$. Define $g_n := \inf_{m\geq n} f_m$, then $g_n \in \mathcal{M}^+(X)$ and $g_n \nearrow \liminf_n f_n$ as $n \to \infty$. By MCT:

$$\lim_{n\to\infty} \int g_n = \int \lim_{n\to\infty} g_n = \int \liminf_n f_n \tag{2}$$

Moreover, $g_n \leq f_m \ \forall m \geq n \implies \int g_n \leq \int f_m \ \forall m \geq n$, thus

$$\int g_n \leq \inf_{m\geq n} \int f_m \leq \sup_n \left(\inf_{m\geq n} \int f_m \right) = \liminf_n \int f_n.$$

Taking limit as $n \to \infty$ and using (2) we obtain the result. □

Remarks.

(1) Equality does not need to hold: $f_n = \frac{1}{n}\chi_{[n,\infty)}$

$$\liminf_n f_n = \lim_{n\to\infty} f_n = 0 \implies \int \liminf_n f_n = 0$$

$$\text{however, } \int f_n = \infty, \ \forall n \implies \liminf_n \int f_n = \infty$$

(2) We cannot replace $\liminf$ with $\limsup$: define f_n as follows

$$f_{2n} = \sum_{k=1}^{n} \frac{1}{2^k}\chi_{(2k,2k+1]}, \quad f_{2n+1} = \sum_{k=1}^{n} \frac{1}{2^k}\chi_{(2k-1,2k]}$$

Then

$$\int f_{2n} = \int f_{2n+1} = \sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^{n+1}} \implies \limsup_n \int f_n = 1$$

OTOH:

$$\limsup_n f_n = \sum_{k=1}^{\infty} \frac{1}{2^k}\chi_{(2k-1,2k+1]} \implies \int \limsup_n f_n = 2$$

Lecture 6

Corollaries.

Definition 20. *Let (X, Σ, μ) be a measure space. We say that a property P holds* **almost everywhere** *in X (a.e. in X) if $\exists E \in \Sigma$ s.t. $\mu(E) = 0$ and P holds $\forall x \in X \backslash E$.*

(1) Let $f \in \mathcal{M}^+(X)$. Then $f = 0$ a.e. iff $\int f d\mu = 0$.

Proof. $\Longrightarrow$: Let $E := \{x \in X : f(x) > 0\}$ and assume that $\mu(E) = 0$ since $f = 0$ a.e. Define $f_n = n\chi_E$, $n \in \mathbb{N}$. Then $\liminf_n f_n = \lim_{n\to\infty} f_n = \begin{cases} \infty, & \text{if } x \in E, \\ 0, & \text{if } x \notin E \end{cases}$
$\Longrightarrow f(x) \le \liminf_n f_n(x)$ $\forall x \in X \Longrightarrow$ (using Fatou's lemma):

$$\int f d\mu \le \int \liminf_n f_n d\mu \le \liminf_n \int f_n d\mu = n\mu(E) = 0.$$

$\Leftarrow$: Let $\int f = 0$, define $E_n := \{x \in X : f(x) > 1/n\}$. Then $f(x) \ge 1/n \chi_{E_n}(x)$ $\forall x \in X$ implies

$$0 = \int f \ge \int \frac{1}{n}\chi_{E_n} = \frac{1}{n}\mu(E_n) \Longrightarrow \mu(E_n) = 0 \ \forall n \in \mathbb{N}$$

Let $E := \{x \in X : f(x) > 0\} = \cup_n E_n \Longrightarrow \mu(E) \le \sum_n \mu(E_n) = 0$. $\square$

(2) Let $f, g \in \mathcal{M}^+(X)$ and $f = g$ a.e. Then $\int f = \int g$.
(What about the reverse statement?)

Proof. Let $E := \{x \in X : f(x) \ne g(x)\}$, $\mu(E) = 0$. Then

$$\int f = \int f(\chi_E + \chi_{E^c}) = \int_E f + \int_{E^c} f = \int_{E^c} f = \int_{E^c} g = \int g,$$

where in the third and fourth equalities we used Corollary (2) of the MCT. $\square$

(3) Let $\{f_n\} \subseteq \mathcal{M}^+(X)$ and $f_n \nearrow f \in \mathcal{M}^+(X)$ a.e. Then

$$\lim_{n\to\infty} \int f_n = \int f.$$

Proof. Let $E := \{x \in X : f$ is not increasing, or $f_n \nrightarrow f\}$, $\mu(E) = 0$. Define $\tilde{f}_n(x) := \begin{cases} f_n(x), & \text{if } x \notin E, \\ 0, & \text{if } x \in E \end{cases} = f_n(x)\chi_{E^c}(x)$, $g := \sup \tilde{f}_n$. Note that $g = f$ a.e. and $\tilde{f}_n \nearrow g$ on X and thus by MCT $\lim_{n\to\infty} \int \tilde{f}_n = \int g$. By (2) $\int \tilde{f}_n = \int f_n$ and $\int g = \int f \Longrightarrow \lim \int f_n = \int f$. $\square$

Integrable functions

Definition 21. *Let $I = I(X, \Sigma, \mu)$ denote the set of all measurable functions $f : X \to \overline{\mathbb{R}}$ s.t. $\int f^+$ and $\int f^-$ are finite. For $f \in I$ define the* **Lebesgue integral of** f *to be*

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu.$$

If $E \in \Sigma$, $\int_E f d\mu = \int f \chi_E d\mu$. $I =$ **integrable functions**.

Properties.

(1) If $f \in \mathcal{M}^+(X)$ then $f \in I$ iff $\int f < \infty$
(2) If $f \in I$ and $\{E_i\}_{i=1}^{\infty}$ are pairwise disjoint and measurable, then

$$\int_{\cup_i E_i} f = \sum_i \int_{E_i} f.$$

(3) Let $f \in \mathcal{M}(X)$. Then $f \in I \iff |f| \in I$.
Proof. $f \in \mathcal{M}(X) \implies |f| \in \mathcal{M}^+(X)$. If $f \in I$, then $\int f^+ < \infty$ and $\int f^- < \infty$.

$$|f|^+ = |f| = f^+ + f^- \implies \int |f|^+ < \infty$$

$$|f|^- = 0 \implies \int |f|^- = 0 < \infty$$

Thus, $|f| \in I$. OTOH if $|f| \in I$ then $\int f^{\pm} \leq \int |f| < \infty \implies f \in I$. □
(4) Let $f \in \mathcal{M}(X)$, $g \in I$, and $|f| \leq |g|$. Then $f \in I$ and

$$\int |f| \leq \int |g|.$$

Proof. $f \in \mathcal{M}(X) \implies |f| \in \mathcal{M}^+(X)$. $|f| \leq |g| \implies \int |f| \leq \int |g|$. $g \in I \implies |g| \in I \implies \int |g| < \infty \implies \int |f| < \infty \implies |f| \in I \implies f \in I$ by (3). □
(5) Let $f \in I$ and $f = f_1 - f_2$ where $f_1, f_2 \in \mathcal{M}^+(X)$ and $\int f_{1,2} < \infty$. Then

$$\int f = \int f_1 - \int f_2.$$

Proof. We have $f = f_1 - f_2 = f^+ - f^- \implies f_1 + f^- = f_2 + f^+$. Integrate both sides, use additivity of $\int$ for $\mathcal{M}^+(X)$ functions and the fact that all integrals are finite to obtain

$$\int f_1 - \int f_2 = \int f^+ - \int f^- = \int f.$$

□

(6) Let $f, g \in I$, $\alpha \in \mathbb{R}$. Then $\alpha f \in I$, $f + g \in I$ and

$$\int \alpha f = \alpha \int f, \quad \int (f+g) = \int f + \int g.$$

Proof. $f + g = f^+ - f^- + g^+ - g^- = (f^+ + g^+) - (f^- + g^-)$, use (5). □

Lecture 7

Dominated convergence theorem

Theorem 22. *Let (X, Σ, μ) be a measure space, $\{f_n\} \subseteq I$ and $f_n \to f$ where f is finite valued. Suppose $\exists g \in I$ s.t. $|f_n(x)| \leq g(x)$ $\forall n \in \mathbb{N}$, $\forall x \in X$. Then $f \in I$ and*

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu. \tag{3}$$

Proof. 1. We have $f_n \to f$ pointwise, f_n measurable $\forall n \implies f \in \mathcal{M}(X)$. Moreover, $|f_n| \leq g$ $\forall n \implies |f| \leq g = |g| \implies f \in I$ by (4).
2. $|f_n| \leq g \implies g + f_n, g - f_n \in \mathcal{M}^+(X)$, thus using Fatou's lemma applied to $g + f_n$ we get

$$\int g + \int f = \int (g+f) = \int \lim_{n\to\infty} (g+f_n) = \int \liminf_n (g+f_n) \leq \liminf_n \int (g+f_n)$$
$$= \int g + \liminf_n \int f_n$$

Similarly for $g - f_n$

$$\int g - \int f \leq \int g + \liminf_n \left(-\int f_n \right) = \int g - \limsup_n \int f_n$$

Exercise: show that for any sequence $\{x_n\} \subseteq \mathbb{R}$, $\liminf_n(-x_n) = -\limsup x_n$
3. We have obtained

$$\limsup_n \int f_n \leq \int f \leq \liminf_n \int f_n$$

OTOH we always have $\liminf \leq \limsup$, and therefore

$$\limsup_n \int f_n = \liminf_n \int f_n \left(= \lim_{n\to\infty f_n} \right) = \int f.$$

$\square$

Example 23. *Existence of a dominating function g is necessary. Consider $f_n = n\chi_{(0,1/n)}$. Then $\lim_{n\to\infty} f_n = 0$, but $\int f_n = 1$.*

Remark 24. *It is sufficient to require $f_n \to f \in \mathcal{M}(X)$ a.e. in X*

Remark 25. *Consider a sequence $\{f_n\}$ satisfying the conditions of Theorem 22. For the sequence $\{h_n\} = \{|f_n - f|\}$ we then have $h_n \to 0$, $\{h_n\} \in I$ since $f_n, f \in I$, and $|h_n| \leq g + |f| \in I$. This shows that $\{h_n\}$ also satisfies all the conditions of Theorem 22 and we have*

$$\lim_{n\to\infty} |f_n - f| d\mu = 0.$$

*In general, this is a stronger statement than (3) (**exercise**).*

Riemann integral

Let $I \subseteq \mathbb{R}^n$ be a bounded rectangle, $f : I \to \mathbb{R}$ a bounded function, and P — a partition of I into subrectangles. Define the **lower Darboux sum**

$$L(f, P) = \sum_{J \in B(P)} \inf_{x \in J} f(x) l(J),$$

and the **upper Darboux sum**

$$U(f, P) = \sum_{J \in B(P)} \sup_{x \in J} f(x) l(J),$$

where $B(P)$ a collection of subrectangles of I given by partition P. The **lower and upper integrals of** f are then defined as follows

$$\underline{\int} f = \sup_P L(f, P), \quad \overline{\int} f = \inf_P U(f, P)$$

Proposition 26. *(1) $L(f, P_1) \leq U(f, P_2)$ for any two partitions P_1 and P_2*
(2) $\underline{\int} f \leq \overline{\int} f$

Definition 27. *If $\underline{\int} f \leq \overline{\int} f$, then f is called* **Riemann integrable***, and $\int f = \underline{\int} f = \overline{\int} f$ — the* **Riemann integral of** *f. If $E \subseteq \mathbb{R}^n$ is bounded, and $f : E \to \mathbb{R}$ is a bounded function, define $\int_E f$ by choosing $I \supseteq E$ and defining $\tilde{f} : I \to \mathbb{R}$ as $\tilde{f}(x) = \begin{cases} f(x), & \text{if } x \in E \\ 0, & \text{if } x \notin E \end{cases}$. Then set $\int_E f = \int_I \tilde{f}$.*

Theorem 28. *Let $I \subseteq \mathbb{R}^n$ a bounded rectangle, and $f : I \to \mathbb{R}$ a bounded function.*

(1) f is Riemann integrable iff f is continuous a.e. (wrt to Lebesgue measure)
(2) If f is Riemann integrable, then it is also Lebesgue integrable, and the integrals coincide.

Examples.

(1) $\chi_{[0,1]\cap\mathbb{Q}} : [0, 1] \to \mathbb{R}$ is bounded, but discontinuous at every point (**exercise**), and thus not Riemann integrable. It is Lebesgue integrable, however, and the integral is equal to $1 \cdot m([0, 1] \cap \mathbb{Q}) = 0$.
(2) Let $f = \chi_{\mathcal{V}}$ where $\mathcal{V}$ is the Vitali set. Then $\{x \in \mathbb{R}^n : f(x) > 0\} = \mathcal{V} \notin \mathcal{L} \implies f \notin \mathcal{M} \implies f \notin \mathcal{I}$.

The space L^1

Definition 29. *Let V be a real vector space. A function $||\cdot|| : V \to \mathbb{R}$ is a* **norm** *if*

(1) $||x|| \geq 0\ \forall x \in V$ *and* $||x|| = 0$ *iff* $x = 0$
(2) $||\alpha x|| = |\alpha| ||x||\ \forall x \in V,\ \forall \alpha \in \mathbb{R}$
(3) $||x + y|| \leq ||x|| + ||y||\ \forall x, y \in V$

Let (X, Σ, μ) be a measure space. Define an equivalence relation $\sim$ on the set $I(X, \Sigma, \mu)$ by $f \sim g$ if $f = g$ a.e. on X. Denote the set of equivalence classes by $L^1(X, \Sigma, \mu)$. Note that if f is integrable, it is finite a.e. (HW), and therefore the arithmetic operations $+$ and $-$ are well defined on L^1. Define the **norm** of $f \in L^1$ by

$$||f|| = ||f||_{L^1} = \int |f| d\mu. \tag{4}$$

It is an **exercise** to check that all the conditions of Definition 29 are satisfied. It follows then that $d(f, g) = ||f - g||_{L^1}$ defines a metric on L^1.

Theorem 30. *The space $L^1 = L^1(X, \Sigma, \mu)$ is complete.*

Proof. 1. Let $\{f_n\} \subseteq L^1$ be a Cauchy sequence, i.e. $||f_n - f_m|| \to 0$ as $n, m \to \infty$. Let $\{f_{n_k}\}$ be a subsequence such that

$$||f_{n_{k+1}} - f_{n_k}|| \leq \frac{1}{2^k}, \quad \forall k \in \mathbb{N}$$

2. Define

$$f(x) = f_{n_1}(x) + \sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x))$$

$$g(x) = |f_{n_1}(x)| + \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|$$

Note that if the second series converges, then the first series converges absolutely and the functions are well-defined. First observe that

$$\int |f_{n_1}| + \sum_{k=1}^{m} \int |f_{n_{k+1}} - f_{n_k}| \leq \int |f_{n_1}| + \sum_{k=1}^{m} \frac{1}{2^k} \leq \int |f_{n_1}| + 1 \quad \forall N \in \mathbb{N},$$

and thus by MCT

$$\int g = \int |f_{n_1}| + \lim_{m \to \infty} \sum_{k=1}^{m} \int |f_{n_{k+1}} - f_{n_k}| \leq \int |f_{n_1}| + 1$$

which shows that g is integrable. Since $|f| \leq g$ this implies that f is integrable as well, which means the series for f converges a.e. (see HW 10 q. 4).

3. We have shown

$$f_{n_m}(x) = f_{n_1}(x) + \sum_{k=1}^{m}(f_{n_{k+1}}(x) - f_{n_k}(x)) \to f(x) \text{ as } m \to \infty \text{ a.e. in X}$$

where f is integrable (and therefore measurable). Using the Dominated Convergence Theorem together with Remarks 24 and 25 we conclude

$$||f_{n_k} - f|| = \int |f_{n_k} - f| d\mu \to 0, \text{ as } k \to \infty$$

4. Finally, recall that if a subsequence of any Cauchy sequence converges to a limit, the whole sequence converges to the same limit (HW). This gives

$$f_n \to f \text{ in } L^1.$$

□

Definition 31. *A complete normed vector space is called a* **Banach space**. *This is an important class of spaces that is studied in the area of functional analysis.*

HILBERT SPACES

Lecture 1

Inner product.

Definition 1. *Let V be a vector space over $\mathbb{C}$ (or $\mathbb{R}$). An* **inner product** *on V is a function $(\cdot,\cdot) : V \times V \to \mathbb{C}(\mathbb{R})$ satisfying*

(1) $(u, v) = \overline{(v, u)}$, $\forall u, v \in V$
(2) $(\alpha u + v, w) = \alpha(u, w) + (v, w)$, $\forall u, v, w \in V$, $\forall \alpha \in \mathbb{C}$
(3) $(u, u) \geq 0$ $\forall u \in V$, *and* $(u, u) = 0$ *iff* $u = 0$

Proposition 2. *If $(\cdot,\cdot)$ is an inner product on V, then $||u|| = (u, u)^{1/2}$, $u \in V$ is a norm on V. It is referred to as the norm* **induced** *by the inner product.*

Proof. **Exercise**. Hint: to prove the triangle inequality, first prove the **Cauchy-Schwartz inequality**

$$|(u, v)| \leq ||u||||v||, \quad \forall u, v \in V.$$

□

Definition 3. *An inner product vector space that is complete with respect to the induced norm is called a* **Hilbert space**. *A Hilbert space is*

- **Infinite dimensional** *if it is not spanned by a finite number of elements*
- **Separable** *if it has a countable dense subset*

Examples.

(1) $\mathbb{R}^n$, $(x, y) = \sum_{i=1}^n x_i y_i$
(2) $\mathbb{C}^n$, $(x, y) = \sum_{i=1}^n x_i \overline{y_i}$
(3) l^2: the set of all real-valued infinite sequences $(x_1, x_2, \dots)$ satisfying $\sum_{k=1}^\infty x_k^2 < \infty$. Define

$$(x, y) = \sum_{k=1}^{\infty} x_k y_k,$$

then $(\cdot,\cdot)$ is an inner product (**exercise**). In HW 4 you showed that l^2 is a complete metric space, where the metric is given by the norm induced by the inner product. This implies that l^2 is a Hilbert space.
(4) L^2, see below

Space L^2

Let (X, Σ, μ) be a measure space. Consider the set $L^2(X, \Sigma, \mu)$ of square integrable functions $f : X \to \mathbb{C}$, i.e. functions satisfying

$$\int |f|^2 d\mu < \infty.$$

As in the case of L^1 we identify functions that are equal a.e. Define the inner product

$$(f, g) = \int f\overline{g}\, d\mu, \quad f, g \in L^2$$

which induces the norm (L^2-norm)

$$||f|| = ||f||_{L^2} = \left(\int |f|^2 d\mu\right)^{\frac{1}{2}}$$

It is straightforward to check that this inner product satisfies all the properties in Definition 1.

Theorem 4. *L^2 is a Hilbert space.*

Proof. The proof is very similar to the proof of completeness of L^1. Let $\{f_n\} \subseteq L^2$ be a Cauchy sequence, i.e. $||f_n - f_m|| \to 0$ as $n, m \to \infty$. Let $\{f_{n_k}\}$ be a subsequence such that

$$||f_{n_{k+1}} - f_{n_k}|| \leq \frac{1}{2^k}, \quad \forall k \in \mathbb{N}$$

Define the following series whose convergence we prove below

$$f(x) = f_{n_1}(x) + \sum_{k=1}^{\infty}(f_{n_{k+1}}(x) - f_{n_k}(x))$$
$$g(x) = |f_{n_1}(x)| + \sum_{k=1}^{\infty}|f_{n_{k+1}}(x) - f_{n_k}(x)|$$

together with their partial sums

$$S_N(f)(x) = f_{n_1}(x) + \sum_{k=1}^{N}(f_{n_{k+1}}(x) - f_{n_k}(x))$$
$$S_N(g)(x) = |f_{n_1}(x)| + \sum_{k=1}^{N}|f_{n_{k+1}}(x) - f_{n_k}(x)|$$

Note that if the second series converges, then the first series converges absolutely and the functions are well-defined. Using the triangle inequality we have

$$||S_N(g)|| \leq ||f_{n_1}|| + \sum_{k=1}^{N} ||f_{n_{k+1}} - f_{n_k}|| \leq ||f_{n_1}|| + \sum_{k=1}^{K} \frac{1}{2^k} \leq ||f_{n_1}|| + 1 \quad \forall N \in \mathbb{N},$$

and thus by MCT

$$\int |g|^2 = \lim_{N\to\infty} \int S_N(g)^2 = \lim_{N\to\infty} ||S_N(g)||^2 \leq (||f_{n_1}|| + 1)^2 < \infty$$

which shows that g is square integrable. Since $|f| \leq g$ this implies that $f \in L^2$ as well, which means the series for f converges a.e. Moreover, since $S_N(f) = f_{n_N}$ we have

$$f_{n_k} \to f \quad a.e.$$

To prove that $f_{n_k} \to f$ in L^2 observe that $|f - S_N(f)|^2 \leq |2g|^2$ so using the Dominated Convergence Theorem we conclude

$$||f_{n_k} - f|| \to 0, \text{ as } k \to \infty, \text{ i.e. } f_{n_k} \to f \text{ in } L^2.$$

Finally, recall that if a subsequence of any Cauchy sequence converges to a limit, the whole sequence converges to the same limit (HW). This gives

$$f_n \to f \text{ in } L^2.$$

□

Lecture 2

Orthogonality.

Definition 5. *Two elements f and g in a Hilbert space H with inner product $(\cdot,\cdot)$ are* **orthogonal** *or* **perpendicular** *if*

$$(f, g) = 0.$$

We then write $f \perp g$.

Proposition 6 (Pythagorean theorem)**.** *If $f \perp g$, then $||f + g||^2 = ||f||^2 + ||g||^2$.*

Proof. Note that $(f, g) = 0$ implies $(g, f) = 0$, and therefore

$$||f + g||^2 = (f + g, f + g) = ||f||^2 + (f, g) + (g, f) + ||g||^2 = ||f||^2 + ||g||^2.$$

□

Definition 7. *A finite or countable subset $\{e_1, e_2, \dots\}$ of a Hilbert space H is* **orthonormal** *if*

$$(e_k, e_j) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

Proposition 8. *If $\{e_k\}_{k=1}^{\infty} \subseteq H$ is orthonormal, and $f = \sum_{k=1}^{n} a_k e_k \in H$, then*

$$||f||^2 = \sum_{k=1}^{n} |a_k|^2.$$

Proof. **Exercise**. Use Proposition 6. □

Hilbert basis.

Definition 9. *Let H be an infinitely dimensional Hilbert space. An orthonormal set $\{e_k\}_{k=1}^{\infty} \subseteq H$ is called a* **Hilbert basis** *or an* **orthonormal basis** *if finite linear combinations of elements in $\{e_k\}_{k=1}^{\infty}$ are dense in H. In other words,*

$$\forall f \in H,\ \forall \varepsilon > 0 \ \ \exists \{a_k\}_{k=1}^{n} \subseteq \mathbb{C} \ s.t. \ \ \left\| \sum_{k=1}^{n} a_k e_k - f \right\| < \varepsilon.$$

Given an orthonormal basis and any $f \in H$ we might expect $f = \sum_{k=1}^{\infty} a_k e_k$ for some constants $a_k \in \mathbb{C}$. Formally, taking an inner product of both sides with e_j and using the fact that $\{e_k\}$ is orthonormal, we obtain

$$(f, e_j) = a_j.$$

Using the notation for Fourier series write $f \sim \sum_{k=1}^{\infty} a_k e_k$, where $a_j = (f, e_j)$ for all $j \in \mathbb{N}$. The next theorem makes this precise.

Theorem 10. *The following properties of an orthonormal set $\{e_k\}_{k=1}^{\infty} \subseteq H$ are equivalent*

(1) $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis
(2) If $f \in H$ and $(f, e_j) = 0 \ \forall j \in \mathbb{N}$, then $f = 0$
(3) If $f \in H$ and $S_N(f) = \sum_{k=1}^{N} a_k e_k$ where $a_k = (f, e_k)$, then $S_N(f) \to f$ as $N \to \infty$ in the norm
(4) **Parseval's identity:** *if $a_k = (f, e_k)$, then $||f||^2 = \sum_{k=1}^{\infty} |a_k|$.*

Proof. (1)$\Longrightarrow$ (2): Let $f \in H$ and $(f, e_j) = 0 \ \forall j \in \mathbb{N}$. Since $\{e_k\}$ is an orthonormal basis, there exists a sequence $\{g_n\}$ s.t. each g_n is a finite linear combination of e_k's and $||g_n - f|| \to 0$ as $n \to \infty$. Since $(f, e_k) = 0 \ \forall k$, we must have $(f, g_n) = 0 \ \forall n$. Therefore, by Cauchy-Schwartz inequality we have

$$||f||^2 = (f, f) = (f, f - g_n) \leq ||f||\,||f - g_n|| \ \ \forall n \in \mathbb{N}.$$

Taking a limit as $n \to \infty$ we obtain $||f|| = 0 \Longrightarrow f = 0$.
(2)$\Longrightarrow$ (3): Let $f \in H$, define

$$S_N(f) = \sum_{k=1}^{N} a_k e_k \text{ where } a_k = (f, e_k).$$

We first show that $S_N(f)$ converges to some element $g \in H$. Note that by definition of a_k and orthogonality of $\{e_k\}$ we have

$$(f, S_N(f)) = \sum_{k=1}^{N} \overline{a_k}(f, e_k) = \sum_{k=1}^{N} |a_k|^2 = (S_N(f), S_N(f)),$$

and therefore $(f - S_N(f)) \perp S_N(f)$. Using Propositions 6 and 8 this gives

$$||f||^2 = ||f - S_N(f)||^2 + ||S_N(f)||^2 = ||f - S_N(f)||^2 + \sum_{k=1}^{N} |a_k|^2, \tag{1}$$

and thus $||f||^2 \geq \sum_{k=1}^{N} |a_k|^2$. Letting $N \to \infty$ we obtain **Bessel's inequality**

$$\sum_{k=1}^{\infty} |a_k|^2 \leq ||f||^2,$$

which implies that the series $\sum_{k=1}^{\infty} |a_k|^2$ converges. This shows that $\{S_N(f)\}_{N=1}^{\infty}$ is a Cauchy sequence in H since

$$||S_N(f) - S_M(f)||^2 = \sum_{k=M+1}^{N} |a_k|^2, \quad \forall N > M.$$

Since H is complete, $\exists g \in H$ s.t. $S_N(f) \to g$ as $N \to \infty$. To show $g = f$ fix $j \in \mathbb{N}$ and note that for $N \geq j$ we have $(f - S_N(f), e_j) = a_j - a_j = 0$. Since $S_N(f) \to g$ as $N \to \infty$, we conclude that

$$(f - g, e_j) = 0 \;\; \forall j \in \mathbb{N}$$

(**exercise**). By assumption (2) this gives $f = g$, and thus $f = \sum_{k=1}^{\infty} a_k e_k$.
(3)$\Longrightarrow$(4): Let $N \to \infty$ in (1) and use $S_N(f) \to f$ to obtain

$$||f||^2 = \sum_{k=1}^{\infty} |a_k|^2.$$

Note that from (1) we have $\sum_{k=1}^{N} |a_k|^2 \leq ||f||^2 \; \forall N$, so the limit is well defined.
(4)$\Longrightarrow$ (1): Finally, if Parseval's identity holds, then taking a limit in (1) we obtain

$$||S_N(f) - f|| \to 0, \text{ as } N \to \infty.$$

Since $S_N(f)$ is a finite linear combination of elements of $\{e_k\}$, this implies $\{e_k\}$ is a Hilbert basis. $\square$

Theorem 11. *Every separable Hilbert space has an orthonormal basis.*

Sketch of proof. H is separable $\Longrightarrow \exists\{h_k\}_{k=1}^{\infty}$ s.t. finite linear combinations of elements in $\{h_k\}$ are dense in H. Select a subcollection of linearly independent vectors $\{f_k\}$ s.t. $Span(\{f_k\}) = Span(\{h_k\})$. Perform the **Gram-Schmidt orthogonalization process**: let $e_1 = f_1/||f_1||$, then induction. Assume $\{e_k\}_{k=1}^{n}$ orthonormal, s.t. $Span(\{e_k\}_{k=1}^{n}) = Span(\{f_k\}_{k=1}^{n})$. Define $e'_{n+1} = f_{n+1} - \sum_{k=1}^{n} a_k e_k$ where $a_k = (f_{n+1}, e_k)$ for $1 \leq k \leq n$, normalize to get $\{e_k\}_{k=1}^{n+1}$. □

Definition 12. *Two Hilbert spaces H_1 and H_2 are* **isometric** *if there exists a linear isomorphism $\varphi : H_1 \to H_2$, i.e. a linear surjective map preserving the norm: $||h||_{H_1} = ||\varphi(h)||_{H_2}$ $\forall h \in H_1$.*

Theorem 13. *Every infinite-dimensional Hilbert space is isomorphic to l^2.*

Proof. From Theorem 11 there exists an orthonormal basis $\{e_k\}_{k=1}^{\infty}$. Define $\varphi : H \to l^2$ by $\varphi(h) = \{(h, e_k)\}$. From Bessel's inequality it follows that φ maps into l^2, and Parseval's identity implies that φ is norm preserving. To show φ is onto, let $\{a_k\} \subseteq l^2$ and define $s_n := \sum_{k=1}^{n} a_k e_k$. Since the series $\sum_{k=1}^{\infty} |a_k|^2$ converges, its sequence of partial sums is Cauchy, and thus

$$||s_n - s_m|| = \sum_{k=m+1}^{n} |a_k| \to 0, \text{ as } n, m \to \infty.$$

This implies $\{s_n\} \subseteq H$ is Cauchy, and therefore converges $s_n \to s \in H$. It is an **exercise** to show that $(s, e_k) = a_k$ $\forall k \in \mathbb{N}$. □

Lecture 3

Fourier series in L^2

Consider $L^2([-\pi, \pi])$ (complex-valued functions s.t. $\int_{-\pi}^{\pi} |f|^2 < \infty$), $[-\pi, \pi]$ is equipped with Lebesgue measure. Using Cauchy-Schwartz inequality we immediately get $L^2([-\pi, \pi]) \subseteq L^1([-\pi, \pi])$, so these functions are integrable. For any $f \in L^1([-\pi, \pi])$ and $n \in \mathbb{Z}$ define the n^{th} **Fourier coefficient** of f by

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx,$$

where dx stands for Lebesgue measure on $[-\pi, \pi]$.

Theorem 14. $\{e_n\}_{n\in\mathbb{Z}} := \left\{ \dfrac{e^{inx}}{\sqrt{2\pi}} \right\}_{n\in\mathbb{Z}}$ *is a Hilbert basis for $L^2([-\pi, \pi])$.*

Proof. **Exercise:** $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal system in $L^2([-\pi, \pi])$. By Theorem 10 (2) it is sufficient to check that the only function satisfying $(f, e_n) = 0$ $\forall n \in \mathbb{Z}$ is $f \equiv 0$. Suppose $f \in L^2([-\pi, \pi])$ and $(f, e_n) = 0$ $\forall n \in \mathbb{Z}$. Then $f \in L^1([-\pi, \pi])$ and $a_n = 0$ for all $n \in \mathbb{Z}$. From part (2) of Proposition 15 $f = 0$ which concludes the proof. □

Proposition 15. *Let $f \in L^1([-\pi,\pi])$, and a_n is the n^{th} Fourier coefficient of f.*

(1) If $a_n = 0$ for all n, then $f(x) = 0$

(2) $\sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{inx} \to f(x)$ for a.e. $x \in [-\pi,\pi]$ as $r \to 1$, $r < 1$.

Proof. Part (1) is an immediate consequence of part (2), thus we will show (2).
1. We have

$$\sum_{-\infty}^{\infty} r^{|n|} e^{iny} = P_r(y) = \frac{1-r^2}{1-2r\cos y + r^2} \quad \text{(Poisson kernel)}$$

Proof: we can write $P_r(y) = \sum_{n=0}^{\infty} w^n + \sum_{n=1}^{\infty} \overline{w}^n$, where $w = re^{iy}$. Since $r < 1$ both (geometric) series converge absolutely, and

$$P_r(y) = \frac{1}{1-w} + \frac{\overline{w}}{1-\overline{w}} = \frac{1-|w|^2}{|1-w|^2} = \frac{1-r^2}{1-2r\cos y + r^2}$$

2. **Exercise:**

$$\frac{1}{2\pi}\int P_r(y)dy = 1.$$

3. Extend f to a periodic function on $\mathbb{R}$ with period 2π. Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)P_r(y)dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)\sum_{-\infty}^{\infty} a_n r^{|n|} e^{iny}dy = \sum_{-\infty}^{\infty} r^{|n|}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)e^{iny}dy$$

where the series and the integral can be switch by the DCT (**exercise**). Moreover, fore each x and n

$$\int_{-\pi}^{\pi} f(x-y)e^{iny}dy = \int_{-\pi+x}^{\pi+x} f(y)e^{in(x-y)}dy = e^{inx}\int_{-\pi}^{\pi} f(y)e^{-iny}dy = e^{inx}2\pi a_n,$$

(by translation invariance and periodicity), and we therefore obtain

$$\sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{inx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)P_r(y)dy$$

4. Using 2. and 3.

$$\begin{aligned}\left| f(x) - \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{inx}\right| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x)-f(x-y))P_r(y)dy\right| \\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)-f(x-y)|P_r(y)dy \qquad (2)\end{aligned}$$

5. Assume f is continuous on $[-\pi,\pi]$. Since $[-\pi,\pi]$ is compact we have that f is *uniformly continuous* on $[-\pi,\pi]$, meaning

$$\forall\varepsilon > 0\ \exists\delta > 0 \text{ s.t. } |x-y| < \delta \Longrightarrow |f(x)-f(y)| < \varepsilon \quad \textbf{(exercise)}$$

Let $\varepsilon > 0$, choose δ that works for $\varepsilon/2$, we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - f(x-y)|P_r(y)dy \\
&\qquad = \frac{1}{2\pi}\int_{|y|<\delta} |f(x) - f(x-y)|P_r(y)dy + \frac{1}{2\pi}\int_{|y|\geq\delta} |f(x) - f(x-y)|P_r(y)dy \\
&\qquad \leq \frac{\varepsilon}{2} + \frac{1-r^2}{2r(1-\cos(\delta))}2M, \qquad (3)
\end{aligned}$$

where in the last inequality we used f is continuous on $[-\pi, \pi]$, and therefore bounded by some $M > 0$, and the bound

$$P_r(x) \leq \frac{1-r^2}{2r(1-\cos(x))}.$$

Taking r sufficiently close to 1 s.t. the second term on the right-hand side of 3 is $< \varepsilon/2$, we obtain from (2) and (3)

$$\left| f(x) - \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{inx} \right| < \varepsilon.$$

6. Finally, if $f \in L^2([-\pi, \pi])$ we use the fact that continuous functions are dense in L^2 i.e. for any $\varepsilon > 0$ and for any $f \in L^2([-\pi, \pi])$ there exists a continuous $h \in C([-\pi, \pi])$ s.t.

$$||f - h||_{L^2} < \varepsilon/2.$$

If $\{d_n\}$ are the Fourier coefficients of h we have from part 5. that for r sufficiently close to 1

$$\sup_{x\in[-\pi,\pi]} |h(x) - \sum d_n r^{|n|} e^{inx}| < \frac{\varepsilon}{4\pi} \Longrightarrow ||h - \sum d_n r^{|n|} e^{inx}||_{L^2} < \frac{\varepsilon}{2}.$$

Therefore,

$$||f - \sum d_n r^{|n|} e^{inx}||_{L^2} < \varepsilon$$

It is an **exercise** to show

$$||f - \sum a_n r^{|n|} e^{inx}||_{L^2} \leq ||f - \sum d_n r^{|n|} e^{inx}||_{L^2}$$

if $\{a_n\}$ are the Fourier coefficients of f. (Hint: first show that $f - \sum_{|n|\leq N} a_n r^{|n|} e^{inx}$ is orthogonal to $\sum_{|n|\leq N}(d_n - a_n) r^{|n|} e^{inx}$, and then use the Pythagoras theorem). $\square$