

A Logistic Model with a Carrying Capacity Driven Diffusion

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Abstract

In this paper, we consider the diffusive logistic model and remark that the introduction of the standard diffusion term Δu (incorporated with the zero Neumann boundary conditions) leads to the situation when the population tends to be equally distributed over the space available, even if the carrying capacity $K(t, x)$ varies significantly with location. To account for this effect, we introduce the model with a K -driven diffusion

$$\frac{\partial u(t, x)}{\partial t} = D\Delta \left(\frac{u(t, x)}{K(t, x)} \right) + r(t, x)u(t, x) \left(1 - \frac{u(t, x)}{K(t, x)} \right).$$

If parameters K and r are time-independent, we obtain that $K(x)$ is a globally stable stationary solution. If K and r are T_0 -periodic in t functions, then there exists an attractive positive periodic solution.

Keywords: logistic equation, diffusion, positive periodic solution, global attractivity, rate of convergence

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1 Introduction

Historically, some of the first applications of mathematics to biology were in the area of population dynamics, such as the unbounded exponential growth predicted by Maltus and the logistic equation proposed by Verhulst which corresponds to the limited carrying capacity of the environment. The first models were either differential or first order difference equations. However, in many cases either spatial distribution or dependence on the previous history of population densities cannot be neglected, thus the adequate models would

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be partial differential [19] or functional differential equations [13]; sometimes both effects should be accounted for. As for spatially distributed populations, their dynamics was usually modeled in the following way. If at each point the growth of the population size N was described by a certain law (logistic, Gompertz or some other)

$$\frac{dN}{dt} = g(N(t)), \quad (1.1)$$

it was assumed that the growth function can be modified to $f(x, N)$ to involve the space variable (for logistic models this may correspond to a non-uniform carrying capacity in the domain), and the diffusion term $D\Delta u$ was added to the right hand side to describe population movements. Usually the zero Neumann boundary conditions were considered (assuming that the population is closed and there is no flux through the isolated boundaries, or that immigration to the domain is compensated by emigration), though other types of boundary conditions could be imposed.

Overall, this led to the parabolic equation

$$\frac{\partial u(t, x)}{\partial t} = D\Delta u(t, x) + f(x, u(t, x)), \quad t > 0, \quad x \in \Omega, \quad (1.2)$$

with, generally, mixed boundary conditions

$$au(t, x) + b \frac{\partial u(t, x)}{\partial n} = 0, \quad x \in \partial\Omega, \quad t \in (0, \infty), \quad (1.3)$$

and the initial condition

$$u(0, x) = u_0(x), \quad x \in \Omega, \quad (1.4)$$

where Ω is an open domain of $\mathbb{R}^d$, $d = 1, 2$ or 3 , n is the exterior normal to the boundary $\partial\Omega$. If in (1.2) we assume $f(x, u) = 0$ (there are no births and no mortality) then eventually there will be a uniform population distribution over the domain. Let us consider the logistic growth

$$\frac{\partial u(t, x)}{\partial t} = D\Delta u(t, x) + r(x)u(t, x) \left(1 - \frac{u(t, x)}{K(x)}\right), \quad t > 0, \quad x \in \Omega, \quad (1.5)$$

where $u(t, x)$ is the population density, $r(x)$ is the intrinsic growth rate, $K(x)$ is the carrying capacity of the environment (generally, both can also be time-dependent). Diffusive systems of this type were considered in many papers, see [1, 2, 3, 6, 8, 10, 11, 14, 19, 20] and references therein. Similar to (1.2), if $r = 0$ or D is very large, the solution of (1.5) tends to be uniformly distributed. This is a reasonable assumption for spatially uniform resources (which means that the carrying capacity K is not x -dependent). However, for a nonuniform resources distribution the model (1.2) and (1.5) suggests that species may move to the regions with lower per capita available resources which doesn't seem to be biologically feasible.

We assume an alternative type of diffusion when ultimately the population tends to have not a uniform distribution over the domain but the uniform per capita available resources.

This means that not u but u/K diffuses. More precisely, we consider the following equation:

$$\frac{\partial u(t, x)}{\partial t} = D\Delta \left(\frac{u(t, x)}{K(x)} \right) + r(x)u(t, x) \left(1 - \frac{u(t, x)}{K(x)} \right), \quad t > 0, \quad x \in \Omega, \quad (1.6)$$

with the Neumann boundary condition

$$\frac{\partial \left(\frac{u}{K} \right)}{\partial n} = 0, \quad x \in \partial\Omega, \quad t \in (0, \infty) \quad (1.7)$$

and the initial condition

$$u(0, x) = u_0(x), \quad x \in \Omega. \quad (1.8)$$

Let us note that for the parabolic equation

$$\frac{\partial u}{\partial t} = \nabla \cdot [D(x)\nabla g(u)] \quad (1.9)$$

the zero Neumann condition for the flux across the boundary $\partial\Omega$ is

$$D(x)\frac{\partial g(u)}{\partial n} = 0, \quad (t, x) \in (0, \infty) \times \partial\Omega, \quad (1.10)$$

for details see [9], p. 490 and [7], p. 30-31. Thus, the modification of the diffusion term in (1.6) compared to (1.1) leads to the Neumann boundary condition (1.7). If $r(x) \equiv 0$, then (1.7) means that the total population number does not change, the domain is either isolated or the incoming population flux equals the outgoing stream.

Let us note that other diffusion types, in particular nonlinear, were considered in population dynamics, mainly based on the experimental data, see, for example, [12, 15, 18]. Some general approach to the equation with a nonlinear diffusion term $\nabla \cdot (D(u)\nabla u)$ was developed in [16]. The type of diffusion in (1.6) was first considered in [5] motivated by the choice of optimal harvesting strategies.

The paper is organized as follows. After introducing relevant definitions and results in Section 2 we proceed to the main results. In Section 3.1 we consider the equation (1.6) with space-dependent K and r and demonstrate that all solutions with a nonnegative initial function u_0 (and positive in some nonempty open domain) are positive. In Section 3.2 global attractivity of the equilibrium solution $K(x)$ of the problem (1.6),(1.7),(1.8) for all solutions with a nonnegative initial function u_0 (and positive in some nonempty open domain) is justified; moreover, there is an estimate of the convergence rate in the terms of the growth rate $r(x)$. In Section 3.3 again the analogue of (1.6) with time-variable T_0 -periodic growth rate $r(t, x)$ and carrying capacity $K(t, x)$ is considered. Under certain conditions, the existence of a positive periodic solution is justified, and convergence of all solutions with a positive initial function to this periodic solution is demonstrated. Finally, in Section 4 we discuss the results and state some open problems.

2 Preliminaries

To prove the existence of a positive bounded solution of the problem (1.6)-(1.8) we will use the method of upper and lower solutions [17].

Denote $Q_T = (0, T] \times \Omega$, $\partial Q_T = (0, T] \times \partial\Omega$ for any $T > 0$ and consider the following nonlinear parabolic problem:

$$\frac{\partial u}{\partial t} - Lu = f(t, x, u), \quad (t, x) \in Q_T, \quad (2.1)$$

$$\frac{\partial u}{\partial n} = 0, \quad (t, x) \in \partial Q_T, \quad (2.2)$$

$$u(0, x) = u_0(x), \quad x \in \Omega, \quad (2.3)$$

where the operator L defined as

$$Lu := \sum_{i,j=1}^n a_{ij}(t, x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(t, x) \frac{\partial u}{\partial x_i} \quad (2.4)$$

is uniformly elliptic, namely, there exist positive numbers λ and Λ such that for every vector $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(t, x) \xi_i \xi_j \leq \Lambda |\xi|^2, \quad (t, x) \in \overline{Q}_T.$$

We assume that the coefficients of L are Hölder continuous in Q_T .

Definition 1. A function $\bar{u}(t, x) \in C(\overline{Q}_T) \cap C^{1,2}(Q_T)$ is called **an upper solution** of (2.1)-(2.3) if it satisfies the following inequalities:

$$\begin{aligned} \frac{\partial \bar{u}}{\partial t} - L\bar{u} &\geq f(t, x, \bar{u}), \quad (t, x) \in Q_T, \\ \frac{\partial \bar{u}}{\partial n} &\geq 0, \quad (t, x) \in \partial Q_T, \\ \bar{u}(0, x) &\geq u_0(x), \quad x \in \Omega. \end{aligned} \quad (2.5)$$

A function $\underline{u}(t, x) \in C(\overline{Q}_T) \cap C^{1,2}(Q_T)$ is called **a lower solution** if it satisfies the following inequalities:

$$\begin{aligned} \frac{\partial \underline{u}}{\partial t} - L\underline{u} &\leq f(t, x, \underline{u}), \quad (t, x) \in Q_T, \\ \frac{\partial \underline{u}}{\partial n} &\leq 0, \quad (t, x) \in \partial Q_T, \\ \underline{u}(0, x) &\leq u_0(x), \quad x \in \Omega. \end{aligned} \quad (2.6)$$

The pair $\bar{u}, \underline{u}$ is said to be ordered if $\bar{u} \geq \underline{u}$ in Q_T . The set of continuous functions $u \in C(\bar{Q}_T)$ such that $\underline{u} \leq u \leq \bar{u}$ is denoted as $\langle \underline{u}, \bar{u} \rangle$. In the sector $\langle \underline{u}, \bar{u} \rangle$ we assume that for some bounded functions $\underline{c} \equiv \underline{c}(t, x)$, $\bar{c} \equiv \bar{c}(t, x)$ the function f satisfies the condition

$$-\underline{c}(u_1 - u_2) \leq f(t, x, u_1) - f(t, x, u_2) \leq \bar{c}(u_1 - u_2), \quad \underline{u} \leq u_2 \leq u_1 \leq \bar{u}, \quad (t, x) \in Q_T. \quad (2.7)$$

Then function $F(u) = \underline{c} \cdot u + f(u)$ is monotone nondecreasing in u for $\underline{u} \leq u \leq \bar{u}$.

Now we are going to construct an upper and a lower sequences for (2.1)-(2.3) which will converge to a unique solution of the problem (2.1)-(2.3).

Definition 2. Consider a sequence $\{u^{(k)}\}_{k=0}^{\infty}$ defined by the following iteration process

$$\begin{aligned} \frac{\partial u^{(k)}(t, x)}{\partial t} - Lu^{(k)}(t, x) + \underline{c}u^{(k)} &= F(t, x, u^{(k-1)}), \quad (t, x) \in Q_T, \\ \frac{\partial u^{(k)}}{\partial n} &= 0, \quad (t, x) \in \partial Q_T, \\ u^{(k)}(0, x) &= u_0(x), \quad x \in \Omega, \end{aligned} \quad (2.8)$$

where $F(t, x, u) = \underline{c} \cdot u + f(t, x, u)$. Denote the sequence with the initial iteration $u^{(0)} = \underline{u}$ by $\{\underline{u}^{(k)}\}_{k=0}^{\infty}$ (**the lower sequence**), and the sequence with $u^{(0)} = \bar{u}$ by $\{\bar{u}^{(k)}\}_{k=0}^{\infty}$ (**the upper sequence**).

Lemma 1. [17] Let condition (2.7) be satisfied. Then the two sequences $\{\underline{u}^{(k)}\}$ and $\{\bar{u}^{(k)}\}$ introduced in Definition 2 are well defined and $\bar{u}^{(k)}, \underline{u}^{(k)}$ are in $C^\alpha(Q_T)$, $0 < \alpha \leq 1$, for each k .

Now we can present the existence result for (2.1)-(2.3).

Lemma 2. [17] Let $\underline{u}, \bar{u}$ be ordered lower and upper solutions of (2.1)-(2.3) introduced in Definition 2 and let $f(t, x, u)$ satisfy (2.7). Then the sequences $\{\underline{u}^{(k)}\}$ and $\{\bar{u}^{(k)}\}$ converge monotonically to a unique solution u of (2.1)-(2.3) and

$$\underline{u} \leq \underline{u}^{(k)} \leq u \leq \bar{u}^{(k)} \leq \bar{u}. \quad (2.9)$$

To prove positivity of solutions we will need the strong maximum principle for parabolic equations [17].

Lemma 3. [17] Let $u(t, x) \in C(\bar{Q}_T) \cap C^{1,2}(Q_T)$ be such that

$$\frac{\partial u}{\partial t} - Lu \geq 0, \quad (t, x) \in Q_T,$$

where L is a uniformly elliptic operator (2.4). If $u(t, x)$ attains a minimum value m_0 at some point in Q_T then $u(t, x) \equiv m_0$ throughout Q_T . If $\partial\Omega$ is in $C^{\alpha+1}$ with $0 < \alpha < 1$ and u attains a minimum at some point (t_0, x_0) on ∂Q_T , then $\partial u / \partial n < 0$ at (t_0, x_0) whenever u is not constant.

For problems with periodic in t parameters we will apply the global attractivity result [20] for the following problem:

$$\frac{\partial u}{\partial t} - Lu = f(t, x, u), \quad (t, x) \in Q, \quad (2.10)$$

$$\frac{\partial u}{\partial n} = 0, \quad (t, x) \in \partial Q, \quad (2.11)$$

$$u(0, x) = u_0(x), \quad x \in \Omega. \quad (2.12)$$

Here $Q = (0, \infty) \times \Omega$, $\partial Q = (0, \infty) \times \partial\Omega$, the function $f(t, x, u)$ is periodic in t with a period T_0 . Then the following result holds [20].

Lemma 4. [20] *Let $\tilde{u} \geq \hat{u}$ be a pair of constant upper and lower solutions of (2.10), (2.11) and let the function $f(t, x, u)$ be Hölder continuous in (t, x) , T_0 -periodic in t and continuously differentiable in u for $u \in [\hat{u}, \tilde{u}]$. Then there exists a pair of T_0 -periodic solutions $\bar{u}$ and $\underline{u}$ of (2.10), (2.11) with $\hat{u} \leq \underline{u} \leq \bar{u} \leq \tilde{u}$. Moreover, for any initial function $u_0(x)$ satisfying $\hat{u} \leq u_0(x) \leq \tilde{u}$ in Ω , the corresponding solution $u(x, t)$ of (2.10)-(2.12) satisfies*

$$\underline{u}(t, x) \leq \liminf_{t \rightarrow \infty} u(t, x) \leq \limsup_{t \rightarrow \infty} u(t, x) \leq \bar{u}(t, x), \quad \text{for any } x \in \bar{\Omega}.$$

Further, if $\underline{u} = \bar{u} \equiv u^*$, then u^* is the unique T_0 -periodic solution in $\langle \hat{u}, \tilde{u} \rangle$ which satisfies

$$\lim_{t \rightarrow \infty} |u(t, x) - u^*(t, x)| = 0, \quad \forall x \in \bar{\Omega}.$$

The proof of a more general result can be found in [20] (Theorem 3.2).

3 Main results

Everywhere in this section we assume that $K(t, x)$ is a Hölder continuous in x and continuously differentiable periodic in t function, $K(t, x) > 0$ for any $(t, x) \in \bar{Q} \equiv [0, \infty) \times \bar{\Omega}$, $r(t, x)$ is continuous in $Q \equiv (0, \infty) \times \Omega$ and is periodic in t , $r(t, x) > 0$ for any $(t, x) \in \bar{Q}$. Ω is a an open nonempty bounded domain with $\partial\Omega \in C^{1+\alpha}$, $0 < \alpha < 1$.

3.1 Existence and uniqueness

We will formulate the existence and uniqueness theorem for the general case of a time-dependent carrying capacity $K(t, x)$ and a growth function $r(t, x)$:

$$\frac{\partial u(t, x)}{\partial t} = D\Delta \left(\frac{u(t, x)}{K(t, x)} \right) + r(t, x)u(t, x) \left(1 - \frac{u(t, x)}{K(t, x)} \right), \quad (t, x) \in Q_T, \quad (3.1)$$

with (1.7) and (1.8) as the boundary and the initial condition, respectively. The result will obviously hold for time independent functions $K(x)$ and $r(x)$.

Theorem 1. *Let $u_0(x) \in C(\Omega)$, $u_0(x) \geq 0$ in Ω and $u_0(x) > 0$ in some open bounded nonempty domain $\Omega_1 \subset \Omega$. Then there exists a unique solution $u(t, x)$ of the problem (3.1), (1.7), (1.8) and it is positive.*

Proof. First we make the following substitution

$$v(t, x) = \frac{u(t, x)}{K(t, x)}.$$

Moreover, since the function K is positive and bounded from above in $\overline{Q_T}$ then $v(t, x)$ is well defined. We obtain that function v satisfies

$$\begin{aligned} \frac{\partial v}{\partial t} + \frac{v}{K} \frac{\partial K}{\partial t} - \frac{D}{K} \Delta v &= r(t, x)v(t, x)(1 - v(t, x)), \quad (t, x) \in Q_T, \\ \frac{\partial v}{\partial n} &= 0 \quad \text{on } \partial Q_T, \\ v(0, x) &= \frac{u_0(x)}{K(0, x)} \quad \text{in } \Omega. \end{aligned} \tag{3.2}$$

It is easy to see that the operator $L := \frac{D}{K} \Delta$ is uniformly elliptic, with Hölder continuous coefficients. Therefore, according to Lemma 2, in order to show the existence of the unique solution of (3.2) we only need to construct an ordered pair of upper and lower solutions of (3.2).

To construct an upper solution denote

$$a(t, x) := \frac{1}{K} \frac{\partial K}{\partial t} \tag{3.3}$$

and consider the function:

$$\bar{v}(t, x) = \max \left\{ \sup_{x \in \bar{\Omega}} (v(0, x)), \sup_{(t, x) \in \overline{Q_T}} \frac{r(t, x) - a(t, x)}{r(t, x)} \right\} > 0$$

Then

$$\begin{aligned} \frac{\partial \bar{v}}{\partial t} - \frac{D}{K} \Delta \bar{v} &= 0, \\ \bar{v}(r - a - r\bar{v}) &\leq 0. \end{aligned}$$

So the first inequality of (2.5) in Definition 1 holds. Further,

$$\frac{\partial \bar{v}}{\partial n} = 0$$

and $\bar{v} \geq v(0, x)$, therefore $\bar{v}$ is an upper solution of (3.2) by Definition 1. The function $\underline{v}(t, x) \equiv 0$ is obviously a lower solution.

Next, the function $f(t, x, v) := r(t, x)v(1 - v) - a(t, x)v$ is continuously differentiable with respect to v and we can denote the maximal derivative of f in v for each (t, x)

$$\underline{c}(t, x) = \sup\{-f_v(t, x, v), \underline{v} \leq v \leq \bar{v}\},$$

$$\bar{c}(t, x) = \sup\{f_v(t, x, v), \underline{v} \leq v \leq \bar{v}\}.$$

Then the Lipschitz condition (2.7) holds and by Lemma 2 there exists a unique solution of the problem (3.2) satisfying $\underline{v} \leq v \leq \bar{v}$. Making the inverse substitution we obtain a unique solution of (3.1),(1.7),(1.8) which is $u(t, x) = v(t, x)/K(t, x)$, $0 \leq u(t, x) \leq \bar{v}/K(t, x)$.

To show that the solution is positive for any nonnegative initial function $u_0(x)$ let us consider

$$Z(t, x) = v(t, x)e^{At} = \frac{u(t, x)}{K(t, x)} e^{At}$$

where $A = \bar{v} \sup_{(t, x) \in \bar{Q}_T} (r(t, x)) + \sup_{(t, x) \in \bar{Q}_T} |a(t, x)|$. Substituting $v(t, x) = Z(t, x)e^{-At}$ into (3.2) we have

$$\begin{aligned} \frac{\partial Z(t, x)}{\partial t} - \frac{D}{K(t, x)} \Delta Z(t, x) &= [r v(1 - v) - av + Av]e^{At} \geq 0, \quad (t, x) \in Q_T, \\ \frac{\partial Z}{\partial n} &= 0, \quad (t, x) \in \partial Q_T, \\ Z(0, x) &= v(0, x) \geq 0, \quad x \in \Omega. \end{aligned} \tag{3.4}$$

First, note that $Z(t, x)$ is nonnegative since $v(t, x) \geq \underline{v} = 0$. Next, assume that $Z(t, x)$ attains a zero value at some point (t_0, x_0) . If $(t_0, x_0) \in Q_T$ then according to the maximum principle (Lemma 3) we obtain $Z(t, x) \equiv 0$ in Q_T . According to the assumption of the theorem $Z(0, x) > 0$ on some open bounded domain $\Omega_1 \subset \Omega$ and $Z(t, x) \in C(\bar{Q}_T)$ since $v(t, x)$ is a solution of (3.2). Therefore $Z(t, x) > 0$ in Ω_1 for some $t > 0$ and $Z(t, x) \not\equiv 0$ in Q_T . Thus $Z(t, x) > 0$ in Q_T . Furthermore, if $Z(t_1, x_1) = 0$ for some $(t_1, x_1) \in \partial Q_T$ then by Lemma 3 we have $\partial Z / \partial n < 0$ at (t_1, x_1) which contradicts (3.4). This concludes the proof. $\square$

3.2 Time-independent carrying capacity

The next theorem establishes that in the case (1.6) when the carrying capacity $K(x)$ and the intrinsic growth rate $r(x)$ are time-independent, $p(x) = K(x)$ is an equilibrium solution of the following elliptic boundary value problem

$$-D\Delta \left(\frac{p(x)}{K(x)} \right) = r(x)p(x) \left(1 - \frac{p(x)}{K(x)} \right), \quad x \in \Omega, \tag{3.5}$$

$$\frac{\partial \left(\frac{p}{K} \right)}{\partial n} = 0, \quad x \in \partial\Omega. \tag{3.6}$$

Moreover, all nonnegative (and not identically equal to zero) solutions converge to this positive equilibrium, and we can estimate the convergence rate.

Theorem 2. *The function $p(x) := K(x)$ is an equilibrium solution of (3.5),(3.6). Moreover, for any $u_0(x) \geq 0$, $u_0(x) \not\equiv 0$ the solution $u(t, x)$ of (1.6)-(1.8) converges to $K(x)$, with the convergence speed estimated as*

$$\int_{\Omega} |u(t, x) - K(x)| dx \leq C e^{-\rho t},$$

where

$$\rho = \inf_{x \in \bar{\Omega}} r(x) > 0, \quad (3.7)$$

and therefore, a positive equilibrium solution is unique.

Proof. The first statement of the theorem is obvious. To prove the second part define the constants

$$F := \max \left\{ M = \sup_{x \in \bar{\Omega}} u_0(x), K = \sup_{x \in \bar{\Omega}} K(x) \right\}$$

$$f_{\tau} := \min \left\{ \inf_{x \in \bar{\Omega}} u(\tau, x), k = \inf_{x \in \bar{\Omega}} K(x) \right\}, \quad \text{for some fixed moment } \tau > 0.$$

Note, that $f_{\tau} > 0$ since $K(x) > 0$ by assumption and $u(\tau, x) > 0$ for any $x \in \bar{\Omega}$, $\tau > 0$ by Theorem 1.

Let $u_F(t, x)$ and $u_{f_{\tau}}(t, x)$ be the solutions of the initial-boundary value problem (1.6), (1.7) satisfying initial conditions $u_F(0, x) = F \geq u_0(x)$ and $u_{f_{\tau}}(\tau, x) = f_{\tau} \leq u(\tau, x)$, respectively. According to Definition 1 the function $u_F(t, x)$ satisfies (2.5) and it is an upper solution of (1.6)-(1.8). The function $u_{f_{\tau}}(t, x)$ is a lower solution of the problem (1.6),(1.7) with the initial condition at the moment τ which is $u(\tau, x)$. Then by Theorem 1 we have $u_F(t, x) \geq u(t, x)$ and $u(t, x) \geq u_{f_{\tau}}(t, x)$ for any $t \geq \tau$, or

$$u_{f_{\tau}}(t, x) \leq u(t, x) \leq u_F(t, x) \quad \text{for any } (t, x) \in [\tau, \infty) \times \Omega,$$

$$u_{f_{\tau}}(t, x) - K(x) \leq u(t, x) - K(x) \leq u_F(t, x) - K(x) \quad \text{for any } (t, x) \in [\tau, \infty) \times \Omega. \quad (3.8)$$

Using the same argument as above we obtain

$$u_{f_{\tau}}(t, x) - K(x) \leq 0, \quad u_F(t, x) - K(x) \geq 0, \quad (t, x) \in [\tau, \infty) \times \Omega. \quad (3.9)$$

Combining (3.8) and (3.9) leads to

$$|u(t, x) - K(x)| \leq \max\{u_F(t, x) - K(x), K(x) - u_{f_{\tau}}(t, x)\}, \quad (t, x) \in [\tau, \infty) \times \Omega,$$

thus integrating both sides yields

$$\int_{\Omega} |u(t, x) - K(x)| dx \leq \max \left\{ \int_{\Omega} (u_F(t, x) - K(x)) dx, \int_{\Omega} (K(x) - u_{f_{\tau}}(t, x)) dx \right\}. \quad (3.10)$$

Next, denote $w := u_F(t, x) - K(x)$, then $w \geq 0$ by (3.9) and using (1.6) we obtain that w satisfies

$$\frac{\partial w}{\partial t} = D\Delta\left(\frac{w}{K(x)}\right) - \frac{r}{K}u_F w, \quad (3.11)$$

where $u_F := u_F(t, x)$. Integrating both sides of (3.11) over Ω and using the Gauss' theorem and the Neumann boundary condition which implies $\int_{\Omega} D\Delta(\omega/K) dx = 0$, we obtain

$$\frac{d}{dt} \int_{\Omega} w dx = - \int_{\Omega} \frac{r}{K} u_F w dx \leq - \int_{\Omega} r w dx \leq -\rho \int_{\Omega} w dx,$$

where ρ was defined in (3.7). Therefore,

$$\int_{\Omega} w(t, x) dx \leq e^{-\rho t} \int_{\Omega} w(0, x) dx = e^{-\rho t} \int_{\Omega} (F - K(x)) dx,$$

which implies

$$\int_{\Omega} (u_F(t, x) - K(x)) dx \leq e^{-\rho t} \int_{\Omega} (F - K(x)) dx. \quad (3.12)$$

To estimate the second integral in (3.10) denote $\omega(t, x) := K(x) - u_{f_{\tau}}(t, x)$ and consider

$$\frac{\partial}{\partial t} \left(\frac{\omega K}{u_{f_{\tau}}} \right) = \frac{\partial}{\partial t} \left(\frac{K^2}{u_{f_{\tau}}} \right) = -\frac{K^2}{u_{f_{\tau}}^2} \frac{\partial u_{f_{\tau}}}{\partial t}.$$

Using equation (1.6) which $u_{f_{\tau}}$ satisfies and integrating over Ω , we obtain

$$\frac{d}{dt} \int_{\Omega} \frac{\omega K}{u_{f_{\tau}}} dx = -D \int_{\Omega} \frac{K^2}{u_{f_{\tau}}^2} \Delta \frac{u_{f_{\tau}}}{K} dx - \int_{\Omega} r \frac{\omega K}{u_{f_{\tau}}} dx.$$

According to the Gauss theorem and the Neumann boundary condition which $u_{f_{\tau}}/K$ satisfies, the first integral on the right becomes

$$-D \int_{\Omega} \frac{K^2}{u_{f_{\tau}}^2} \Delta \frac{u_{f_{\tau}}}{K} dx = D \int_{\Omega} \nabla \frac{K^2}{u_{f_{\tau}}^2} \cdot \nabla \frac{u_{f_{\tau}}}{K} dx = -D \int_{\Omega} 2 \frac{K^3}{u_{f_{\tau}}^3} \left| \nabla \frac{u_{f_{\tau}}}{K} \right|^2 dx \leq 0.$$

Thus we have

$$\frac{d}{dt} \int_{\Omega} \frac{\omega K}{u_{f_{\tau}}} dx \leq - \int_{\Omega} r \frac{\omega K}{u_{f_{\tau}}} dx \leq -\rho \int_{\Omega} \frac{\omega K}{u_{f_{\tau}}} dx.$$

Thus

$$\int_{\Omega} (K(x) - u_{f_{\tau}}(t, x)) dx = \int_{\Omega} \omega dx \leq \int_{\Omega} \frac{\omega K}{u_{f_{\tau}}} dx \leq e^{-\rho t} e^{\rho \tau} \int_{\Omega} (K(x) - f_{\tau}) \frac{K}{f_{\tau}} dx. \quad (3.13)$$

Finally, from (3.10), (3.12), (3.13) it follows

$$\int_{\Omega} |u(t, x) - K(x)| dx \leq C e^{-\rho t} \quad \text{for any } t \in [\tau, \infty),$$

with ρ defined in (3.7) and

$$C = \max \left\{ \int_{\Omega} (F - K(x)) dx, e^{\rho\tau} \cdot \int_{\Omega} (K(x) - f_{\tau}) \frac{K}{f_{\tau}} dx \right\}.$$

Let $t \rightarrow \infty$, then

$$\lim_{t \rightarrow \infty} \int_{\Omega} |u(t, x) - K(x)| dx = 0 \quad (3.14)$$

for any nonnegative initial function $u_0(x)$ (which also takes positive values in Ω). Equality (3.14) implies the uniqueness of the positive equilibrium solution. $\square$

Remark 1. It is the existence and the uniqueness of a *positive* equilibrium solution that Theorem 2 states, since there is also an equilibrium solution $p(x) \equiv 0$.

3.3 Periodic carrying capacity

In the case when the carrying capacity is a time-dependent function, generally, there is no more a positive equilibrium solution of (3.1), (1.7), (1.8). However, if both carrying capacity $K(t, x)$ and intrinsic growth $r(t, x)$ are periodic bounded functions with the same period T_0 , under certain conditions we can establish existence of an attracting positive periodic solution.

Once again we substitute $v(t, x) = u(t, x)/K(t, x)$. $K(t, x)$ is assumed to be strictly positive, bounded, Hölder continuous in x , periodic and at least $C^1(\mathbb{R}_+)$ in t . Then the problem (3.1), (1.7), (1.8) becomes

$$\frac{\partial v(t, x)}{\partial t} - \frac{D}{K} \Delta v(t, x) = r(t, x) v(t, x) (1 - v(t, x)) - a(t, x) v(t, x), \quad (t, x) \in Q, \quad (3.15)$$

$$\frac{\partial v}{\partial n} = 0 \quad \text{on } \partial Q \equiv (0, \infty) \times \partial \Omega, \quad (3.16)$$

$$v(0, x) = u_0(x)/K(0, x) \quad \text{in } \Omega, \quad (3.17)$$

where $a(t, x) = (1/K) \cdot (\partial K / \partial t)$ as in (3.3). We also denote

$$g_{min} = \inf \{g(t, x), (t, x) \in \overline{Q}\}, \quad g_{max} = \sup \{g(t, x), (t, x) \in \overline{Q}\}$$

for any bounded continuous periodic function $g(t, x)$.

Theorem 3. Suppose that $r_{\min} - a_{\max} > 0$ and

$$2r_{\min}(r_{\min} - a_{\max}) - r_{\max} \left(r_{\max} - \frac{a_{\min}}{2} \right) > 0, \quad (3.18)$$

then there exists a unique periodic solution $v^*(t, x)$ of (3.15), (3.16) in the interval

$$\left\langle \frac{r_{\min} - a_{\max}}{r_{\max}}, \frac{r_{\max} - a_{\min}}{r_{\min}} \right\rangle.$$

Moreover, for any initial function $v_0(x)$ from this interval the corresponding solution of (3.15)-(3.17) satisfies

$$v(t, x) \rightarrow v^*(t, x) \text{ as } t \rightarrow \infty \text{ for } x \in \bar{\Omega}.$$

Proof. Denote

$$\hat{v} = \frac{r_{\min} - a_{\max}}{r_{\max}}, \quad \tilde{v} = \frac{r_{\max} - a_{\min}}{r_{\min}}.$$

It is easy to check that

$$\frac{\partial \hat{v}}{\partial t} - \frac{D}{K} \Delta \hat{v} = 0 = \hat{v}(r_{\min} - a_{\max} - r_{\max} \hat{v}) \leq \hat{v}(r(t, x) - a(t, x) - r(t, x) \hat{v}),$$

$$\frac{\partial \tilde{v}}{\partial t} - \frac{D}{K} \Delta \tilde{v} = 0 = \tilde{v}(r_{\max} - a_{\min} - r_{\min} \tilde{v}) \geq \tilde{v}(r(t, x) - a(t, x) - r(t, x) \tilde{v}),$$

these inequalities imply that $\hat{v}$ and $\tilde{v}$ are the constant lower and upper solutions of (3.15)-(3.17) with the initial functions $v(0, x) = \hat{v}$ and $\tilde{v}$, respectively. Then according to Lemma 4 there exists a pair of T_0 -periodic solutions $\underline{v}$ and $\bar{v}$ satisfying

$$\hat{v} \leq \underline{v}(t, x) \leq \bar{v}(t, x) \leq \tilde{v}, \quad (t, x) \in Q,$$

$$\frac{\partial \underline{v}}{\partial t} - \frac{D}{K} \Delta \underline{v} = \underline{v}(r(t, x) - a(t, x) - r(t, x) \underline{v}), \quad (t, x) \in Q, \quad (3.19)$$

$$\frac{\partial \bar{v}}{\partial t} - \frac{D}{K} \Delta \bar{v} = \bar{v}(r(t, x) - a(t, x) - r(t, x) \bar{v}), \quad (t, x) \in Q, \quad (3.20)$$

$$\frac{\partial \underline{v}}{\partial n} = \frac{\partial \bar{v}}{\partial n} = 0, \quad (t, x) \in \partial Q.$$

Now we will show that $\underline{v} \equiv \bar{v}$ if condition (3.18) holds. Denote $w := (\bar{v} - \underline{v})\sqrt{K}$, we subtract (3.19) from (3.20) and multiply the result by $w\sqrt{K}$:

$$w \frac{\partial}{\partial t} w = D \frac{w}{\sqrt{K}} \Delta \frac{w}{\sqrt{K}} + w^2 \left(r - \frac{a}{2} \right) - w^2 r (\bar{v} + \underline{v}).$$

Due to the periodicity of K , $\bar{v}$ and $\underline{v}$, the function w is also T_0 -periodic, therefore

$$0 = \int_0^{T_0} \int_{\Omega} w \frac{\partial}{\partial t} w \, dx \, dt = - \int_0^{T_0} \int_{\Omega} D \left| \nabla \frac{w}{\sqrt{K}} \right|^2 \, dx \, dt + \int_0^{T_0} \int_{\Omega} w^2 \left[\left(r - \frac{a}{2} \right) - r(\bar{v} + \underline{v}) \right] \, dx \, dt$$

$$\leq - \int_0^{T_0} \int_{\Omega} D \left| \nabla \frac{w}{\sqrt{K}} \right|^2 dx dt - \left[\frac{2r_{\min}(r_{\min} - a_{\max})}{r_{\max}} - (r_{\max} - \frac{a_{\min}}{2}) \right] \int_0^{T_0} \int_{\Omega} w^2 dx dt \leq 0.$$

In the first inequality of the previous line we used $\bar{v} + \underline{v} \geq 2\hat{v}$; the last inequality follows from the condition (3.18). Therefore $\bar{v} - \underline{v} = 0$ and we obtain a unique periodic solution $v^* \equiv \bar{v} = \underline{v}$ of (3.15)-(3.17) in the interval $\langle \hat{v}, \tilde{v} \rangle$. Moreover, according to Lemma 4 for any initial function $v_0(x) \in \langle \hat{v}, \tilde{v} \rangle$ the corresponding solution of (3.15)-(3.17) satisfies $v(t, x) \rightarrow v^*(t, x)$ as $t \rightarrow \infty$ for $x \in \bar{\Omega}$. $\square$

Remark 2. The proof of Theorem 3 is similar to the proof of Theorem 2.1 in [2].

Corollary 1. *Let the hypotheses of the Theorem 3 hold. Then there exists a unique periodic solution $u^*(t, x)$ of (3.1), (1.7) in the interval*

$$\left\langle \frac{r_{\min} - a_{\max}}{r_{\max}} K_{\min}, \frac{r_{\max} - a_{\min}}{r_{\min}} K_{\max} \right\rangle.$$

If $u(t, x)$ is a solution of (3.1), (1.7), (1.8) with an initial function $u_0(x)$ from the interval

$$\left\langle \frac{r_{\min} - a_{\max}}{r_{\max}} K(0, x), \frac{r_{\max} - a_{\min}}{r_{\min}} K(0, x) \right\rangle,$$

then it satisfies the following property

$$u(t, x) \rightarrow u^*(t, x) \text{ as } t \rightarrow \infty \text{ for } x \in \bar{\Omega}.$$

4 Discussion and Open Problems

First, we summarize the results of the paper and discuss some of conditions, restrictions and possible generalizations.

1. The model with a carrying capacity driven diffusion was considered in this paper, and some basic results were obtained. In particular, it was demonstrated that a solution of the Neumann problem exists and is unique, it is positive for a nonnegative initial function which is positive in a nonempty open domain. For the model with time-independent carrying capacity all such solutions of (1.6) converge to the positive equilibrium (which coincides with the carrying capacity). Moreover, if the growth rate $r(x)$ is continuous and does not vanish on $\bar{\Omega}$, then a certain convergence rate can be guaranteed.
2. However, if parameters of (3.1) are positive periodic in t functions rather than t -independent parameters, then certain restrictions on $r(t, x)$ and $K(t, x)$ were imposed in Theorem 3. The condition that the infimum of the intrinsic growth rate exceeds the supremum of the relative change of the carrying capacity in time $\frac{1}{K} \frac{\partial K}{\partial t}$ means

that the growth rate can compensate for the change in the carrying capacity, this restriction frequently occurs in the problems of optimal harvesting. Though used in the proof of Theorem 3, the restrictions on $r(t, x)$ and $K(t, x)$ most probably are not sharp for pointwise convergence.

3. We have demonstrated stability in L^1 sense for the time-independent steady state and the pointwise convergence to the periodic positive solution (once the carrying capacity is time-dependent). Typically, these are two most common stability types studied for spatial models of population dynamics [1, 7]. However, stronger stability types can be considered, for example, the uniform (C^0) convergence. Since convergence and estimates are usually constructed inside the domain, establishing conditions of C^0 convergence may be a challenging problem.
4. Optimal harvesting for (1.5) and for a similar model with a Gilpin-Ayala growth law was considered in [1, 2]. Using different argument, the optimal harvesting policy for (3.1) was justified in [5]. Applying the methods of [1] we can obtain the same result as in [5] for optimal harvesting.

Finally, let us formulate some open problems.

1. Consider the model with a carrying capacity driven diffusion and other boundary conditions of type (1.3). As a special case, we can study the Dirichlet boundary conditions

$$u(t, x) = K(x), \quad x \in \partial\Omega. \quad (4.1)$$

Then $K(x)$ is a steady state and its asymptotic stability can be investigated. The Dirichlet boundary conditions (4.1) can also be introduced for the variable carrying capacity, once $K(t, x)$ is constant at the boundary. For arbitrary boundary conditions and $K(t, x)$ periodic in t establish the existence of a positive periodic solution. Does it attract all positive solutions?

2. For the logistic equation (without diffusion) with a variable carrying capacity the average of the periodic solution does not exceed the average periodic carrying capacity (and is less, if the carrying capacity is essentially non-constant over time) [4], it is also possible to estimate how the variability of $K(t, x)$ over time period T_0 influences the solution average [4]. Prove or disprove that the average of a periodic solution u of (3.1) is less than the average in $K(t, x)$ (unless $K(t, x)$ is time-independent), where the average is computed over the domain Ω and time period T_0 .
3. Biologically, one of the main open problems is how the value of the diffusion coefficient D in (3.1) influences the average population value for a positive periodic solution. For models with a variable carrying capacity and diffusion Δu the usual conclusion was that, generally, high diffusion does not favour higher average population values (chaotic movement can lead to massive migration to the areas with low carrying capacity). Does the situation change with carrying capacity driven diffusion?

4. The estimate of the convergence rate presented in Theorem 2 is based on the fact that the intrinsic growth rate $r(x)$ is a positive function separated from zero. If we assume that $r(x)$ is nonnegative (and is positive on an open nonempty subset of Ω), can we deduce any specific rate of convergence (probably in average) to the equilibrium solution?
5. We believe that if $r(t, x)$ is a time-dependent function satisfying $r(t, x) \geq \rho$ for any $x \in \Omega$, $t \geq 0$, then changing the statement and the proof of Theorem 2 is an easy exercise. However, it is not trivial to obtain sharp conditions on $r(t, x)$ (it can tend to zero as $t \rightarrow \infty$) leading to convergence.
6. In addition to (3.1) models with different growth laws:

$$\frac{\partial u(t, x)}{\partial t} = D\Delta \left(\frac{u(t, x)}{K(t, x)} \right) + r(t, x)f(u(t, x), K(t, x)), \quad (4.2)$$

where $f(u, K)(K - u) > 0$ for $u \neq K$, can be considered. They can describe populations with the logistic, Gilpin-Ayala or Gompertz growth laws.

7. Some systems are subject to short-time harvesting when the duration of the harvesting event is negligible compared to the whole growth process. Such models are adequately described by the impulsive equation, when, for example, (1.6) or (3.1) is complemented by the linear impulsive conditions

$$u(nT^+, x) = \alpha(x)u(nT, x), \quad \text{where } 0 < \alpha(x) < 1. \quad (4.3)$$

Define the optimal harvesting strategy for equation (1.6),(4.3), i.e., find the optimal $\alpha(x)$ and the initial function u_0 . Consider the same problem for (3.1).

8. If the carrying-capacity dependent diffusion is introduced in the Hutchinson (delay logistic) equation

$$\frac{\partial u(t, x)}{\partial t} = D\Delta \left(\frac{u(t, x)}{K(x)} \right) + r(x)u(t, x) \left(1 - \frac{u(t - \tau, x)}{K(x)} \right), \quad (4.4)$$

establish positivity of solutions and obtain conditions for the global attractivity of $K(x)$.

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