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Formulation and Well-Posedness of the Cauchy Problem for a Diffusion Equation with Discontinuous Degenerating Coefficients

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Abstract—The choice of a differential diffusion operator with discontinuous coefficients that corresponds to a finite flow velocity and a finite concentration is substantiated. For the equation with a uniformly elliptic operator and a nonzero diffusion coefficient, conditions are established for the existence and uniqueness of a solution to the corresponding Cauchy problem. For the diffusion equation with degeneration on a half-line, it is proved that the Cauchy problem with an arbitrary initial condition has a unique solution if and only if there is no flux from the degeneration domain to the ellipticity domain of the operator. Under this condition, a sequence of solutions to regularized problems is proved to converge uniformly to the solution of the degenerate problem in $L_1(R)$ on each interval.

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1. INTRODUCTION

In this paper, we analyze the well-posedness of the Cauchy problem for a one-dimensional diffusion equation with discontinuous and degenerating coefficients. Additionally, we study passage to the limit as the diffusion coefficient tends to zero on a subset of the coordinate space. The interest in these problems is motivated by the study of transfer phenomena in nonuniform media, for example, when the dynamics of the charge carrier density in semiconductors are described (see [1, 2]). We investigate the problem of ordering the operators of differentiation and multiplication by a discontinuous function, which is also addressed in the quantum dynamics of charge carriers in semiconductors (see [3]).

One of the goals of this work is to establish the relation between the form of the differential operator of the problem and the Markov diffusion process corresponding to a diffusion phenomenon. This relation was addressed in [4–6]. We study this relation for operators whose domain contains discontinuous functions and for semigroups other than Feller.

The differential diffusion operator under study is associated with a Markov process. Its choice is dictated by the following requirements: (i) the Markov process is least deviating from a standard Wiener process in a homogeneous medium; (ii) the local properties of the process in the homogeneity domain of the medium are independent of the medium's properties in the complement of the homogeneity domain; and (iii) the process is regular; i.e., the flow velocities of diffusing particles and their concentration are finite. The requirement that the flow velocity of diffusing particles be finite determines conditions on the limiting values of the particle distribution function on the boundary of the homogeneity domains, while the finiteness of particle distribution function values determines the relation between the boundary values of the derivatives of the distribution function. It is these conditions on the boundary values of the functions that specify the domain of the diffusion equation operator.

Obtained according to the above requirements, the operator of the diffusion equation with discontinuous coefficients is associated with a Markov process whose transition probabilities are derived from those of a standard Wiener process with the help of a continuous piecewise differentiable transformation of the coordinate space.

Boundary value problems with nonsmooth coefficients for a second-order parabolic differential equation are a classical object of study. The theory of such problems was addressed in [7]. At present, a theory has been constructed of uniformly elliptic Schrödinger operators in which the singular potentials and magnetic field are measures (see [8, 9]). On the other hand, a theory of boundary value problems for degenerating differential operators with smooth coefficients was developed, and results were obtained on the convergence of a sequence of solutions to problems with a small parameter multiplying the highest derivative as this parameter tends uniformly to zero (see [10]). The problem studied in this paper occupies an intermediate position and combines the nonuniform degeneration of the differential operator with the lack of smoothness of its coefficients. These properties of the operator lead to certain difficulties in the well-posedness analysis of such problems (see [11, 12]).

In this paper, we study the asymptotics of a sequence of solutions to the diffusion equation with a small (in the degeneration domain) parameter multiplying the highest derivative and prove that this sequence converges to the solution of the problem with a degenerate operator if the latter exists. Thus, the solution to the Cauchy problem for the diffusion equation with degenerate coefficients is shown to be stable in $L_1(\mathbb{R})$ with respect to small perturbations in the coefficients of the diffusion operator.

2. STATEMENT OF THE PROBLEM

2.1. Cauchy Problem

We study diffusion in the one-dimensional space $\mathbb{R}^1$ under the assumption that the diffusion and drift coefficients are piecewise constant functions on $\mathbb{R}^1$ with a discontinuity at the point $x = 0$:

$$D(x) = \begin{cases} 0, & x \in G_1 \triangleq \mathbb{R}_-, \\ D, & x \in G_2 \triangleq \mathbb{R}_+, \end{cases}$$

$$a(x) = \begin{cases} a, & x \in G_1, \\ 0, & x \in G_2. \end{cases}$$

If the diffusion and drift coefficients are sufficiently smooth functions (satisfy the Lipschitz condition [6]), the dynamics of the substance concentration $U(t, x)$ is determined by the following Cauchy problem for $U(t, x)$:

$$\begin{gathered} \frac{\partial U}{\partial t} = \mathbf{L}U, \quad t > 0, \quad x \in \mathbb{R}, \\ U(t, x)|_{t=0} = U_0(x), \quad x \in \mathbb{R}. \end{gathered} \tag{1}$$

The initial concentration $U_0(x)$ is a nonnegative function from $L_1(\mathbb{R})$. The generating operator $\mathbf{L}$ of the Cauchy problem is a second-order linear differential operator in the Banach space $L_1(\mathbb{R})$ with coefficients determined by the diffusion and drift coefficients.

In the case of discontinuous coefficients, we assume that the dynamics of $U(t, x)$ is also determined by a Cauchy problem of form (1). To complete the formulation of the Cauchy problem, we have to define the operator $\mathbf{L}$. The restrictions of $U_0(x)$ to the domains G_1 and G_2 are denoted by $U_{01}(x)$ and $U_{02}(x)$, respectively. Let Ω_1 and Ω_2 denote the ranges $\mathbb{R}_+ \times G_1$ and $\mathbb{R}_+ \times G_2$ of variables (t, x).

Following the results of classical works on diffusion processes (see [6, 4, 5]), we make the following assumption about the dynamics of the concentration of the diffusing substance.

Assumption 1. In the smoothness domains Ω_1 and Ω_2 of the diffusion and drift coefficients, $U(t, x)$ satisfies the standard diffusion equation with constant diffusion and drift coefficients

$$\frac{\partial U_i}{\partial t} = D_i \frac{\partial^2 U_i}{\partial x^2} + a_i \frac{\partial U_i}{\partial x}, \quad i = \overline{1, 2}, \quad (t, x) \in \Omega_i,$$

where $U_i = U|_{\Omega_i}$.

Then the action of the operator $\mathbf{L}$ on the concentration function is defined as

$$(\mathbf{L}U)|_{G_1} = -a \frac{\partial U_1}{\partial x},$$

$$(\mathbf{L}U)|_{G_2} = D\frac{\partial^2 U_2}{\partial x^2},$$

and its domain consists of functions from $W_1^2(\mathbb{R}_\pm)$ satisfying on the boundary $x = 0$ conditions ensuring that the flow velocity across the boundary is finite and the number of particles is preserved. A detailed derivation of these conditions is given in the next section.

Along with degenerate problem (1), we consider the regularized Cauchy problem

$$\begin{aligned}\frac{\partial U_\varepsilon}{\partial t} &= \mathbf{L}_\varepsilon U_\varepsilon, \quad t > 0, \quad x \in \mathbb{R},\\ U_\varepsilon(t, x)|_{t=0} &= U_0(x), \quad x \in \mathbb{R},\end{aligned} \tag{2}$$

with the same initial conditions and the operator $\mathbf{L}_\varepsilon$

$$(\mathbf{L}_\varepsilon U_\varepsilon)|_{G_1} = \varepsilon\frac{\partial^2 U_{\varepsilon 1}}{\partial x^2} - a\frac{\partial U_{\varepsilon 1}}{\partial x},$$

$$(\mathbf{L}_\varepsilon U_\varepsilon)|_{G_1} = D\frac{\partial^2 U_{\varepsilon 2}}{\partial x^2}.$$

The domain of $\mathbf{L}_\varepsilon$ consists of functions from $W_1^2(\mathbb{R}_-) \oplus W_1^2(\mathbb{R}_+)$ whose boundary values ensure that the flow velocity is finite and the number of particles is preserved.

Below, we analyze the existence of a solution to problem (1) and the convergence of a sequence of regularized-problem solutions to the solution of the degenerate problem.

2.2. Boundary Conditions

To complete the formulation of Cauchy problem (1), (2), we consider the choice of boundary conditions for functions from the domains of $\mathbf{L}$ and $\mathbf{L}_\varepsilon$.

As was noted above, these conditions are determined by the following conditions:

(i) The flow velocity of particles across the boundary $x = 0$ is finite.

(ii) The number of particles is a constant at any time $t \geq 0$.

In [13] the problem was considered in detail of choosing an operator of the equation

$$\mathbf{L}U = \begin{cases} D_1\dfrac{\partial^2 U_1}{\partial x^2}, & x < 0,\\[2ex] D_2\dfrac{\partial^2 U_2}{\partial x^2}, & x > 0,\end{cases}$$

with the domain consisting of functions ensuring conditions (i) and (ii). We study the problem of specifying boundary conditions at $x = 0$ and choosing an operator for the more general case of the equation with a drift:

$$\begin{aligned}\frac{\partial U_1}{\partial t} &= D_1\frac{\partial^2 U_1}{\partial x^2} - a_1\frac{\partial U_1}{\partial x}, \quad x \in G_1,\\ \frac{\partial U_2}{\partial t} &= D_2\frac{\partial^2 U_2}{\partial x^2} - a_2\frac{\partial U_2}{\partial x}, \quad x \in G_2.\end{aligned} \tag{3}$$

Then boundary conditions and the operators of Eqs. (1) and (2) are obtained by setting $D_1 = a_2 = 0$ and $D_1 = \varepsilon$, $a_2 = 0$, respectively.

To formulate conditions (i) and (ii) mathematically, the diffusion process is treated as a Markov process and the following assumptions are made.

Assumption 2. The diffusion of particles with discrete time $t \in \{n\tau, n \in Z_+\}$, where $\tau > 0$ is the time step, is a stationary Markov process with the transition probability $P(x, \tau, B)$ from a point $x \in \mathbb{R}$ to a set $B \subset \mathbb{R}$. For the transition from a point $x \in G_i$ to a set $B \in G_i$ in the time τ, the transition probability is given by the formula

$$\mathbf{P}(x, \tau, B) = \int_B \frac{1}{2\sqrt{\pi D_i \tau}} \exp\left(-\frac{(x - y - a_i\tau)^2}{4D_i\tau}\right) dy.$$

Assumption 3. The flow velocity of particles from G_1 to G_2 at time t is given by the formula

$$J_2(t) = \lim_{\tau \to 0}\left\{\frac{1}{\tau}[N_{12}(t, \tau) - N_{21}(t, \tau)]\right\},$$

where N_{12} and N_{21} are the numbers of particles passing from G_1 to G_2 and from G_2 to G_1, respectively, in the time τ starting at t.

Assumption 4. The diffusion of particles on a straight line with the diffusion coefficient $D(x)$ occurs so that the flow velocity of particles is finite.

Assumption 5. The number $N(t)$ of particles is preserved:

$$\frac{dN(t)}{dt} = 0. \tag{4}$$

Using these assumptions, we prove the following theorem, which gives boundary conditions on the probability function and its derivative at $x = 0$.

Theorem 1. *Suppose that the diffusion of particles with coefficient* $D(x)$ *satisfies Assumptions* 2 *and* 3 *and the function* $U(t, x)$ *satisfies the condition* $U_i(t, x) \in C(\mathbb{R}_+, W_1^2(G_i)) \cap C^1(\mathbb{R}_+, L_1(G_i))$, $i = 1, 2$. *Then the particle flow velocity across the boundary* $x = 0$ *at time t is finite if and only if the particle concentration at t satisfies*

$$U_1(t, 0)\sqrt{D_1} = U_2(t, 0)\sqrt{D_2}. \tag{5}$$

Condition (4) (*the conservation of the total number of particles*)

$$\frac{d}{dt}\int_{\mathbb{R}} U(t, x)dx = 0$$

is equivalent to the condition

$$D_1\frac{\partial U_1}{\partial x}(t, 0) - a_1U_1(t, 0) = D_2\frac{\partial U_2}{\partial x}(t, 0) - a_2U_2(t, 0). \tag{6}$$

The theorem is proved using the following result.

Lemma 1. *Let* $f(x) \in W_2^2(\mathbb{R}_\pm)$. *Then* $\lim_{x \to \pm\infty} f(x) = 0$.

Proof of the theorem. The number of particles passing from G_1 to G_2 over the time τ starting at t can be estimated as follows. According to Assumption 2, each particle involved in the diffusion process passes from a point $x \in \mathbb{R}$ to a point $y \in \mathbb{R}$ over the time τ starting at the time t. If the particle is at a point $x \in G_1$ at the time t, then the probability of the particle remaining in G_1 after the time τ is given by

$$\mathbf{P}(\tau, x, G_1) = \int_{-\infty}^{0} \frac{1}{2\sqrt{\pi D_1\tau}} \exp\left(-\frac{(x - y - a_1\tau)^2}{4D_1\tau}\right) dy.$$

Then the probability of the particle passing to G_2 is

$$\mathbf{P}(\tau, x, G_2) = 1 - \mathbf{P}(\tau, x, G_1).$$

Therefore, the number $N_{12}(t, \tau)$ of particles passing from G_1 to G_2 over the time τ starting at t is independent of the particle concentration in G_2 and is determined only by the concentration $U_1(t, x)$ in G_1. Hence, the number of particles passing from G_1 to G_2 over the time τ is given by the formula

$$N_{12}(t, \tau) = \int_{-\infty}^{0} U_1(t, x)\left[1 - \frac{1}{2\sqrt{\pi D_1 \tau}}\int_{-\infty}^{0} \exp\left(\frac{-(x-y-a_1\tau)^2}{4D_1\tau}\right)dy\right]dx.$$

Using the relation

$$\int_{-\infty}^{+\infty}\left[\frac{1}{2\sqrt{\pi D_1\tau}}\exp\left(\frac{-(x-y-a_1\tau)^2}{4D_1\tau}\right)\right]dy = 1,$$

we obtain

$$N_{12}(t, \tau) = \int_{-\infty}^{0}\left[\frac{1}{2\sqrt{\pi D_1\tau}}\int_{0}^{+\infty} U_1(t, x)\exp\left(\frac{-(x-y-a_1\tau)^2}{4D_1\tau}\right)dy\right]dx. \tag{7}$$

Applying similar reasoning to the calculation of particles passing from G_2 to G_1 yields

$$N_{21}(t, \tau) = \int_{0}^{+\infty}\left[\frac{1}{2\sqrt{\pi D_2\tau}}\int_{-\infty}^{0} U_2(t, x)\exp\left(\frac{-(x-y-a_2\tau)^2}{4D_2\tau}\right)dy\right]dx. \tag{8}$$

To analyze how integrals (7) and (8) depend on time, we make the transformation

$$\begin{aligned} N_{12}(t, \tau) &= \frac{1}{2\sqrt{\pi D_1\tau}}\int_{-\infty}^{0}\int_{0}^{\infty} U_1(t, x)\exp\left(\frac{-(x-y-a_1\tau)^2}{4D_1\tau}\right)dydx \\ &= \frac{\sqrt{D_1\tau}}{2\sqrt{\pi}}\int_{-\infty}^{0}\int_{0}^{\infty} U_1(t, \xi\sqrt{D_1\tau})\exp\left(\frac{-(\xi-\eta-a_1\sqrt{\tau/D_1})^2}{4}\right)d\eta d\xi, \end{aligned} \tag{9}$$

where $\xi = \dfrac{x}{\sqrt{D_1\tau}}$ and $\eta = \dfrac{y}{\sqrt{D_1\tau}}$. The limit $\lim\limits_{\tau\to 0}\dfrac{N_{12}(\tau, t)}{\sqrt{\tau}}$ is calculated as follows. If the integrand is continuous on Ω_1 and integral (9) converges uniformly, then $N_{12}(\tau, t)$ is a continuous function of τ and we can pass to the limit under the integral sign.

The concentration function $U_1(t, x)$ $(x \in \mathbb{R})$ is continuous, absolutely integrable, and bounded for every $t \geq 0$, while the function

$$g(\xi, \tau) = \int_{0}^{\infty}\exp\left(\frac{-\left(\xi - \eta - a_1\sqrt{\dfrac{\tau}{D_1}}\right)^2}{4}\right)d\eta$$

is continuous and bounded and, for all sufficiently small τ, is uniformly bounded from above by the function

$$G(\xi) = \int_{-|a_1|}^{\infty}\exp\left(\frac{-(\xi-\eta)^2}{4}\right)d\eta.$$

Therefore, the integral in (9) converges uniformly according to the Weierstrass test. Therefore, $N_{12}(t, \tau)/\sqrt{\tau}$ is a continuous function of τ and we can pass to the limit

$$\lim_{\tau\to 0}\frac{N_{12}(\tau, t)}{\sqrt{\tau}} = A\sqrt{D_1}U_1(t, 0),$$

where

$$A = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{0}\int_{0}^{\infty}\exp\left(\frac{-(\xi-\eta)^2}{4}\right)d\eta d\xi = \frac{1}{\sqrt{\pi}}.$$

Thus, $N_{12}(\tau, t) = A\sqrt{D_1}U_1(t, 0)\sqrt{\tau} + o(\sqrt{\tau})$. Now let us find the limit

$$\lim_{\tau\to 0}\frac{N_{12}(\tau, t) - A\sqrt{D_1}U_1(t, 0)\sqrt{\tau}}{\tau}.$$

For this purpose, we make the substitution $p = \sqrt{\tau}$ and use the L'Hopital's rule:

$$\lim_{\tau\to +0}\frac{N_{12}(\tau, t) - A\sqrt{D_1}U_1(t, 0)\sqrt{\tau}}{\tau}$$

$$= \lim_{p\to +0}\left(p^{-1}\left[\frac{1}{2}\sqrt{\frac{D_1}{\pi}}\int_{-\infty}^{0}\int_{0}^{\infty}U_1(t, \xi\sqrt{D_1}p)\exp\left(\frac{-\left(\xi-\eta\frac{a_1 p}{\sqrt{D_1}}\right)^2}{4}\right)d\eta d\xi - A\sqrt{D_1}U_1(t, 0)\right]\right) \tag{10}$$

$$= \lim_{p\to +0}\left(\frac{1}{2}\sqrt{\frac{D_1}{\pi}}\frac{\partial}{\partial p}\int_{-\infty}^{0}\int_{0}^{\infty}U_1(t, \xi\sqrt{D_1}p)\exp\left(\frac{-(\xi-\eta)^2}{4}\right)d\eta d\xi\right).$$

To differentiate under the integral sign, the integral

$$\frac{1}{2}\sqrt{\frac{D_1}{\pi}}\int_{-\infty}^{0}\int_{0}^{\infty}U_1(t, \xi\sqrt{D_1}p)\exp\left(\frac{-\left(\xi-\eta-\frac{a_1 p}{\sqrt{D_1}}\right)^2}{4}\right)d\eta d\xi \tag{11}$$

has to converge $\forall p \ge 0$, while the integral

$$\frac{1}{2}\frac{D_1}{\sqrt{\pi}}\int_{-\infty}^{0}\int_{0}^{\infty}\left(U_1'(t, \xi\sqrt{D_1}p)\xi + U_1(t, \xi\sqrt{D_1}p)\left(\xi-\eta-\frac{a_1 p}{\sqrt{D_1}}\right)\frac{a_1}{2D_1}\right)\exp\left(\frac{-(\xi-\eta-a_1 p/\sqrt{D_1})^2}{4}\right)d\eta d\xi \tag{12}$$

of the differentiated integrand has to converge uniformly with respect to p on the interval $[0; +\infty)$. The convergence of integral (11) $\forall p \ge 0$ follows from the uniform convergence of integral (9), which was shown above. The uniform convergence of integral (12) can be shown in a similar manner.

Since

$$B = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{0}\int_{0}^{\infty}\xi\exp\left(\frac{-(\xi-\eta)^2}{4}\right)d\eta d\xi = -\frac{1}{2} < 0,$$

the desired limit (10) is equal to $-\frac{1}{2}D_1U_1'(t, 0) + \frac{1}{2}a_1U_1(t, 0)$. Consequently,

$$N_{12}(\tau, t) = \frac{1}{\sqrt{\pi}}\sqrt{D_1}U_1(t, 0)\sqrt{\tau} - \frac{1}{2}D_1U_1'(t, 0)\tau + \frac{1}{2}a_1U_1(t, 0)\tau + o(\tau).$$

Similarly, we obtain

$$N_{21}(\tau, t) = \frac{1}{\sqrt{\pi}}\sqrt{D_1}U_2(t, 0)\sqrt{\tau} + \frac{1}{2}D_2U_2'(t, 0)\tau - \frac{1}{2}a_2U_2(t, 0)\tau + o(\tau).$$

Then the desired velocity is given by

$$J_2 = \lim_{\tau \to 0} \frac{A(U_1(t,0)\sqrt{D_1} - U_2(t,0)\sqrt{D_2})}{\sqrt{\tau}} - \frac{1}{2}[D_1 U_1'(t,0) + D_2 U_2'(t,0) - a_1 U_1(t,0) + a_2 U_2(t,0)];$$

i.e., this velocity is finite at the time t if and only if $U_1(t,0)\sqrt{D_1} = U_2(t,0)\sqrt{D_2}$. This immediately yields an expression for the flow velocity under condition (9):

$$J_2 = -\frac{1}{2}[D_1 U_1'(t,0) + D_2 U_2'(t,0) - a_1 U_1(t,0) + a_2 U_2(t,0)].$$

To derive the second boundary condition, the derivative of the total number of particles is expressed as

$$\frac{dN(t)}{dt} = \frac{d}{dt}\int_{-\infty}^{+\infty} U(t,x)dx = \int_{-\infty}^{0} D_1 \frac{\partial^2 U_1}{\partial x^2}dx - \int_{-\infty}^{0} a_1 \frac{\partial U_1}{\partial x}dx + \int_{0}^{+\infty} D_2 \frac{\partial^2 U_2}{\partial x^2}dx - \int_{0}^{+\infty} a_2 \frac{\partial U_2}{\partial x}dx$$

$$= D_1 \frac{\partial U_1}{\partial x}(t,0) - a_1 U_1(t,0) - D_2 \frac{\partial U_2}{\partial x}(t,0) + a_2 U_2(t,0)$$

(by Lemma 1, the concentration function and its derivative vanish at infinity). Then

$$\frac{dN(t)}{dt} = 0 \Leftrightarrow D_1 \frac{\partial U_1}{\partial x}(t,0) - a_1 U_1(t,0) = D_2 \frac{\partial U_2}{\partial x}(t,0) - a_2 U_2(t,0) = 0;$$

i.e., boundary condition (6) is satisfied. The theorem is proved.

The linear manifold of functions

$$D(\mathbf{L}) = \{U(x) \in L_1(\mathbb{R}) : U(x)|_{\mathbb{R}_\pm} \in W_1^2(\mathbb{R}_\pm);\ \sqrt{D_1}U(-0) = \sqrt{D_2}U(+0),$$
$$D_1 U'(-0) - a_1 U(-0) = D_2 U'(+0) - a_2 U(+0)\}$$

forms the domain of the linear operator $\mathbf{L}$ defined in the Banach space $L_1(\mathbb{R})$ by the differential expression

$$\mathbf{L}U = \frac{\partial}{\partial x}\left(\sqrt{D(x)}\frac{\partial}{\partial x}(\sqrt{D(x)}U)\right) - \frac{\partial}{\partial x}(a(x)U),$$

where

$$D(x) = \begin{cases} D_1, & x < 0, \\ D_2, & x > 0 \end{cases} \qquad a(x) = \begin{cases} a_1, & x < 0, \\ a_2, & x > 0. \end{cases}$$

Indeed, $U(x) \in L_1(\mathbb{R})$; therefore, $a(x)U(x) \in L_1(\mathbb{R})$ and, hence, $\frac{\partial}{\partial x}(a(x)U) \in W_1^{-1}(\mathbb{R})$. By the condition $U(-0)\sqrt{D}(-0) = U(+0)\sqrt{D}(+0)$, we have $D(x)W(x) \in W_1^1(\mathbb{R})$. Therefore, $\frac{\partial}{\partial x}(\sqrt{D(x)}U) \in L_1(\mathbb{R})$ and $\sqrt{D(x)}\frac{\partial}{\partial x}(\sqrt{D(x)}U) \in L_1(\mathbb{R})$. Consequently, the operator $\mathbf{L}$ defined on $D(\mathbf{L})$ is a linear operator from $L_1(\mathbb{R})$ to $W_2^{-1}(\mathbb{R})$. However, by the condition

$$D_1 \frac{\partial U}{\partial x}(-0) - a_1 U(-0) = D_2 \frac{\partial U}{\partial x}(-0) - a_2 U(-0),$$

the generalized function (distribution) $\mathbf{L}U(x) \in W_2^{-1}$ is regular and is an element of $L_1(\mathbb{R})$. Thus, $\mathbf{L}$ is a linear operator in $L_1(\mathbb{R})$.

Conversely, the domain of the linear operator $\mathbf{L}$ in $L_1(\mathbb{R})$ defined by the above differential expression consists of functions from $L_1(\mathbb{R})$ whose restrictions to the half-lines have second distributional derivatives from $L_1(\mathbb{R})$. Therefore, functions from $D(\mathbf{L})$ and their derivatives have limiting values $U(\pm 0)$ and $U'(\pm 0)$. If $U(x)$ and its restrictions to the half-lines satisfy the above inclusions, but the first boundary condition

$\sqrt{D(-0)}U(-0) = \sqrt{D(+0)}U(+0)$ is violated, then there is a fundamental sequence of functions $h_n(x)$ in $W_1^1(\mathbb{R})$ such that the sequence of values of the integrals

$$\int_R \left(\sqrt{D(x)}\frac{\partial}{\partial x}(\sqrt{D(x)}U) + a(x)U(x)\right)\frac{\partial h_n}{\partial x}(x)dx$$

is defined and diverges. Therefore, the relation $\sqrt{D(-0)}U(-0) = \sqrt{D(+0)}U(+0)$ is necessary for the inclusion $U(x) \in D(\mathbf{L})$. In view of this, the second relation is the regularity condition for the generalized function $\mathbf{L}U(x) \in L_1(\mathbb{R})$.

Corollary 1. If diffusion with coefficients $a(x)$ and $D(x)$ satisfies Assumptions 1–5 and defines a semigroup $\mathbf{V}_t$ $(t > 0)$ of transformations of the space $L_1(\mathbb{R})$, then the generator of this semigroup is the operator $\mathbf{L}$.

Proof. According to Assumption 1, if $U(x)$ belongs to the domain of the generator of $\mathbf{V}_t$ in $L_1(\mathbb{R})$, then its restrictions to the half-lines $\mathbb{R}_\pm$ lie in $W_1^2(\mathbb{R}_\pm)$ and, hence, have limiting values $U(\pm 0)$ and $U'(\pm 0)$. By Theorem 1 and Assumptions 2–4, conditions (5) and (6) hold. Therefore, $U_0 \in D(\mathbf{L})$. Corollary 1 is proved.

Thus, the evolution of the concentration function $U(t, x) = \mathbf{V}_t U_0(x)$ in the diffusion defined by Assumptions 1–5 is described by the Cauchy problem

$$\frac{\partial U}{\partial t} = \mathbf{L}U, \quad t > 0, \quad x \in \mathbf{R},$$
$$U|_{t=0} = U_0(x) = \begin{cases} U_{10}(x), & x < 0, \\ U_{20}(x), & x > 0. \end{cases} \tag{13}$$

3. SOLUTION TO A REGULARIZED PROBLEM

In Section 2, we formulated the Cauchy problem for the diffusion equation with a drift in the case of discontinuous coefficients D and a. First, we study the family of regularized Cauchy problems (2) with the coefficients $D_1 = \varepsilon$, $D_2 = D$, $a_1 = a$, and $a_2 = 0$, where $\varepsilon \in (0, 1)$ is the regularization parameter:

$$\frac{\partial U}{\partial t} = \mathbf{L}_\varepsilon U = \frac{\partial}{\partial x}\sqrt{D_\varepsilon(x)}\frac{\partial}{\partial x}\sqrt{D_\varepsilon(x)}U - \frac{\partial}{\partial x}a(x)U, \quad t > 0, \quad x \in \mathbb{R}, \tag{14}$$

$$D(\mathbf{L}_\varepsilon) = \{U(x) \in L_1(\mathbf{R}) : U(x)|_{\mathbf{R}_\pm} \in W_1^2(\mathbf{R}_\pm);$$
$$\sqrt{\varepsilon}U(-0) = \sqrt{D}U(+0), \varepsilon U'(-0) - aU(-0) = DU'(+0)\}, \tag{15}$$

$$D_\varepsilon(x) = \begin{cases} \varepsilon, & x < 0, \\ D, & x > 0, \end{cases} \tag{16}$$
$$U(t, x)|_{t=0} = U_0(x).$$

The domain $D(\mathbf{L}_\varepsilon)$ is a linear manifold that is dense in the Banach space $H = L_1(\mathbb{R})$. Our goal is to show that the regularized problems are well-posed and to analyze the asymptotics of their solutions as the regularization parameter tends to zero.

Definition 1. The solution to problem (14)–(16) is a function $U(t, x)$ with uniformly continuous restrictions to Ω_i having continuous derivatives $\frac{\partial U_i}{\partial t}$ and $\frac{\partial^2 U_i}{\partial x^2}$ on $\Omega_i$and continuous limiting values я $\frac{\partial U}{\partial x}(t, \pm 0)$ $(t > 0)$ on the half-line $\mathbb{R}_+$ that satisfies Eq. (14) almost everywhere on $\Omega_1 \cup \Omega_2$, satisfies initial condition (16) in the sense that $\lim_{t \to 0}\|U(t, x) - U_0(x)\|_H = 0$, and satisfies boundary conditions (15) identically for $t > 0$.

Definition 2. A function $U(t, x) \in C([0, T], L_1(\mathbb{R})))$ is called a weak solution to problem (14), (16) if there exists a sequence of initial data $\{U_{0,n}(x)\}$ with $\lim_{n \to \infty}\|U_{0,n}(x) - U_0(x)\|_{L_1(\mathbb{R})} = 0$ such that, for every

$n \in \mathbb{N}$, problem (14) with the initial data $U_{0,n}(x)$ has a solution $U_n(t, x)$ and $\lim_{n\to\infty}\|U_n(t, x) - U(t, x)\|_{C([0, T], L_1(\mathbb{R}))} = 0$.

Definition 3. A function $U(t, x) \in C(\mathbb{R}_+, C^2(\mathbb{R}_\pm)) \cap C^1(\mathbb{R}_+, C(\mathbb{R}_\pm))$ is called a classical solution to problem (14), (16) if it satisfies Eq. (14) everywhere on $\mathbb{R}_+ \times \mathbb{R}$ and satisfies initial conditions (16) in the sense that $\lim_{t\to 0}\|U(t, x) - U_0(x)\|_{C(\mathbb{R})} = 0$.

Let C_{00}^{∞} denote the linear space of compactly supported infinitely differentiable functions with a support not containing zero.

Lemma 2. *If* $U_0(x) \in C_{00}^{\infty}$, *then problem* (2) *has a unique classical solution.*

Proof. The function U is transformed as follows:

$$U(t, x) = V(t, x)f(x), \quad f(x) = \begin{cases} \exp\left(\dfrac{a}{2\varepsilon}x\right), & x < 0, \\ 1, \quad x > 0. \end{cases} \tag{17}$$

Substituting $U(t, x) = V(t, x)f(x)$ into (14), we obtain for $V(t, x)$ the equation

$$\frac{\partial V}{\partial t} = \tilde{\mathbf{L}}V \tag{18}$$

with the operator

$$\tilde{\mathbf{L}}V = \frac{\partial}{\partial x}\sqrt{D_\varepsilon(x)}\frac{\partial}{\partial x}\sqrt{D_\varepsilon(x)}V + \frac{a(x)}{2\varepsilon}\sqrt{D_\varepsilon(x)}\frac{\partial}{\partial x}\sqrt{D_\varepsilon(x)}V - \frac{\partial}{\partial x}\frac{a(x)D_\varepsilon(x)}{2\varepsilon}V - \frac{a^2(x)}{4\varepsilon}V \tag{19}$$

and the initial conditions

$$V_0(x) = \frac{1}{f(x)}U_0(x).$$

Note that, according to this transformation, if the initial function $U_0(x)$ is in C_{00}^{∞}, then $V_0(x)$ is also in this space.

Let us first prove that problem (19) has a solution in the weighted space $L_{2,\rho}$ (the weight ρ is chosen so that the operator $\tilde{\mathbf{L}}$ is self-adjoint). In this space, the operator $\mathbf{L}$ is defined by differential expression (19) on the linear manifold

$$D(\tilde{\mathbf{L}}) = \left\{V(x) \in L_2(\mathbb{R}) : V(x)|_{\mathbb{R}_\pm} \in W_2^2(\mathbb{R}_\pm);\ \sqrt{\varepsilon}V(-0) = \sqrt{D}V(+0);\right.$$
$$\left.\varepsilon V'(-0) - \frac{a}{2}V(-0) = DV'(+0)\right\}. \tag{20}$$

By direct calculations, we can show that the operator $\tilde{\mathbf{L}}$ is densely defined and symmetric in the weighted space $L_{2,\sqrt{D_\varepsilon(x)}}$.

Let us show that $\tilde{\mathbf{L}}$ is self-adjoint in the space $L_{2,\sqrt{D_\varepsilon(x)}}$) with the inner product defined as

$$(f, g)_{L_{2,\rho}} = \int_{-\infty}^{+\infty} f(x)g(x)\rho(x)dx.$$

The operator $\tilde{\mathbf{L}}$ is densely defined in $L_{2,\sqrt{D_\varepsilon(x)}}$ and has an adjoint operator $\tilde{\mathbf{L}}^*$. By the definition of the adjoint operator, its domain $D(\tilde{\mathbf{L}}^*)$ consists of functions $V(x) \in L_{2,\sqrt{D_\varepsilon(x)}}$ such and only such that the following conditions are satisfied:

(i) The restrictions $V(x)|_{\mathbb{R}_\pm}$ have second distributional derivatives from the space $L_{2,\sqrt{D_\varepsilon(x)}}$ (consequently, $V(x)$ and $V'(x)$ have one-sided limiting values).

(ii) These limiting values satisfy the relations

$$\sqrt{\varepsilon}V(-0) = \sqrt{D}V(+0),$$

$$\varepsilon V'(-0) - \frac{a}{2}V(-0) = DV'(+0).$$

Moreover, any $V(x) \in D(\tilde{\mathbf{L}}^*)$ satisfies $V(x) \in D(\tilde{\mathbf{L}})$ and $\tilde{\mathbf{L}}^* V(x) = \tilde{\mathbf{L}}V(x)$. Therefore, $\tilde{\mathbf{L}}^* = \tilde{\mathbf{L}}$.

Since $\tilde{\mathbf{L}}$ is self-adjoint in $L_{2,\sqrt{D_\varepsilon(x)}}$, it is the generator of a semigroup that is representable, according to the spectral theorem, as

$$e^{t\tilde{\mathbf{L}}} = \int_{-\infty}^{\infty} e^{\lambda t} d\mathbf{E}_\lambda,$$

where $\tilde{\mathbf{L}} = \int_{-\infty}^{\infty} \lambda d\mathbf{E}_\lambda$ is the spectral expansion of $\tilde{\mathbf{L}}$.

Therefore, if $V_0(x) \in D(\tilde{\mathbf{L}})$, then problem (19) has a unique solution representable as

$$V(t, x) = e^{t\tilde{\mathbf{L}}}V_0(x) = \int_{-\infty}^{+\infty} e^{\lambda t} d\mathbf{E}_\lambda V_0(x). \tag{21}$$

Moreover, (21) is a weak solution if $V_0(x) \in \mathbf{L}_2$ and is a classical solution if $V_0(x) \in C_{00}^\infty$.

By virtue of transformation (17), problem (2) also has a classical solution $U(t, x) \in D(\tilde{\mathbf{L}})$. The lemma is proved.

To prove that $U(t, x)$ is in L_1, this function must be nonnegative, which can be shown with the help of some transformation.

Lemma 3. *If $U_0(x) \in C_{00}^\infty$, then the classical solution to problem* (2) *belongs to $L_1(\mathbb{R})$.*

Proof. The following one-to-one transformation depending on three real parameters α, β, $b \in \mathbb{R}$ is applied to the solution $V(t, x)$ of problem (19):

$$V(t, x) = \begin{cases} e^{bt}e^{-\alpha x^2 + \beta x}\sqrt{\dfrac{D}{\varepsilon}}W\left(t, x\sqrt{\dfrac{D}{\varepsilon}}\right), & x < 0, \\ e^{bt}W(t, x), & x > 0. \end{cases} \tag{22}$$

Note that, according to (22), $V(t, x) \in W_2^2(\mathbb{R}_\pm)$ if and only if $W(t, x) \in W_2^2(\mathbb{R}_\pm)$, and the boundary values $W(t, \pm 0)$ and $\dfrac{\partial W}{\partial x}(t, \pm 0)$ are determined by the condition $W(t, x) \in D(\hat{\mathbf{L}})$, where the domain of the operator $\hat{\mathbf{L}}$ is the linear manifold

$$D(\hat{\mathbf{L}}) = \left\{W(x) \in L_2(\mathbb{R}) : W(x)|_{\mathbb{R}_\pm} \in W_2^2(\mathbb{R}_\pm);\ W(t, -0) = W(t, +0);\ \frac{\partial W}{\partial x}(t, -0) = \frac{\partial W}{\partial x}(t, +0)\right\}. \tag{23}$$

Therefore, $V(t, x)$ solves Eq. (18) with the initial data $V_0(x) \in C_{00}^{\infty}$ if and only if the function $W(t, x)$ is a solution to the equation

$$\frac{\partial W}{\partial t} = \hat{\mathbf{L}} W = D\frac{\partial^2 W}{\partial x^2} + f(x)\frac{\partial W}{\partial x} + g(x)W,$$

$$g(x) = \begin{cases} -\dfrac{a^2}{4\varepsilon}\left(\beta - \dfrac{2\varepsilon^2}{a^2}\right)^2 + \dfrac{\varepsilon^3}{a^2} + \dfrac{\alpha a^2}{2\varepsilon} - b, & x < 0, \\ -b, & x > 0, \end{cases} \tag{24}$$

with the initial data $W_0(x) \in C_{00}^{\infty}$ obtained from V_0 by transformation (22). Here, the function $f(x)$ is real-valued, piecewise continuous, and bounded. As will be seen later, its particular representation is of no importance.

Thus, we have obtained an equation to which the generalized maximum principle [14] can be applied. For this purpose, the function $g(x)$ must be nonnegative. Setting

$$\alpha = \frac{2\varepsilon^4}{a^4}, \quad b = \frac{3\varepsilon^3}{a^2},$$

yields the required condition. Then, according to [14], $W(t, x)$ cannot have a negative minimum or a positive maximum inside the domain $(0, T) \times (-L, L)$. Taking into account that $W \longrightarrow 0$ at infinity according to Lemma 1 and that this function is initially nonnegative, we conclude that $W(t, x)$ cannot take negative values.

Note that transformations (17) and (22) preserve the nonnegativity of functions. Therefore, the classical solution $U(t, x)$ to problem (2) is also nonnegative.

Integrating both sides of the equation

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(\sqrt{D_\varepsilon(x)}\frac{\partial}{\partial x}(\sqrt{D_\varepsilon(x)}U)\right) - \frac{\partial}{\partial x}(a(x)U)$$

with respect to x from $-\infty$ to $+\infty$ and using the relation

$$\int \frac{\partial U}{\partial t}dx = \frac{d}{dt}\int U dx = \frac{d}{dt}\int |U| dx = \frac{d}{dt}\|U\|_{L_1},$$

which holds since $U \geq 0$, we obtain

$$\frac{d}{dt}\|U\|_{L_1} = \varepsilon\frac{\partial U}{\partial x}(t, -0) - D\frac{\partial U}{\partial x}(t, +0) - aU(t, -0) = 0.$$

Note that the limiting values $U(t, \pm 0)$ and $\dfrac{\partial U}{\partial x}(t, \pm 0)$ exist and are uniformly bounded on any interval of the half-line $u > 0$ since $U_0(x) \in C_{00}^{\infty}$ and, consequently, $W_0 \in C_{00}^{\infty}$ and, hence, $W(t, x) \in C([0, +\infty), D(\hat{\mathbf{L}}))$. Therefore, $U(t, x) \in D(\mathbf{L}_\varepsilon)$ and $\|U(t, x)\|_{L_1} = \|U_0(x)\|_{L_1} < \infty$. The lemma is proved.

This result is extended to the case of $U_0 \in L_1(\mathbb{R})$ as follows.

Theorem 2. *For any* $U_0(x) \in L_1(\mathbb{R})$, *problem* (25) *has a unique weak solution.*

Since the space C_{00}^{∞} is dense in L_1, we conclude that, for any function $U_0(x) \in L_1$, there exists a fundamental sequence $\{U_{0n}(x)\}$ from C_{00}^{∞} such that $\lim_{n \to \infty}\|U_{0n}(x) - U_0(x)\|_{L_1(R)} = 0$. According to Lemmas 2 and 3, for any $U_{0n}(x) \in C_{00}^{\infty}$, problem (25) has a unique solution $U_n(t, x)$. Let us show that the sequence $\{U_n(t, x)\}$ of solutions to the Cauchy problem with initial data $U_{0,n}(x)$ is fundamental.

If $U_n(t, x)$ and $U_m(t, x)$ are the solutions to problem (25) with initial data $U_{0,n}(x)$ and $U_{0,m}(x)$, respectively, then $U_n(t, x) - U_m(t, x)$ is the solution with the initial data $U_{0,n}(x) - U_{0,m}(x)$. Then, for any $t > 0$,

$$\|U_n(t, x) - U_m(t, x)\|_{L_1(R)} = \|U_{0,n}(x) - U_{0,m}(x)\|_{L_1(R)}$$

(the proof of this relation is similar to the end of the proof of Lemma 2). Therefore, $\{U_n(t, x)\}$ is a fundamental sequence and has a limit $U(t, x)$ in $L_1(\mathbb{R})$ that is independent of the approximating sequence of initial data. The theorem is proved.

Note that the solution $U(t, x)$ to the problem depends continuously on the initial data $U_0(x)$ in $L_1(\mathbb{R})$ by virtue of the estimate $\|U(t, x)\|_{L_1} = \|U_0(x)\|_{L_1}$.

Thus, the regularized problem is well-posed in the class L_1 of initial data. This implies (see [17]) that the solution to the regularized problem defines a semigroup of linear continuous operators

$$\mathbf{V}_t^{\varepsilon} : L_1 \longrightarrow L_1.$$

This fact is used later in the convergence analysis of a sequence of solutions to a regularized problem.

4. SOLUTION TO THE DEGENERATE PROBLEM

In Section 2.2, we found an operator of the diffusion equation with a drift such that the flow velocity across the boundary is finite and the total number of particles is preserved. By using this operator, problem (1) can be rewritten as

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(\sqrt{D(x)}\frac{\partial}{\partial x}(\sqrt{D(x)}U)\right) - \frac{\partial}{\partial x}(a(x)U), \quad t > 0, \quad x \in \mathbb{R},$$

where

$$D(x) = \begin{cases} 0, & x < 0, \\ D, & x > 0, \end{cases} \qquad a(x) = \begin{cases} a, & x < 0, \\ 0, & x > 0, \end{cases}$$

$$D(\mathbf{L}) = \{U(x) \in L_2(\mathbb{R}) \cap L_1(\mathbb{R}) : U(x)|_{\mathbb{R}_\pm} \in W_2^2(\mathbb{R}_\pm);\ U(+0) = 0, -aU(-0) = DU'(+0)\}, \tag{25}$$

$$U_{t=0} = U_0(t, x) = \begin{cases} U_{10}(x), & x < 0, \\ U_{20}(x), & x > 0. \end{cases}$$

Consider the problem of finding $U(t, x)$ in either of the quadrants $\mathbb{R}_+ \times \mathbb{R}_-$ and $\mathbb{R}_+ \times \mathbb{R}_+$ taking into account the found boundary conditions in the cases of $a \le 0$ and $a > 0$.

1. **Case** $a \le 0$:

$$\begin{aligned} \frac{\partial U_1}{\partial t} &= -a\frac{\partial U_1}{\partial x}, \\ U_1(0, x) &= U_{10}(x), \\ U_1(t, 0) &= -\frac{D}{a}\frac{\partial U_2}{\partial x}(t, 0); \end{aligned} \tag{26}$$

$$\begin{aligned} \frac{\partial U_2}{\partial t} &= D\frac{\partial^2 U_2}{\partial^2 x}, \\ U_2(0, x) &= U_{20}(x), \\ U_2(t, 0) &= 0. \end{aligned} \tag{27}$$

The first system is solved by the function

$$U_1(t, x) = \begin{cases} U_{10}(x - at), & x < at, \\ f(x - at), & at < x < 0, \end{cases} \tag{28}$$

where $f(x)$ is a function from the domain of $\mathbf{L}$ that satisfies the matching conditions

$$f(0) = U_{10}(0), \quad f'(0) = U'_{10}(0) \tag{29}$$

and the boundary condition

$$f(-at) = -\frac{D}{a}\frac{\partial U_2}{\partial x}(t, 0). \tag{30}$$

Following [18], the solution to system (27) is given by

$$U_2(t, x) = \frac{1}{2\sqrt{\pi D t}}\int_0^{+\infty} U_{20}(\xi)\left[\exp\left(-\frac{(\xi - x)^2}{4Dt}\right) - \exp\left(-\frac{(\xi + x)^2}{4Dt}\right)\right]d\xi. \tag{31}$$

Then condition (30) implies

$$f(-at) = -\frac{2\sqrt{D}}{\sqrt{\pi D t}}\int_0^{+\infty} U_{20}(\xi)\xi\exp\left(-\frac{\xi^2}{4Dt}\right)d\xi. \tag{32}$$

Therefore,

$$f(y) = 2\sqrt{\frac{D}{-a\pi}}\frac{1}{y^{1/2}}\int_0^{+\infty} U_{20}\left(2p\sqrt{\frac{Du}{-a}}\right)\exp(-p^2)p\,dp. \tag{33}$$

This expression implies that $f(0)$ is finite if $U_{20}(0) = 0$. Similarly, for $f'(0)$ to be finite, it suffices that

$$U_{20}(0) = U'_{20}(0) = U''_{20}(0) = 0.$$

Therefore, if $U(x) \in C^{\infty}_{00}$, then $f(0) = f'(0)$; then (29) implies that $U_{10}(0) = U'_{10}(0) = 0$.

Thus, we have shown that problem (25) with $a \le 0$ has a solution for initial data from the class C^{∞}_{00}. To prove that the problem is well-posed, we have to show that, if the sequence $\{U_{0n}\}$ with values in C^{∞}_{00} converges to zero in $L_1(R)$, then the corresponding sequence $\{U_n(t)\}$ of classical solutions satisfies

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|U_n(t) - U(t)\|_{L_1(\mathbb{R})}.$$

for every $T > 0$. According to (28) and (33), the restriction of the solution to the left half-line tends to zero as $U_{20} \longrightarrow 0$. The fact that the restriction to the right half-line tends to zero follows from (31). Therefore, we have proved the following result.

Lemma 4. *Let $a < 0$. Then, for any $U_0(x) \in C^{\infty}_{00}$, Cauchy problem* (25) *has a unique solution $U(t, x)$ satisfying the estimate $\|U(t, x)\|_{L_1(R)} \le \|U_0(x)\|_{L_1(R)}$ for any $t > 0$.*

Below is the generalization of this result to $U_0(x) \in L_1$.

Theorem 3. *For any $U_0(x) \in L_1(\mathbb{R})$, Cauchy problem* (25) *has a unique weak solution that depends continuously on the initial data.*

The proof of the theorem is entirely similar to that of Theorem 2. The continuous dependence on the initial data is obvious. The theorem is proved.

2. Consider the case of $a > 0$.

Theorem 4. *If $a > 0$, then Cauchy problem* (25) *has no nontrivial solutions.*

Proof. If $a > 0$, the boundary condition

$$U_1(t, 0) = -\frac{D}{a}\frac{\partial U_2}{\partial x}(t, 0) \tag{34}$$

implies that $\frac{\partial U_2}{\partial x}(t, 0) \le 0$. Since $U_2(t, 0) = 0$ and $U_2(t, x) \ge 0$ $\forall x > 0$, it is necessary that

$$\frac{\partial U_2}{\partial x}(t, 0) = 0.$$

In view of (34), we have

$$U_1(t, 0) = 0. \tag{35}$$

For $a > 0$, the solution to system (26) can be represented as

$$U_1(t, x) = U_{10}(x - at), \quad x < at.$$

Taking into account (35) yields $U_{10}(-at) = 0$, which implies that system (26) has no nontrivial solutions. The theorem is proved.

5. CONVERGENCE OF THE SEQUENCE OF SOLUTIONS TO REGULARIZED PROBLEMS

Now we analyze the convergence of solutions of regularized problems (2) to the solution of the degenerate problem as $\varepsilon \longrightarrow 0$ when the latter solution exists, i.e., for $a < 0$. For this purpose, we define the function

$$Z_\varepsilon = U + P_\varepsilon \in D(L_\varepsilon),$$

where U solves problem (1) and P_ε compensated for the difference between the boundary conditions for $\mathbf{L}$ and $\mathbf{L}_\varepsilon$. For Z_ε to belong to the domain of $\mathbf{L}_\varepsilon$, it is necessary that P_ε satisfy the boundary conditions

$$\begin{gathered} \sqrt{\varepsilon}[U(t,-0) + P_\varepsilon(t,-0)] = \sqrt{D}P_\varepsilon(t,+0), \\ \varepsilon\left(\frac{\partial}{\partial x}U(t,-0) + \frac{\partial}{\partial x}P_\varepsilon(t,-0)\right) = D\frac{\partial}{\partial x}P_\varepsilon(t,+0). \end{gathered} \tag{36}$$

We search for a function $P_\varepsilon(t, x)$ such that

$$P_\varepsilon(t,-0) = \frac{\partial}{\partial x}P_\varepsilon(t,-0) = 0.$$

In view of this relation, boundary conditions (36) are written as

$$\sqrt{\varepsilon}U(t,-0) = \sqrt{D}P_\varepsilon(t,+0),$$

$$\varepsilon\frac{\partial}{\partial x}U(t,-0) = D\frac{\partial}{\partial x}P_\varepsilon(t,+0).$$

This function can be defined, for example, as

$$P_\varepsilon(t,x) = \begin{cases} U(t,-0)\exp(-x^2)\sqrt{\dfrac{\varepsilon}{D}} + \dfrac{\varepsilon}{D}\dfrac{\partial}{\partial x}U(t,-0)x\exp(-x^2), & x > 0, \\ 0, \quad x < 0. \end{cases}$$

Define the functions

$$V_\varepsilon = U_\varepsilon - U, \quad W_\varepsilon = U_\varepsilon - Z_\varepsilon = V_\varepsilon - P_\varepsilon.$$

Then W_ε satisfies the equation

$$\frac{\partial W_\varepsilon}{\partial t} = \mathbf{L}_\varepsilon U_\varepsilon - \mathbf{L}U - \frac{\partial P_\varepsilon}{\partial t} = \mathbf{L}_\varepsilon(U_\varepsilon - Z_\varepsilon) + (\mathbf{L}_\varepsilon Z_\varepsilon - \mathbf{L}U) - \frac{\partial P_\varepsilon}{\partial t} = \mathbf{L}_\varepsilon W_\varepsilon + f_\varepsilon, \tag{37}$$

where

$$f_\varepsilon = (\mathbf{L}_\varepsilon Z_\varepsilon - \mathbf{L}U) - \frac{\partial P_\varepsilon}{\partial t}. \tag{38}$$

Let us prove that f_ε converges to zero in the L_1-norm uniformly with respect to t on any interval $[0, T]$.

Consider the function

$$\frac{\partial P_\varepsilon}{\partial t} = \begin{cases} \dfrac{\partial U(t,-0)}{\partial t}\exp(-x^2)\sqrt{\dfrac{\varepsilon}{D}} + \dfrac{\varepsilon}{D}\dfrac{\partial}{\partial t}\left(\dfrac{\partial U}{\partial x}(t,-0)\right)x\exp(-x^2), & x > 0, \\ 0, \quad x < 0. \end{cases}$$

Let us show that

$$\left\|\frac{\partial P_\varepsilon}{\partial t}\right\|_{C([0,T],L_1(\mathbb{R}))} = O(\sqrt{\varepsilon}). \tag{39}$$

Indeed, we have the estimate

$$\left\|\frac{\partial P_\varepsilon}{\partial t}\right\|_{C([0,T],L_1(\mathbb{R}))} \le \sqrt{\frac{\varepsilon}{D}}\left\|\int_0^{+\infty}\frac{\partial U(t,-0)}{\partial t}\exp(-x^2)dx\right\|_{C([0,T],L_1(\mathbb{R}))}$$
$$+\frac{\varepsilon}{D}\left\|\int_0^{+\infty}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}U(t,-0)\right)x\exp(-x^2)dx\right\|_{C([0,T],L_1(\mathbb{R}))}.$$

Using expressions (28) and (32) for $U(t, -0)$, differentiating, and taking into account $U_0(x) \in C_{00}^{\infty}$, we obtain (39).

Let us show that the restrictions of $\mathbf{L}_\varepsilon Z_\varepsilon - \mathbf{L}U$ to the half-line $\mathbb{R}_-$ $\mathbb{R}_+$ converge to zero in the L_1-norm. Note that $\mathbf{L}_\varepsilon Z_\varepsilon \in L_1$ and $\mathbf{L}U \in L_1$. Then

$$(\mathbf{L}_\varepsilon Z_\varepsilon - \mathbf{L}U)|_{\mathbb{R}_\pm} = (\mathbf{L}_\varepsilon U - \mathbf{L}_\varepsilon P_\varepsilon - \mathbf{L}U)|_{\mathbb{R}_\pm} = [(\mathbf{L}_\varepsilon - \mathbf{L})U + \mathbf{L}_\varepsilon P_\varepsilon]|_{\mathbb{R}_\pm},$$

$\mathbf{L}_\varepsilon P_\varepsilon|_{\mathbb{R}_-} = 0$,@ так как $P_\varepsilon = 0$ при $x < 0$,

$$\mathbf{L}_\varepsilon P_\varepsilon|_{\mathbb{R}_+} = D\frac{\partial^2 P_\varepsilon}{\partial^2 x},$$
$$(\mathbf{L}_\varepsilon - \mathbf{L})U|_{\mathbb{R}_-} = \varepsilon\frac{\partial^2 U}{\partial^2 x}, \tag{40}$$
$$(\mathbf{L}_\varepsilon - \mathbf{L})U|_{\mathbb{R}_+} = 0,$$

since $\mathbf{L}_\varepsilon U|_{\mathbb{R}_+} = \mathbf{L}U|_{\mathbb{R}_+}$.

The relation $\left\|\dfrac{\partial^2 P_\varepsilon}{\partial^2 x}\right\|_{C([0,T],L_1(\mathbb{R}))} = O(\sqrt{\varepsilon})$ is proved in the same manner as for $\left\|\dfrac{\partial P_\varepsilon}{\partial t}\right\|_{C([0,T],L_1(\mathbb{R}))}$. Moreover, for every $t > 0$, we have $\left\|\dfrac{\partial^2 U(t,x)|_{\Omega_i}}{\partial^2 x}\right\|_{L_1(G_i)} < \mathrm{const}(U_0)$ for any $U_0(x) \in C_{00}^{\infty}$.

Thus, it follows from (38)–(40) that $\|b_\varepsilon\|_{C([0,T],L_1(\mathbb{R}))} = O(\sqrt{\varepsilon})$.

Let us return to Eq. (37). In Section 3, it was shown that the Cauchy problem with the operator $\mathbf{L}_\varepsilon$ is well-posed for any initial data $U_0 \in L_1$. Therefore, the solution to Eq. (37) defines a semigroup of linear continuous operators

$$\mathbf{V}_t : L_1(\mathbb{R}) \longrightarrow L_1(\mathbb{R}).$$

Moreover, $\mathbf{L}_\varepsilon$ is isometric in the L_1-norm (see the proof of Theorem 2). Therefore (see [19]), the solution to Eq. (37) can be represented as

$$W_\varepsilon(t, x) = \mathbf{V}_t W_\varepsilon(0, x) + \int_0^t \mathbf{V}_{t-s} f_\varepsilon(s) ds = \int_0^t \mathbf{V}_{t-s} f_\varepsilon(s) ds.$$

Then

$$\|W_\varepsilon\|_{C([0,T], L_1(\mathbb{R}))} = \left\| \int_0^t \mathbf{V}_{t-s} f_\varepsilon(s) ds \right\|_{C([0,T], L_1(\mathbb{R}))} \le \sup_{t \in [0,T]} \int_0^t \|\mathbf{V}_{t-s}\|_{B(L_1)} \|f_\varepsilon(s)\|_{L_1} ds$$

$$= \sup_{t \in [0,T]} \int_0^t \|f_\varepsilon(s)\|_{L_1} ds \le \|f_\varepsilon(s)\|_{C([0,T], L_1(\mathbb{R}))} T = TO(\sqrt{\varepsilon}).$$

Finally, for any $T > 0$, the function $U_\varepsilon - U$ for $a < 0$ (i.e., when U exists) satisfies

$$\|U_\varepsilon - U\|_{C([0,T], L_1(\mathbb{R}))} = \|W_\varepsilon + P_\varepsilon\|_{C([0,T], L_1(\mathbb{R}))} \le \|W_\varepsilon\|_{C([0,T], L_1(\mathbb{R}))} + \|P_\varepsilon\|_{C([0,T], L_1(\mathbb{R}))} = O(\sqrt{\varepsilon}). \tag{41}$$

In the proof of convergence, we assumes that $U_0(x) \in C_{00}^\infty$. To prove the convergence of weak solutions, we note that

$$\begin{aligned} \|U_\varepsilon - U\|_{C([0,T], L_1(\mathbb{R}))} &= \|U_\varepsilon - U_{\varepsilon,n} + U_{\varepsilon,n} - U_n + U_n - U\|_{C([0,T], L_1(\mathbb{R}))} \\ \le \|U_\varepsilon - U_{\varepsilon,n}\|_{C([0,T], L_1(\mathbb{R}))} &+ \|U_{\varepsilon,n} - U_n\|_{C([0,T], L_1(\mathbb{R}))} + \|U_n - U\|_{C([0,T], L_1(\mathbb{R}))}, \end{aligned} \tag{42}$$

where U_ε and U are weak solutions to Cauchy problems (14) and (25), respectively, and $U_{\varepsilon,n}$ and U_n are solutions to the Cauchy problems with initial data from C_{00}^∞.

By the definition of a weak solution, the first and third terms in (42) are small. The second term satisfies estimate (41).

Thus, we have proved the following result.

Theorem 5. *Let $a < 0$. If $U_0(x) \in L_1(\mathbb{R})$, then the sequence of solutions to regularized problem* (2) *converges to the solution of degenerate problem* (1) *in the sense that*

$$\lim_{\varepsilon \to 0} \|U_\varepsilon(t, x) - U(t, x)\|_{C([0,T], L_1(\mathbb{R}))} = 0 \quad \forall T > 0.$$

We have established that, in the lack of degeneration, diffusion with discontinuous coefficients satisfying Assumptions 1–5 defines a semigroup of transformations of $L_1(\mathbb{R})$ with the generator $\mathbf{L}$. Whether diffusion with discontinuous coefficients satisfying Assumptions 1–5 defines a semigroup when the diffusion coefficient is degenerate depends on the sign of the transfer coefficient. If a semigroup with a degenerate generator exists, it is the limit of a sequence of nondegenerate regularized semigroups in the strong operator topology.

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