Hypoellipticity of Certain Infinitely Degenerate Second Order Operators

Lyudmila Korobenko

University of Calgary
Calgary, Alberta
lkoroben@ucalgary.ca

Cristian Rios*

University of Calgary
Calgary, Alberta
crios@ucalgary.ca

Abstract

In this paper we establish the hypoellipticity without loss of weak derivatives of second order linear operators comprised by a linear combination, with infinite vanishing coefficients, of subelliptic operators in separate spaces. This generalizes previously known results.

Keywords: hypoellipticity, subellipticity, infinite vanishing, loss of derivatives
2000 MSC: 35H10, 35H20, 35S05, 35G05, 35B65, 35A18

1. Introduction

A linear differential operator L acting on $\mathcal{D}'(\mathbb{R}^n)$, the space of distributions, is said to be hypoelliptic if whenever $u \in \mathcal{D}'(\mathbb{R}^n)$ and $Lu \in C^\infty(\mathbb{R}^n)$ then $u \in C^\infty(\mathbb{R}^n)$. A sufficient condition for a differential operator to be hypoelliptic is subellipticity: L is subelliptic if there exists some $\varepsilon, C > 0$ such that

$$||u||_\varepsilon^2 \leq C\left(|(Lu, u)| + \|u\|^2\right) \qquad \text{for all } u \in C_0^\infty(\mathbb{R}^n); \tag{1}$$

*Author partially supported by the National Science and Engineering Research Council of Canada under operating Grant 733901.

$\|\cdot\|_s$ denotes the Sobolev norm of order $s \in \mathbb{R}$ (see Definition 2.1 below), and $\|\cdot\| = \|\cdot\|_0$ is the L^2 norm in $\mathbb{R}^n$. Some necessary and sufficient conditions for subellipticity have been established in terms of associated vector fields by Hörmander in his pivotal paper [4]; and in terms of subunit metric balls by Fefferman and Phong [3]. Subelliptic operators may have ellipticity vanishing to at most a finite order.

A linear differential operator with infinitely vanishing ellipticity is not subelliptic, such operators do not satisfy the Hörmander condition. The first known hypoellipticity results for infinitely degenerate operators are due to Fediĭ [2], where the simplest example is $P = \partial_x^2 + k(x)\partial_y^2$ with $k(x) > 0$ for $x \neq 0$, $\sqrt{k}$ is smooth and it may vanish to any order at the origin. A different criterion for hypoellipticity was developed by Morimoto in Section 2 of [12], where the seminal techniques from [2] were generalized. Other sufficient conditions for hypoellipticity were obtained by the same author in [13], where the left hand side on the subellipticity condition (1) was replaced by logarithmic Sobolev norms.

The hypoellipticity of semilinear operators with principal part satisfying the Hörmander condition was established in [24]. Quasilinear operators with infinitely vanishing ellipticity of the "Fediĭ" form $\partial_x^2 + \partial_y k(x,z)\partial_y$ have been studied in two dimensions by Sawyer and Wheeden motivated by applications to Monge-Ampère equations [21, 22]; Rios et al extended these results to a wider class of infinitely vanishing quasilinear equations in higher dimensions [19, 20]. However, hypoellipticity was only obtained for continuous solutions. Previous nonlinear hypoellipticity results had also required extra hypothesis on the solutions; in [25] quasilinear subelliptic systems were considered, and hypoellipticity was obtained for continuous solutions; in [14, 15] hypoellipticity was obtained for bounded solutions of certain infinitely degenerate quasilinear equations.

Returning to the linear case, Kusuoka and Stroock extended Fediĭ's two dimensional result to the case when only k was required to be smooth and vanish to any order at the origin [9]; however, the authors also showed that in higher dimensions hypoellipticity may fail for certain linear differential operators depending on the vanishing order of the coefficients. In fact, a quite spectacular characterization of hypoellipticity for $Q = \partial_x^2 + k(x)\partial_y^2 + \partial_z^2$ was obtained: Q is hypoelliptic if and only if $\lim_{x\to 0} x \log k(x) = 0$. The proofs in [9] rely on the Malliavin Calculus. In [12], Morimoto, using non-probabilistic methods, extended Kusuoka and Strook's result to pseudodifferential operators of the form $R = a\,(x, y, D_x) + g\,(x')\, b\,(x, y, D_y)$ in $\mathbb{R}^n = \mathbb{R}^{n_1}_x \times \mathbb{R}^{n_2}_y$, where a and b are strongly elliptic pseudodifferential operators, $x = (x', x'') \in \mathbb{R}^{n_1} = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, g is smooth, $g\,(x') > 0$ for $x' \neq 0$ and $\lim_{x' \to 0} |x'|\, |\log g\,(x')| = 0$.

Since the operator Q is obtained from P by adding an "elliptic term" ∂_z^2, the Kusuoka and Strook's operator Q seems to show that Fediĭ's result is not easily extended to higher dimensions. However, Fediĭ's two dimensional result does extend to operators in higher dimensions regardless of the order of vanishing if their structure is similar that of the two dimensional linear differential operator P. Indeed, P may be written in the form $P = L_1 + k\,(x)\, L_2$ in $\mathbb{R} \times \mathbb{R}$, where $L_1 = \partial_x^2$ and $L_2 = \partial_y^2$ are (one dimensional) elliptic operators and the coefficient k does not depend on the second variable. With this perspective, Morimoto generalized Fediĭ's result to pseudodifferential operators of the from $R = a\,(x, y, D_x) + g\,(x)\, b\,(x, y, D_y)$ in $\mathbb{R}^n = \mathbb{R}^{n_1}_x \times \mathbb{R}^{n_2}_y$, where a and b are strongly elliptic pseudodifferential operators, $g\,(x) > 0$ for $x \neq 0$, g is smooth and it can vanish at any order at the origin [11]. Over a decade later, Kohn [6] proved the hypoellipticity of R in the case that $a\,(x, D_x) = L_1$ and $b\,(y, D_y) = L_2$ are assumed to be linear differential operators

$$L_k = -\sum_{i,j=1}^{n_k} a_{ij}^k \left(x^k\right) \frac{\partial^2}{\partial x_i^k \partial x_j^k} + \sum_{i=1}^{n_k} b_i^k \left(x^k\right) \frac{\partial}{\partial x_i^k} + c^k \left(x^k\right), \tag{2}$$

which are subelliptic in $\mathbb{R}^{n_k}$, $k = 1, 2$, respectively.

The purpose of this paper is to generalize Kohn's result to an arbitrary finite number of subelliptic linear differential operators in separate variables, extending the Fediĭ's type structure modeled in [11, 6].

We now define the concepts of subellipticity and hypoellipticity that will be used in this work.

Definition 1.1 (Subelliptic operator)**.** *Let L be a linear differential operator defined by*

$$L = -\sum_1^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_1^n b_i(x) \frac{\partial}{\partial x_i} + c(x) \tag{3}$$

where $a_{ij},\ b_i,\ c \in C^\infty(U)$ and $(a_{ij})_{i,j=1}^n \geq 0$. Then L is subelliptic *at $x_0 \in \mathbb{R}^n$ if there exists a neighborhood U of x_0 and positive constants ε and C such that (1) holds for all $u \in C_0^\infty(\mathbb{R}^n)$. L is called* subelliptic *if it is subelliptic at each point of $\mathbb{R}^n$.*

Definition 1.2 (Hypoelliptic operator)**.** *A linear differential operator L acting on distributions in $\mathbb{R}^n$ is* hypoelliptic *if and only if whenever $Lu \in C^\infty(\mathbb{R}^n)$ for some distribution u, then $u \in C^\infty(\mathbb{R}^n)$.*

In [7] the author made precise the concept of loss or gain of derivatives. Given a linear differential operator L which is hypoelliptic in $\mathbb{R}^n$, L must satisfy the following:

1. Given open sets $U \Subset U' \subset \mathbb{R}^n$, a nonnegative integer p and $s_0 \in \mathbb{R}$, there exist an integer q and a constant $C = C(U, p, q, s_0)$ such that for all $u \in C_0^\infty(\mathbb{R}^n)$,
$$\sum_{|\alpha|\le p} \sup_{x\in U} |D^\alpha u(x)| \le C \left(\sum_{|\beta|\le q} \sup_{x\in U} |D^\alpha Lu(x)| + \|u\|_{-s_0} \right). \tag{4}$$
2. Given $\varphi, \varphi' \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi \prec \varphi'$ (see Definition 2.5), and $s_0, s_1 \in \mathbb{R}$, there exists $s_2 \in \mathbb{R}$ and constant $C = C(\varphi, \varphi', s_0, s_1, s_2)$ such that for all $u \in C_0^\infty(\mathbb{R}^n)$,
$$\|\varphi u\|_{s_1} \le C \left(\|\varphi' Lu\|_{s_2} + \|u\|_{-s_0} \right). \tag{5}$$

Following [7], we say that L *gains* $p - q$ derivatives in the sup norms if (4) holds for the smallest possible integer q (and large enough s_0) and $q \le p$; reciprocally, if $p \le q$ we say that L *loses* $q - p$ derivatives in the sup norms. Similarly, if s_2 is the smallest real number for which (5) holds (for large s_0), then if $s_1 \ge s_2$ we say that L *gains* $s_1 - s_2$ derivatives in the Sobolev norms, and if $s_1 \le s_2$, and we say that L *loses* $s_2 - s_1$ derivatives in the Sobolev norms.

When $p = q$ or $s_1 = s_2$ in the previous concepts, there is no loss of derivatives in the respective norms. We adopt the following definition:

Definition 1.3 (Hypoellipticity without loss of weak derivatives)**.** *A linear differential operator L acting on distributions in $\mathbb{R}^n$ is said to be* hypoelliptic without loss of weak derivatives *if given any open set $U \subset \mathbb{R}^n$ such that $\zeta Lu \in H^s(\mathbb{R}^n)$ for all $\zeta \in C_0^\infty(U)$, then $\zeta u \in H^s(\mathbb{R}^n)$ for all $\zeta \in C_0^\infty(U)$.*

Because of Sobolev imbeddings, a linear differential operator which is hypoelliptic without loss of weak derivatives it is also hypoelliptic. The loss of derivatives question have been studied by several authors. In [17] Parenti and Parmeggiani gave necessary and sufficient conditions for the hypoellipticity with loss of an arbitrary prescribed number r of derivatives, for a class of pseudodifferential operators with nontrivial symplectic characteristics. In [7] Kohn considered complex vector fields $\{X_1, \dots, X_m\}$ and showed that if they satisfy a bracket condition of order one at the origin, that is, if $\{X_i, [X_i, X_j]\}_{i,j=1,\dots,m}$ span the complex tangent space at the origin, then the operator $\sum X_i^2$ is subelliptic in a neighbourhood of the origin and

$$\|u\|_\varepsilon^2 \leq C\left(\sum \|X_j u\|^2 + \|u\|^2\right), \tag{6}$$

with $\varepsilon = \frac{1}{2}$. Surprisingly, if the vector fields satisfy a bracket condition (i.e.: the Hörmander condition in the real case) of order higher than one, then the operator $\sum X_i^2$ is not necessarily subelliptic. In fact, in [7] there are examples of complex vector fields $\{X_{1k}, X_2\}$, for each integer $k > 0$, such that they satisfy a bracket condition of order $k+1$ at the origin and $E_k = X_{1k}^* X_{1k} + X_2^* X_2$ does not satisfy (6) for any $\varepsilon > 0$, i.e., E_k is not subelliptic. Moreover, E_k loses k derivatives in the sup norm and $k-1$ derivatives in the Sobolev norms. On the other hand, in that same paper it was also shown that the operators E_k are indeed hypoelliptic. In [1] Christ showed that the operators $-\frac{\partial^2}{\partial s^2} + E_K$ on $\mathbb{R}^4$ are not hypoelliptic for $k > 0$. Other examples of operators with a prescribed number of loss of derivatives were provided in [18]. These results make evident that further study is needed to better understand the subellipticity and hypoellipticity of operators obtained as sums of squares of complex vector fields and, more generally, linear operators with complex coefficients. Some of these questions have been addressed in the recent work [8], where sufficient conditions were given for subellipticity, and the existence of solutions was established for operators arising as sum of squares of special classes of complex vector fields.

In this work we prove the hypoellipticity without loss of weak derivatives of linear differential operators L obtained by "gluing" subelliptic operators L_k in different variables x^k with Fediĭ's type "glue functions" λ_k. To make this precise we introduce some definitions. For fixed positive integers n_k, $k = 1, \cdots, m$, we denote $x \in \prod_{k=1}^m \mathbb{R}^{n_k}$ as $x = (x^1, \dots, x^{n_m}) \in \mathbb{R}^n$, with

$$x^k = \left(x_1^k, \dots, x_{n_k}^k\right) \in \mathbb{R}^{n_k}, \quad k = 1, \cdots, m, \quad n = \sum_{k=1}^m n_k,$$

and we let $\overline{x^k}$ be the vector obtained from x by omitting x^k, i.e.

$$\overline{x^k} = \left(x^1, \ \dots \ , x^{k-1}, x^{k+1}, \ \dots \ x^m\right).$$

Our main result is the following:

Theorem 1.4. *Suppose that* L_k *as in (2) are subelliptic, and* $\lambda_k = \lambda_k(\overline{x^k}) \geq 0$*,* $k = 1, \cdots, m$*, are smooth functions. Assume* $\lambda_1 \equiv 1$*, and that for* $2 \leq k \leq m$*,* $\lambda_k\left(\overline{x^k}\right) > 0$ *for* $\overline{x^k} \neq 0$*. Then the operator* L *defined by*

$$L = \sum_{k=1}^{m} \lambda_k L_k \tag{7}$$

is hypoelliptic without loss of weak derivatives in $\mathbb{R}^n$*.*

The important cases of the above result are when some of the coefficients λ_k have a zero of infinite order at the n_k-dimensional subspaces $\overline{x^k} = 0$ in $\mathbb{R}^n$. Note that no restrictions are imposed on the order of vanishing of the coefficients λ_k.

Note that in the case $m = 2$ considered by Kohn [6] the (nonnegative) coefficient $\lambda = \lambda_2$ was allowed to have zeroes of finite order outside $\overline{x_2} = 0$. In this case, the operator $L(x^1, x^2) = L_1(x^1) + \lambda(x^1) L_2(x^2)$ is subelliptic whenever λ has a zero of finite order and L_1, L_2 are subelliptic. We must clarify this concept further; in this case, $\lambda(x^1) = \lambda\left(x_1^1, \ldots, x_{n_1}^1\right)$ may depend on less variables than $x_1^1, \ldots, x_{n_1}^1$. So, for example, $\lambda(x^1) = (x_1^1)^2$ would be considered as having a zero of order 2. However, as a function of $\left(x_1^1, x_2^1, \ldots, x_{n_1}^1\right)$, λ has a zero of infinite order at $\left(0, x_2^1, \ldots, x_{n_1}^1\right)$. This structure allows to maintain subellipticity because L_1 is assumed to be subelliptic in $\mathbb{R}^{n_1}$, and hence the individual vector fields $\partial_{x_j^1}$ are "available" via a finite number of commutators. However, when $m \geq 3$ it is no longer true that if the coefficients λ_k have zeroes of finite order in the previous sense then L as in (7) is subelliptic, or even hypoelliptic. Indeed, the operators $M_1 = -\partial_x^2 - \partial_y^2 \exp\left(-|x|^{-\frac{1}{2}}\right) - \exp\left(-|y|^{-\frac{1}{2}}\right)\partial_z^2$ and $M_2 = -\partial_x^2 - z^2\partial_y^2 - y^2\partial_z^2$ are not subelliptic in $\mathbb{R}^3$ since they are sum of the squares of vector fields which do not satisfy the Hörmander condition. Now, M_1 is hypoelliptic in $\mathbb{R}^3$ while M_2 is not. See Theorem 1 in [16] to check the first assertion. The proof in [16] relies on the special structure of M_1, in which the vanishing order of the coefficients is restricted. We consider a different structure, where the degeneracy is localized in space but there is no restrictions to the order of vanishing. On the other hand, to check that M_2 is not hypoelliptic, it is enough to note its action on the distribution $u = \delta_{yz}$, where δ_{yz} is the

Dirac delta function at the origin in $\mathbb{R}^2$. Since M_2 is self-adjoint, for any test function $\varphi \in C_0^\infty(\mathbb{R}^3)$ we have

$$\langle M_2 u, \varphi\rangle = -\left\langle \delta_{yz}, \varphi_{xx} + z^2\varphi_{yy} + y^2\varphi_{zz}\right\rangle = -\int_{\mathbb{R}} \varphi_{xx}(x,0,0)\ dx = 0.$$

These examples illustrate some of the difficulties in generalizing Kohn's result to a structure (7) including more than two summands.

On the other hand,because of the local nature of Theorem 1.4, our results easily generalize to the case when $\sum_{k=1}^m \lambda_k > 0$, and λ_k has *isolated* zeroes (of any order) in $\prod_{j\neq k} \mathbb{R}^{n_j}$, $k = 1, \ldots, m$.

We broadly follow the line of the proof established by Kohn [6], the presence of more than one function λ_i prevents however of a straightforward adaptation of proofs and requires a more delicate analysis. The paper is organized as follows. First, in Section 2 we introduce some notation and establish some preliminary lemmas that are used further in Section 3 to prove the main a-priori estimate, Lemma 3.7. The main result is proved in Section 4 using families of smoothing operators.

We gratefully acknowledge the references, corrections, and suggestions provided by the referee, which contributed to the clarity and the relevance of this work.

2. Preliminaries

In this section we give basic definitions and establish some preliminary results. The Fourier transform of an integrable function u is defined by

$$\hat{u}(\xi) = \int\limits_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\ dx.$$

The inverse Fourier transform is given by

$$f^\vee(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\ d\overline{\xi},$$

where $d\overline{\xi} = (2\pi)^{-n}\, d\xi$. Note that $f^\vee(x) = (2\pi)^{-n}\,\widehat{\widetilde{f}}(x)$ where $\widetilde{f}(\xi) = f(-\xi)$.

Definition 2.1. *For any $s \in \mathbb{R}$ we define an operator Λ^s by the identity*

$$\widehat{\Lambda^s u}(\xi) = \left(1 + |\xi|^2\right)^{s/2} \hat{u}(\xi) \tag{8}$$

and the norm $||\cdot||_s$ *by*
$$||u||_s = ||\Lambda^s u||_{L^2(\mathbb{R}^n)} \tag{9}$$
For any vector function $\mathbf{u} = (u_1, \ldots, u_N)$ *we define* $\Lambda^s \mathbf{u}$ *by the identity* $\widehat{\Lambda^s \mathbf{u}} = \left(\widehat{\Lambda^s u_1}, \ldots, \widehat{\Lambda^s u_N}\right)$*, with the norm*

$$||\mathbf{u}||_s = \left(\sum_{p=1}^{N} ||\Lambda^s u_p||^2_{L^2(\mathbb{R}^n)}\right)^{1/2}.$$

We recall that, more generally, a pseudodifferential operator P with symbol $p(x,\xi)$ is given by

$$Pf(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\,\widehat{u}(\xi)\, d\bar{\xi}.$$

Note that if $p(x,\xi) = i\xi_j$, then $P = \dfrac{\partial}{\partial x_j}$.

Definition 2.2. *Given* $u \in C_0^\infty(\mathbb{R}^n)$ *define the* partial Fourier transform $\mathfrak{F}_{x^k} u(\overline{x^k}, \xi^k)$ *by*
$$\mathfrak{F}_{x^k} u(\overline{x^k}, \xi^k) = \int\limits_{\mathbb{R}^{n_k}_{x^k}} e^{-ix^k\cdot\xi^k} u(x) dx^k$$
For vector functions $\mathbf{u} = (u_1, \ldots, u_N),\ u_i \in C_0^\infty(\mathbb{R}^n)$ *we set*

$$\mathfrak{F}_{x^k}\mathbf{u}\left(\overline{x^k}, \xi^k\right) = \left(\mathfrak{F}_{x^k} u_1\left(\overline{x^k}, \xi^k\right), \ldots, \mathfrak{F}_{x^k} u_N\left(\overline{x^k}, \xi^k\right)\right).$$

Definition 2.3. *For* $s \in \mathbb{R}$ *define the partial operators* $\Lambda^s_{x^k}$ *by*

$$\mathfrak{F}_{x^k}(\Lambda^s_{x^k} u)(\overline{x^k}, \xi^k) = (1 + |\xi^k|^2)^{s/2} \mathfrak{F}_{x^k} u(\overline{x^k}, \xi^k)$$

Similarly, for vector functions $\mathbf{u}$*, we set* $\mathfrak{F}_{x^k}(\Lambda^s_{x^k}\mathbf{u})(\overline{x^k}, \xi^k) = (1+|\xi^k|)^{s/2}\mathfrak{F}_{x^k}\mathbf{u}(\overline{x^k}, \xi^k)$.

The next lemma is the classical result on a composition of pseudodifferential operators (see for example [23]). In what follows S^m denotes the usual classes $S^m_{1,0}$, of symbols $p(x,\xi)$ satisfying

$$\left|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)\right| \leq B_{\alpha,\beta}\left(1 + |\xi|^2\right)^{\frac{1}{2}(m-|\beta|)}, \qquad \text{for all } x, \xi,\ \alpha, \beta. \tag{10}$$

Lemma 2.4. *Let $p(x,\xi) \in S^m$ and $q(x,\xi) \in S^k$ then*

$$p(x,D)\, q(x,D) = r(x,D) \in S^{m+k}$$

with

$$r(x,\xi) \sim \sum_{\alpha \geq 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha p(x,\xi)\, D_x^\alpha q(x,\xi),$$

in the sense that

$$\left(p(x,\xi) - \sum_{|\alpha|<N} D_\xi^\alpha p(x,\xi)\, D_x^\alpha q(x,\xi) \right) \in S^{m+k-N}, \text{ for all } N \geq 0.$$

We will localize our estimates by multiplication with suitable cutoff functions. The following concepts will be useful in our microlocal analysis.

Definition 2.5 (Cutoff functons, supporting relation)**.** *We say that φ is a cutoff function in $\mathbb{R}^n$ if $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $0 \leq \varphi \leq 1$. Given two measurable functions φ, ψ we introduce the notation $\varphi \prec \psi$, and we say that ψ* supports *φ if ψ is a cutoff function and $\psi \equiv 1$ in a neighbourhood of* supportφ.

The following lemma [6] is an important tool for estimating inner products involving pseudodifferential operators and ordinary derivatives. Roughly speaking, it allows us to lower the order of differentiation in an inner product using integration by parts and standard pseudodifferential calculus.

Lemma 2.6. *Let P and Q be pseudodifferential operators of orders p and q, respectively. Assume that $P - P^*$ and $Q - Q^*$ are of orders (at most) $p-1$ and $q-1$, respectively. Let $\zeta, \eta \in C_0^\infty(\mathbb{R}^n)$, such that $\zeta_{x_i} \prec \eta$. Then there exists $C > 0$ such that for all $\in C^\infty(\mathbb{R}^n)$*

$$|(P\zeta u_{x_i}, Q\zeta u)| \leq C \left(\|\zeta u\|^2_{(p+q)/2} + \|\eta u\|^2_{(p+q)/2} \right). \tag{11}$$

Moreover, if $u \in H^r(\mathbb{R}^n)$ with $r = \max\{p+2, q+1\}$, then the same estimate holds.

Proof. The desired estimate has been already shown for $u \in C^\infty(\mathbb{R}^n)$ [6]. In case $u \in H^r(\mathbb{R}^n)$ we find an approximating sequence $\{u_n\}_{n=1}^\infty \subset C^\infty$ such that $\lim_{n\to\infty} ||u_n - u||_r = 0$. One can check that u_n defined by $\widehat{u_n}(\xi) = \exp(-|\xi|^2/n^2)\widehat{u}(\xi)$ satisfies the desired properties for all $p,\ q\ \in \mathbb{R}$. By the definition of r it follows that $\lim_{n\to\infty} ||u_n - u||_{(p+q)/2} = 0$ and, moreover, by Arzela-Ascoli theorem (replacing $\{u_n\}$ by an appropriate subsequence, which we dub again $\{u_n\}$) $||P\zeta\partial_{x_i}(u_n - u)|| \to 0$ and $||Q\zeta(u_n - u)|| \to 0$ as $n \to \infty$. Therefore, applying (11) to u_n and taking the limit as $n \to \infty$ we obtain the desired result. $\square$

Definition 2.7. *We will henceforth use special families of cutoff functions satisfying the following properties.*

- *Let $\sigma_k,\ \tilde{\sigma}_k,\ \sigma'_k,\ \sigma''_k \in C_0^\infty(\mathbb{R}^{n_k}),\ k = 1, \cdots, m$ be cutoff functions such that $\sigma_k = 1$ in a neighbourhood of $0 \in \mathbb{R}^{n_k}$, and $\sigma_k \prec \tilde{\sigma}_k \prec \sigma'_k \prec \sigma''_k$. Let $\zeta(x) = \prod_{k=1}^{m} \sigma_k(x^k)$, with $\tilde{\zeta}(x)$, $\zeta'(x)$, and $\zeta''(x)$ are similarly defined.*
- *Fix neighborhoods of the origin U_0^k and U^k in $\mathbb{R}^{n_k}$ such that $\overline{U}_0^k \subset U^k$ and $\sigma_k = 1$ on U^k. Let $\sigma_0^k,\ \tilde{\sigma}_0^k$ be cutoff functions in $\mathbb{R}^{n_k}$ with $\mathrm{support}(\sigma_0^k) \cap U_0^k = \emptyset$, and $|\nabla_{x^k}\sigma_k| \prec \sigma_0^k \prec \tilde{\sigma}_0^k$. Set $\zeta_0^k = \sigma_0^k \prod_{l=1, l\neq k}^{m} \tilde{\sigma}_l(x^l)$, and $\tilde{\zeta}_0^k = \tilde{\sigma}_0^k \prod_{l=1, l\neq k}^{m} \sigma'_l(x^l)$. Note that $\zeta_0^k \zeta_{x_i^k} = \zeta_{x_i^k}$.*
- *Further, choose the cutoffs functions so that they satisfy $\sigma_k^0 \prec \tilde{\sigma}_k$, $\tilde{\sigma}_k^0 \prec \sigma'_k$. Hence $\zeta_0^k \prec \tilde{\zeta}$ and $\tilde{\zeta}_0^k \prec \zeta'$.*

Next, we define a partially smoothing pseudodifferential operator that will be useful in our calculations.

Definition 2.8. *For $\delta > 0$ we define S_δ by*

$$\widehat{S_\delta u}\,(\xi) = \frac{1}{\left(1 + \delta^2 |\xi|^2\right)^{3/2}} \widehat{u}\,(\xi) = s_\delta\,(\xi)\,\widehat{u}\,(\xi)\,. \tag{12}$$

Note: In our proofs all the pseudodifferential operators considered are applied to compactly supported functions or distributions.

Lemma 2.9. *The operator S_δ has the following properties:*

i. *$S_\delta \in S^0$ uniformly in δ for $0 \le \delta \le 1$, where S^0 is the symbol class defined by (10) with $m = 0$. More precisely, for any $s \in \mathbb{R}$*

$$\sup_{0<\delta\le 1} \|S_\delta u\|_s = \|u\|_s\,.$$

ii. *$S_\delta : H^s \longrightarrow H^s$ is a bounded operator, with bounds independent of δ.*

iii. *If $u \in H^{s_0}$ for some $s_0 \in \mathbb{R}$ then $S_\delta u \in H^{s_0+3}$.*

iv. *If for any $s \in \mathbb{R}$, $u \in H^{s-3}$ and $\lim_{\delta\to 0^+} \|S_\delta u\|_s \leq C$, then $u \in H^s$.*

v. *If $a \in C_0^\infty(\mathbb{R}^n)$, then*

$$[a, S_\delta] \;=\; \textstyle\sum_{k=1}^m \sum_{j=1}^{n_k} \left(a_{x_j^k} \frac{\partial}{\partial x_j^k}\right) R_\delta^{-2} S_\delta + Q_\delta^{-2} S_\delta$$

where R_δ^{-2} and Q_δ^{-2} are families of pseudodifferential operators of order -2 uniformly in δ for $0 \leq \delta \leq 1$.

Proof. We have

$$\begin{aligned}
\|S_\delta u\|_s^2 &= \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^s \left(\frac{1}{\left(1+\delta^2 |\xi|^2\right)^{3/2}}\right)^2 |\widehat{u}|^2 \; d\xi \\
&\leq \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^s |\widehat{u}|^2 \; d\xi = \|u\|_s^2
\end{aligned}$$

This proves (i) and (ii). Property (iii) follows easily from the definition of S_δ.

On the other hand, if $u \in H^{s-3}$ and $\lim_{\delta\to 0^+} \|S_\delta u\|_s \leq C$ then

$$\begin{aligned}
\|u\|_s &= \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^s |\widehat{u}|^2 \; d\xi \\
&= \lim_{\delta\to 0^+} \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^s \left(\frac{1}{\left(1+\delta^2 |\xi|^2\right)^{3/2}}\right)^2 |\widehat{u}|^2 \; d\xi \\
&= \lim_{\delta\to 0^+} \|\Lambda_s S_\delta u\|^2 \leq C^2.
\end{aligned}$$

So $||u||_s < C$, which shows (iv). To prove property (v) we first note that has been shown in [6, Lemma 3.3] using Lemma 2.4 that the principal symbol of $[a, S_\delta]$ has the form

$$[a, S_\delta] \sim \sum_{k=1}^m \sum_{j=1}^{n_k} -2i a_{x_j^k} \xi_j^k \frac{3/2\delta^2}{1+\delta^2|\xi|^2} S_\delta$$

By differentiating $\xi_j^k \frac{3/2\delta^2}{1+\delta^2|\xi|^2} S_\delta$ with respect to ξ_r^l one can check that the lower order terms have the form $Q_\delta^{-2} S_\delta$. $\square$

3. Apriori estimates

Lemma 3.1. *Suppose that L is given by (3) is subelliptic at x_0 and that $\zeta, \tilde{\zeta} \in C_0^\infty(U)$, with U a neighbourhood of x_0 as in (1), and $\zeta \prec \tilde{\zeta}$. Then, for all $u \in C^\infty(\mathbb{R}^n)$,*

$$\begin{aligned} \|\zeta u\|_\varepsilon^2 &\le C\left\{\left|\sum_{i,j=1}^n \left(a_{ij}\zeta u_{x_i}, \zeta u_{x_j}\right)\right| + \left\|\tilde{\zeta} u\right\|^2\right\} \\ &\le C\left\{|(\zeta L u, \zeta u)| + \left\|\tilde{\zeta} u\right\|^2\right\}. \end{aligned} \tag{13}$$

Proof. From (1) we have

$$\|\zeta u\|_\varepsilon^2 \le C\{|(L\zeta u, \zeta u)| + \|\zeta u\|^2\}. \tag{14}$$

Next,

$$(L\zeta u, \zeta u) = -\sum_1^n \left(a_{ij}(\zeta u)_{x_i x_j}, \zeta u\right) + \sum_1^n \left(b^i(\zeta u)_{x_i}, \zeta u\right) + (c\zeta u, \zeta u)\,. \tag{15}$$

Integrating by parts the first term on the right, we have that

$$\begin{aligned} -\left(a_{ij}(\zeta u)_{x_i x_j}, \zeta u\right) &= \sum_1^n \Big(\left(a_{ij}\right)_{x_j}(\zeta u)_{x_i}, \zeta u\Big) + \sum_1^n \Big(a_{ij}(\zeta u)_{x_i}, (\zeta u)_{x_j}\Big) \\ &= \sum_1^n \Big(\left(a_{ij}\right)_{x_j}(\zeta u)_{x_i}, \zeta u\Big) + 2\sum_1^n \left(a_{ij}\zeta u_{x_j}, \zeta_{x_i} u\right) \\ &\quad + \sum_1^n \left(a_{ij}\zeta_{x_i} u, \zeta_{x_j} u\right) + \sum_1^n \left(a_{ij}\zeta u_{x_i}, \zeta u_{x_j}\right). \end{aligned}$$

By Lemma 2.6 the first and second terms on the right are bounded by $C||\tilde{\zeta}u||^2$, while it is clear that the third term on the right also satisfies the same bounds. The same applies to the last two terms on the right of (15), This and (14) yield

$$\|\zeta u\|_\varepsilon^2 \le C\{\left|\sum_{i,j=1}^n \left(a_{ij}\zeta u_{x_i}, \zeta u_{x_j}\right)\right| + \|\zeta u\|^2\}.$$

prove the first inequality in (13).

To prove the second inequality in (13), integrating by parts we write

$$\begin{aligned}
\sum\left(a_{ij}\zeta u_{x_i},\zeta u_{x_j}\right) &= -\sum\left(a_{ij}\zeta u_{x_ix_j},\zeta u\right)-\sum\left(\left(a_{ij}\zeta^2\right)_{x_j}\tilde{\zeta}u_{x_i},\tilde{\zeta}u\right)\\
&= (\zeta Lu,\zeta u)-(c\zeta u,\zeta u)-\sum_1^n\left(b^i\zeta u_{x_i},\zeta u\right)\\
&\quad -\sum\left(\left(a_{ij}\zeta^2\right)_{x_j}\tilde{\zeta}u_{x_i},\tilde{\zeta}u\right),
\end{aligned}$$

and apply Lemma 2.6 the the last two terms on the right. □

The following lemma is a generalization of previous lemma to the operators of the form (7).

Lemma 3.2. *Let L be defined by (7), with L_k subelliptic at $x_0^k\in\mathbb{R}^{n_k}$ and $0\le\lambda_k\in C^\infty\left(\mathbb{R}^{n_k}\right)$, $k=1,\cdots,m$. Let $U_k\subset\mathbb{R}^{n_k}$ be neighbourhoods of x_0^k such that (1) holds with ε_k for L_k in U_k. Then if $\zeta,\tilde{\zeta}\in C_0^\infty\left(\prod_{k=1}^m U_k\right)$, with $\zeta\prec\tilde{\zeta}$, for $\varepsilon=\min_k\varepsilon_k$ we have*

$$\begin{aligned}
\sum_{k=1}^m\left\|\sqrt{\lambda_k}\Lambda^\varepsilon_{x^k}\left(\zeta u\right)\right\|^2 &\le C\sum_{k=1}^m\sum_{i,j=1}^{n_k}\left(\zeta\lambda_k a^k_{ij}u_{x^k_i},\zeta u_{x^k_j}\right)\\
&\quad +C\left\|\tilde{\zeta}u\right\|^2 \qquad (16)\\
&\le C\left\{\left|(\zeta Lu,\zeta u)\right|+\left\|\tilde{\zeta}u\right\|^2\right\}.
\end{aligned}$$

Moreover, the same estimate holds when $u\in H^2(\mathbb{R}^n)$.

Proof. First, consider $u\in C^\infty(\mathbb{R}^n)$. Since for each k the operator L_k is subelliptic it follows from (13) that for each fixed $\overline{x^k}\in\mathbb{R}^{n_k}$ we have

$$\begin{aligned}
&\lambda_k(\overline{x^k})\int_{\mathbb{R}^{n_k}}\left|\Lambda^\varepsilon_{x^k}(\zeta u)(x)\right|^2dx^k\\
&\quad\le C\sum_{i,j=1}^{n_k}\int_{\mathbb{R}^{n_k}}\lambda_k(\overline{x^k})a^k_{ij}(x^k)\zeta(x)u_{x^k_i}(x)\zeta(x)u_{x^k_j}(x)\,dx^k\\
&\quad +C\int_{\mathbb{R}^{n_k}}\left|\tilde{\zeta}\left(x\right)u\left(x\right)\right|^2\,dx^k.
\end{aligned}$$

Integrating the above inequality with respect to $\overline{x^k}$ and summing over $k=1,\cdots,m$ we obtain the first part of (16). To show that the inequality holds for $u \in H^2(\mathbb{R}^n)$ we perform an approximation in the same way it has been done in the proof of Lemma 2.6.

To prove the second part consider each term of the triple sum in the first inequality of the lemma and integrate by parts

$$(\lambda_k a_{ij}^k \zeta u_{x_i^k}, \zeta u_{x_j^k}) = -\left(\lambda_k a_{ij}^k \zeta u_{x_i^k x_j^k}, \zeta u\right) - \left(\left(\lambda_k a_{ij}^k \zeta^2\right)_{x_j^k} u_{x_i^k}, \tilde{\zeta} u\right). \qquad (17)$$

We then have

$$\begin{aligned}\sum_{k=1}^{m}\sum_{i,j=1}^{n_k}\left(\lambda_k a_{ij}^k \zeta u_{x_i^k}, \zeta u_{x_j^k}\right) &= (\zeta L u, \zeta u) - \sum_{k=1}^{m}\sum_{i=1}^{n_k}\left(\lambda_k b_i^k \zeta u_{x_i^k}, \zeta u\right) \\ &\quad - \sum_{k=1}^{m}\left(\lambda_k c^k \zeta u, \zeta u\right) - \sum_{k=1}^{m}\sum_{i,j=1}^{n_k}\left(\left(\lambda_k a_{ij}^k \zeta^2\right)_{x_j^k} u_{x_i^k}, \tilde{\zeta} u\right).\end{aligned}$$

The second inequality of the Lemma 3.2 then follows from Lemma 2.6. $\square$

We now formulate the main technical result that will be used to treat commutators $[L, \Lambda^s \zeta]$. The proof relies on Lemma 2.6 and Lemma 2.4.

Lemma 3.3. *Given $s \in \mathbb{R}$, there exists $C > 0$ such that*

$$\left|\left(\tilde{\zeta}\,[L, \Lambda^s S_\delta \zeta]\,u, \tilde{\zeta}\Lambda^s S_\delta \zeta u\right)\right| \le C\left\{\left\|\tilde{\zeta} S_\delta \zeta u\right\|_s^2 + \sum_{k=1}^{m}\left\|\lambda_k \zeta_0^k S_\delta \zeta_0^k u\right\|_s^2 + \left\|\zeta' S_\delta \zeta' u\right\|_{s-1/2}^2\right\}$$

for all functions $u \in H^{s-2}(\mathbb{R}^n)$ and all $0 < \delta \le 1$. Here $\zeta, \tilde{\zeta}$, and ζ_0^k are the cutoff functions defined above.

Proof. We have

$$\begin{aligned}[L, \Lambda^s S_\delta \zeta]\,u &= [L, \Lambda^s S_\delta]\,\zeta u + \Lambda^s S_\delta\,[L, \zeta]\,u \\ &= \sum_{k=1}^{m}[\lambda_k L_k, \Lambda^s S_\delta]\,\zeta u + \sum_{k=1}^{m}\Lambda^s S_\delta\,[\lambda_k L_k, \zeta]\,u. \end{aligned}\qquad (18)$$

Next, for any $k = 1, \cdots, m$,

$$\begin{aligned}[\lambda_k L_k, \Lambda^s S_\delta] &= -\sum_{i,j=1}^{n_k}\left[\lambda_k a_{ij}^k, \Lambda^s S_\delta\right]\frac{\partial^2}{\partial x_i^k \partial x_j^k} \\ &\quad + \sum_{i=1}^{n_k}\left[\lambda_k b_i^k, \Lambda^s S_\delta\right]\frac{\partial}{\partial x_i^k} + \left[\lambda_k c^k, \Lambda^s S_\delta\right].\end{aligned}\qquad (19)$$

and

$$
\begin{aligned}
[\lambda_k L_k, \zeta] &= \lambda_k [L_k, \zeta] \\
&= \lambda_k \left(-\sum_{i,j=1}^{n_k} a_{ij}^k \zeta_{x_i^k x_j^k} + \sum_{i=1}^{n_k} b_i^k \zeta_{x_i^k} \right) - 2\lambda_k \sum_{i,j=1}^{n_k} a_{ij}^k \zeta_{x_i^k} \frac{\partial}{\partial x_j^k},
\end{aligned} \tag{20}
$$

where we used the symmetry of a^k on the last term. It follows that

$$
\begin{aligned}
& \left(\tilde{\zeta}\, [L, \Lambda^s S_\delta \zeta]\, u, \tilde{\zeta} \Lambda^s S_\delta \zeta u\right) \\
= & -\sum_{k=1}^{m} \sum_{i,j=1}^{n_k} \left(\tilde{\zeta} \left[\lambda_k a_{ij}^k, \Lambda^s S_\delta\right] \frac{\partial^2}{\partial x_i^k \partial x_j^k} \zeta u, \tilde{\zeta} \Lambda^s S_\delta \zeta u \right) \\
& + \sum_{k=1}^{m} \sum_{i=1}^{n_k} \left(\tilde{\zeta} \left[\lambda_k b_i^k, \Lambda^s S_\delta\right] \frac{\partial}{\partial x_i^k} \zeta u, \tilde{\zeta} \Lambda^s S_\delta \zeta u \right) \\
& + \sum_{k=1}^{m} \left(\tilde{\zeta} \left[\lambda_k c^k, \Lambda^s S_\delta\right] \zeta u, \tilde{\zeta} \Lambda^s S_\delta \zeta u \right) \\
& - \sum_{k=1}^{m} \sum_{i,j=1}^{n_k} \left(\tilde{\zeta} \Lambda^s S_\delta \lambda_k a_{ij}^k \zeta_{x_i^k x_j^k} u, \tilde{\zeta} \Lambda^s S_\delta \zeta u \right) \\
& + \sum_{k=1}^{m} \sum_{i=1}^{n_k} \left(\tilde{\zeta} \Lambda^s S_\delta \lambda_k b_i^k \zeta_{x_i^k} u, \tilde{\zeta} \Lambda^s S_\delta \zeta u \right) \\
& - 2 \sum_{k=1}^{m} \sum_{i,j=1}^{n_k} \left(\tilde{\zeta} \Lambda^s S_\delta \lambda_k a_{ij}^k \zeta_{x_i^k} \frac{\partial}{\partial x_j^k} u, \tilde{\zeta} \Lambda^s S_\delta \zeta u \right) \\
= & \; I + II + III + IV + V + VI.
\end{aligned} \tag{21}
$$

Using Lemma 2.4 and property (v) of the operator S_δ, for a smooth function f we have

$$
[\Lambda^s S_\delta, f] = \widetilde{R}_{s,\delta}^0 \Lambda^{s-1} f_0 S_\delta = R_{s,\delta}^0 \Lambda^{s-2} \sum_{r=1}^{m} \sum_{\ell=1}^{n_k} f_{x_\ell^r} \frac{\partial}{\partial x_\ell^r} S_\delta + Q_{s,\delta}^0 \Lambda^{s-2} f_0 S_\delta, \tag{22}
$$

where $\widetilde{R}_{s,\delta}^0, R_{s,\delta}^0, Q_{s,\delta}^0$ are pseudodifferential operators of order 0 uniformly in $0 \le \delta \le 1$, and f_0 is a cutoff function such that $f_0 \succ |\nabla f|$. Moreover, $R_{s,\delta}^0 - \left(R_{s,\delta}^0\right)^*$ is of degree -1 uniformly in $0 \le \delta \le 1$.

We use this to estimate the first term on the right in (19), we obtain

$$\begin{aligned}|I| &\leq \sum_{k,r=1}^{m}\sum_{i,j,\ell=1}^{n_k}\left|\left(\tilde{\zeta}R^0_{s,\delta}\Lambda^{s-2}\left(\lambda_k a^k_{ij}\right)_{x^r_\ell}\frac{\partial^2}{\partial x^k_i\partial x^k_j}\tfrac{\partial}{\partial x^r_\ell}S_\delta\zeta u,\tilde{\zeta}\Lambda^s S_\delta\zeta u\right)\right| \\ &\quad+\sum_{k=1}^{m}\sum_{i,j=1}^{n_k}\left|\left(\tilde{\zeta}Q^{-0}_{s,\delta}\Lambda^{s-2}\left(\lambda_k a^k_{ij}\right)_0\frac{\partial^2}{\partial x^k_i\partial x^k_j}S_\delta\zeta u,\tilde{\zeta}\Lambda^s S_\delta\zeta u\right)\right|. \end{aligned} \tag{23}$$

We apply to each term on the the first sum Lemma 2.6 with

$$P=\tilde{\zeta}R^0_{s,\delta}\Lambda^{s-2}\left(\lambda_k a^k_{ij}\right)_{x^r_\ell}\frac{\partial^2}{\partial x^k_i\partial x^k_j},\qquad\text{and}\qquad Q=\tilde{\zeta}\Lambda^s.$$

Note, that both operators P and Q have order s, and since $u\in H^{s-2}$ it follows that $S_\delta u\in H^{s+1}$ and therefore Lemma 2.6 is applicable. Since the second term on the right of (23) is dominated by $C\left\{\left\|\tilde{\zeta}S_\delta\zeta u\right\|^2_s+\|S_\delta\zeta u\|^2_{s-1}\right\}$, we obtain

$$|I|\leq C\left(\left\|\tilde{\zeta}S_\delta\zeta u\right\|^2_s+\|\zeta' S_\delta\zeta u\|^2_{s-1/2}\right). \tag{24}$$

By the first identity in (22) it follows that

$$\begin{aligned}|II|+|III| &\leq \sum_{k=1}^{m}\sum_{i=1}^{n_k}\left|\left(\tilde{\zeta}\widetilde{R}^0_{s,\delta}\Lambda^{s-1}\left(\lambda_k b^k_i\right)_0\frac{\partial}{\partial x^k_i}S_\delta\zeta u,\tilde{\zeta}\Lambda^s S_\delta\zeta u\right)\right| \\ &\quad+\sum_{k=1}^{m}\left|\left(\tilde{\zeta}\widetilde{R}^0_{s,\delta}\Lambda^{s-1}\left(\lambda_k c^k\right)_0 S_\delta\zeta u,\tilde{\zeta}\Lambda^s S_\delta\zeta u\right)\right| \\ &\leq C\left(\left\|\tilde{\zeta}S_\delta\zeta u\right\|^2_s+\|\zeta' S_\delta\zeta u\|^2_{s-1}\right). \end{aligned} \tag{25}$$

Using that $\zeta_{x^k_i x^k_j}=\zeta^k_0\ \zeta_{x^k_i x^k_j}$ and $\zeta_{x^k_i}=\zeta^k_0\zeta_{x^k_i}$, and (22), it follows that

$$\begin{aligned}|IV|+|V| &\leq \sum_{k=1}^{m}\sum_{i,j=1}^{n_k}\left|\left(\tilde{\zeta}\Lambda^s S_\delta a^k_{ij}\zeta_{x^k_i x^k_j}\lambda_k\zeta^k_0 u,\tilde{\zeta}\Lambda^s S_\delta\zeta u\right)\right| \\ &\quad+\sum_{k=1}^{m}\sum_{i=1}^{n_k}\left|\left(\tilde{\zeta}\Lambda^s S_\delta b^k_i\zeta_{x^k_i}\lambda_k\zeta^k_0 u,\tilde{\zeta}\Lambda^s S_\delta\zeta u\right)\right| \\ &\leq C\left(\left\|\tilde{\zeta}S_\delta\zeta u\right\|^2_s+\sum_{k=1}^{m}\left\|\lambda_k\zeta^k_0 S_\delta\zeta^k_0 u\right\|^2_s+\left\|\tilde{\zeta}S_\delta\tilde{\zeta}u\right\|^2_{s-1}\right). \end{aligned} \tag{26}$$

Now, for each term in VI we commute the functions $\lambda_k a_{ij}^k \zeta_{x_i^k}$ and ζ, and carry out an integration by parts. We obtain the identity

$$
\begin{aligned}
VI^{kij} &= 2\left(\tilde{\zeta}\Lambda^s S_\delta \lambda_k a_{ij}^k \zeta_{x_i^k} \frac{\partial}{\partial x_j^k} u, \tilde{\zeta}\Lambda^s S_\delta \zeta u\right) \\
&= -\left(\tilde{\zeta}\Lambda^s S_\delta \zeta u, \tilde{\zeta}\Lambda^s S_\delta \left(\lambda_k a_{ij}^k \zeta_{x_i^k}\right)_{x_j^k} u\right) \\
&\quad -2\left(\tilde{\zeta}\Lambda^s S_\delta \zeta u, \tilde{\zeta}_{x_j^k} \Lambda^s S_\delta \lambda_k a_{ij}^k \zeta_{x_i^k} u\right) \\
&\quad -\left(\tilde{\zeta}\Lambda^s S_\delta \zeta_{x_j^k} u, \tilde{\zeta}\Lambda^s S_\delta \lambda_k a_{ij}^k \zeta_{x_i^k} u\right) \\
&\quad +\left(\tilde{\zeta}\left[\zeta, \Lambda^s S_\delta\right] \tilde{\zeta} \frac{\partial}{\partial x_j^k} u, \tilde{\zeta}\Lambda^s S_\delta \lambda_k a_{ij}^k \zeta_{x_i^k} u\right) \\
&\quad +\left(\tilde{\zeta}\Lambda^s S_\delta \tilde{\zeta} \frac{\partial}{\partial x_j^k} u, \tilde{\zeta}\left[\Lambda^s S_\delta, \zeta\right] \lambda_k a_{ij}^k \zeta_{x_i^k} u\right) \\
&\quad +\left(\tilde{\zeta}\Lambda^s S_\delta \tilde{\zeta} \frac{\partial}{\partial x_j^k} u, \tilde{\zeta}\left[\lambda_k a_{ij}^k \zeta_{x_i^k}, \Lambda^s S_\delta\right] \zeta u\right) \\
&\quad +\left(\tilde{\zeta}\left[\Lambda^s S_\delta, \lambda_k a_{ij}^k \zeta_{x_i^k}\right] \tilde{\zeta} \frac{\partial}{\partial x_j^k} u, \tilde{\zeta}\Lambda^s S_\delta \zeta u\right) \\
&= VI_1^{kij} + VI_2^{kij} + \cdots + VI_7^{kij}.
\end{aligned} \tag{27}
$$

We now consider each term. We have

$$
\begin{aligned}
\left|VI_1^{kij}\right| &\leq \left|\left(\tilde{\zeta}\Lambda^s S_\delta \zeta u, \tilde{\zeta}\Lambda^s S_\delta \lambda_k \left(a_{ij}^k \zeta_{x_i^k}\right)_{x_j^k} u\right)\right| \\
&\quad + \left|\left(\tilde{\zeta}\Lambda^s S_\delta \zeta u, \tilde{\zeta}\Lambda^s S_\delta \left(\lambda_k\right)_{x_j^k} a_{ij}^k \zeta_{x_i^k} u\right)\right| \\
&\leq C\left\{\left\|\tilde{\zeta} S_\delta \zeta u\right\|_s^2 + \left\|\zeta' S_\delta \zeta u\right\|_{s-1}^2 + \left\|\lambda_k \zeta_0^k S_\delta \zeta_0^k u\right\|_s^2\right. \\
&\quad \left. + \left\|\left(\lambda_k\right)_{x_j^k} \zeta_0^k S_\delta \zeta_{x_i^k} u\right\|_s^2 + \left\|\tilde{\zeta} S_\delta \tilde{\zeta} u\right\|_{s-1}^2\right\}.
\end{aligned} \tag{28}
$$

We will use the following Wirtinger-type inequality (see e.g. Appendix in [22]): If $\phi \in C^2(U)$ with U open in $\mathbb{R}^n$, ϕ nonnegative, then for any compact subset $F \subset U$ there exists a constant C depending on $\|D^2\phi\|_{L^\infty(V)}$, with V

open and $F \subset V \Subset U$, and $\operatorname{dist}(F, \partial V) > 0$ such that

$$|D\phi(x)|^2 \le C\phi(x). \tag{29}$$

We consider the penultimate term in (28),

$$\left\|(\lambda_k)_{x_j^k} \zeta_0^k S_\delta \zeta_{x_i^k} u\right\|_s^2 = \left(\Lambda^s (\lambda_k)_{x_j^k} \zeta_0^k S_\delta \zeta_{x_i^k} u, \Lambda^s (\lambda_k)_{x_j^k} \zeta_0^k S_\delta \zeta_{x_i^k} u\right).$$

We commute $\zeta_{x_i^k}$ from the right into the left and $(\lambda_k)_{x_j^k}$ from the left into the right. We obtain

$$\begin{aligned}
&\left\|(\lambda_k)_{x_j^k} \zeta_0^k S_\delta \zeta_{x_i^k} u\right\|_s^2 \\
= &\left(\Lambda^s \zeta_0^k S_\delta \left(\zeta_{x_i^k}\right)^2 u, \Lambda^s \left((\lambda_k)_{x_j^k}\right)^2 \zeta_0^k S_\delta \zeta_0^k u\right) \\
&+ \left(\Lambda^s \zeta_0^k S_\delta \left(\zeta_{x_i^k}\right)^2 u, \left[(\lambda_k)_{x_j^k}, \Lambda^s\right] (\lambda_k)_{x_j^k} \zeta_0^k S_\delta \zeta_0^k u\right) \\
&+ \left(\left[\Lambda^s, (\lambda_k)_{x_j^k}\right] \zeta_0^k S_\delta \left(\zeta_{x_i^k}\right)^2 u, \Lambda^s (\lambda_k)_{x_j^k} \zeta_0^k S_\delta \zeta_0^k u\right) \\
&+ \left(\Lambda^s (\lambda_k)_{x_j^k} \zeta_0^k \left[\zeta_{x_i^k}, S_\delta\right] \zeta_{x_i^k} u, \Lambda^s (\lambda_k)_{x_j^k} \zeta_0^k S_\delta \zeta_0^k u\right) \\
&+ \left(\left[\zeta_{x_i^k}, \Lambda^s\right] (\lambda_k)_{x_j^k} \zeta_0^k S_\delta \zeta_{x_i^k} u, \Lambda^s (\lambda_k)_{x_j^k} \zeta_0^k S_\delta \zeta_0^k u\right) \\
&+ \left(\Lambda^s (\lambda_k)_{x_j^k} \zeta_0^k S_\delta \zeta_{x_i^k} u, \left[\Lambda^s, \zeta_{x_i^k}\right] (\lambda_k)_{x_j^k} \zeta_0^k S_\delta \zeta_0^k u\right) \\
&+ \left(\Lambda^s (\lambda_k)_{x_j^k} \zeta_0^k S_\delta \zeta_{x_i^k} u, \Lambda^s (\lambda_k)_{x_j^k} \zeta_0^k \left[S_\delta, \zeta_{x_i^k}\right] \zeta_0^k u\right) \\
\le &\left\|\zeta_0^k S_\delta \left(\zeta_{x_i^k}\right)^2 u\right\|_s^2 + \left\|\left((\lambda_k)_{x_j^k}\right)^2 \zeta_0^k S_\delta \zeta_0^k u\right\|_s^2 + C\left\|\tilde{\zeta} S_\delta \tilde{\zeta} u\right\|_{s-1/2}^2.
\end{aligned} \tag{30}$$

The first term on the right is bounded by

$$\begin{aligned}
\left\|\zeta_0^k S_\delta \left(\zeta_{x_i^k}\right)^2 u\right\|_s^2 \le\ &\left\|\left(\zeta_{x_i^k}\right)^2 \Lambda^s \zeta_0^k S_\delta \tilde{\zeta} u\right\|^2 + \left\|\left[\Lambda^s, \left(\zeta_{x_i^k}\right)^2\right] \zeta_0^k S_\delta \tilde{\zeta} u\right\|^2 \\
&+ \left\|\Lambda^s \zeta_0^k \left[S_\delta, \left(\zeta_{x_i^k}\right)^2\right] \tilde{\zeta} u\right\|^2.
\end{aligned}$$

By the Wirtinger inequality (29), it follows that

$$\left\|\zeta_0^k S_\delta \left(\zeta_{x_i^k}\right)^2 u\right\|_s^2 \le C\left\{\left\|\zeta \Lambda^s \zeta_0^k S_\delta \tilde{\zeta} u\right\|^2 + \left\|\tilde{\zeta} S_\delta \tilde{\zeta} u\right\|_{s-1}^2\right\}$$

$$\leq \quad C\left\{\left\|\tilde{\zeta}S_\delta\zeta u\right\|_s^2+\left\|\tilde{\zeta}S_\delta\tilde{\zeta}u\right\|_{s-1}^2\right\}. \qquad (31)$$

Similarly, the second term on the right of (30) is bounded by

$$\left\|\left((\lambda_k)_{x_j^k}\right)^2\zeta_0^k S_\delta\zeta_0^k u\right\|_s^2 \leq C\left\{\left\|\lambda_k\zeta_0^k S_\delta\zeta_0^k u\right\|_s^2+\left\|\tilde{\zeta}S_\delta\tilde{\zeta}u\right\|_{s-1}^2\right\}.$$

We obtain

$$\left\|(\lambda_k)_{x_j^k}\zeta_0^k S_\delta\zeta_{x_i^k}u\right\|_s^2 \leq C\left\{\left\|\tilde{\zeta}S_\delta\zeta u\right\|_s^2+\left\|\lambda_k\zeta_0^k S_\delta\zeta_0^k u\right\|_s^2+\left\|\tilde{\zeta}S_\delta\tilde{\zeta}u\right\|_{s-1/2}^2\right\}.$$

Plugging this estimate on the right of (28) yields

$$\begin{aligned}\left|VI_1^{kij}\right| \quad &\leq \quad C\left\{\left\|\tilde{\zeta}S_\delta\zeta u\right\|_s^2+\left\|\zeta' S_\delta\zeta u\right\|_{s-1}^2+\left\|\lambda_k\zeta_0^k S_\delta\zeta_0^k u\right\|_s^2\right\} \qquad (32)\\ &\quad +C\left\|\tilde{\zeta}S_\delta\tilde{\zeta}u\right\|_{s-1/2}^2.\end{aligned}$$

It easily follows that

$$\begin{aligned}\left|VI_2^{kij}\right| \quad &\leq \quad 2\left|\left(\tilde{\zeta}\Lambda^s S_\delta\zeta u,\tilde{\zeta}_{x_j^k}\Lambda^s S_\delta\lambda_k a_{ij}^k\zeta_{x_i^k}u\right)\right|\\ &\leq \quad \left\{\left\|\tilde{\zeta}S_\delta\zeta u\right\|_s^2+\left\|\lambda_k\zeta_0^k S_\delta\zeta_0^k u\right\|_s^2+\left\|\zeta' S_\delta\tilde{\zeta}u\right\|_{s-1}^2\right\}. \qquad (33)\end{aligned}$$

Commuting $\zeta_{x_i^k}$ into the left, we have that

$$\begin{aligned}\left|VI_3^{kij}\right| \quad &\leq \quad \left|\left(\tilde{\zeta}\Lambda^s S_\delta\zeta_{x_j^k}\zeta_{x_i^k}u,\tilde{\zeta}\Lambda^s S_\delta\lambda_k a_{ij}^k\left(\zeta_0^k\right)^2u\right)\right|\\ &\quad +\left|\left(\tilde{\zeta}\left[\zeta_{x_i^k},\Lambda^s S_\delta\right]\zeta_{x_j^k}u,\tilde{\zeta}\Lambda^s S_\delta\lambda_k a_{ij}^k\left(\zeta_0^k\right)^2u\right)\right|\\ &\quad +\left|\left(\tilde{\zeta}\Lambda^s S_\delta\zeta_{x_j^k}\left(\zeta_0^k\right)^2u,\tilde{\zeta}\left[\Lambda^s S_\delta,\zeta_{x_i^k}\right]\lambda_k a_{ij}^k u\right)\right|\\ &\leq \quad C\left\{\left\|\tilde{\zeta}S_\delta\left(\zeta_{x_j^k}\right)^2u\right\|_s^2+\left\|\lambda_k\zeta_0^k S_\delta\zeta_0^k u\right\|_s^2+\left\|\tilde{\zeta}S_\delta\tilde{\zeta}u\right\|_{s-1}^2\right\}.\end{aligned}$$

Applying (31) to the first term on the right we obtain

$$\left|VI_3^{kij}\right| \leq C\left\{\left\|\tilde{\zeta}S_\delta\zeta u\right\|_s^2+\left\|\lambda_k\zeta_0^k S_\delta\zeta_0^k u\right\|_s^2+\left\|\zeta' S_\delta\tilde{\zeta}u\right\|_{s-1}^2\right\}. \qquad (34)$$

Using the identity (22) for $[\zeta, \Lambda^s S_\delta]$, we obtain

$$\begin{aligned}
\left|VI_4^{kij}\right| &\leq \sum_{r=1}^{m}\sum_{\ell=1}^{n_k}\left|\left(\tilde{\zeta}R^0_{s,\delta}\Lambda^{s-2}\zeta_{x^r_\ell}\frac{\partial}{\partial x^r_\ell}S_\delta\tilde{\zeta}\frac{\partial}{\partial x^k_j}u, \tilde{\zeta}\Lambda^s S_\delta\lambda_k a^k_{ij}\zeta_{x^k_i}u\right)\right| \\
&\quad +\left|\left(\tilde{\zeta}Q^{-0}_{s,\delta}\Lambda^{s-2}\zeta_0 S_\delta\tilde{\zeta}\frac{\partial}{\partial x^k_j}u, \tilde{\zeta}\Lambda^s S_\delta\lambda_k a^k_{ij}\zeta_{x^k_i}u\right)\right| \\
&\leq \sum_{r=1}^{m}\sum_{\ell=1}^{n_k}\left|\left(\tilde{\zeta}R^0_{s,\delta}\Lambda^{s-2}\zeta_{x^r_\ell}\frac{\partial}{\partial x^r_\ell}S_\delta\tilde{\zeta}\frac{\partial}{\partial x^k_j}u, \tilde{\zeta}\Lambda^s S_\delta\lambda_k a^k_{ij}\zeta_{x^k_i}u\right)\right| \qquad (35) \\
&\quad +C\left\{\left\|\lambda_k\zeta^k_0 S_\delta\zeta^k_0 u\right\|^2_s + \|\zeta' S_\delta\zeta' u\|^2_{s-1}\right\}.
\end{aligned}$$

We treat the first term on the right of (35) in a similar way as we obtain (30), we commute $\zeta_{x^k_i}$ into the left. We proceed in the same way with VI_5^{kij}, to obtain

$$\left|VI_4^{kij}\right| + \left|VI_5^{kij}\right| \leq C\left\{\left\|\tilde{\zeta}S_\delta\zeta u\right\|^2_s + \left\|\lambda_k\zeta^k_0 S_\delta\zeta^k_0 u\right\|^2_s + \|\zeta' S_\delta\zeta' u\|^2_{s-1/2}\right\}. \qquad (36)$$

The treatment of VI_6^{kij} and VI_7^{kij} is similar. Applying (22) to $\left[\lambda_k a^k_{ij}\zeta_{x^k_i}, \Lambda^s S_\delta\right]$, we obtain

$$\begin{aligned}
\left|VI_6^{kij}\right| &\leq \sum_{r=1}^{m}\sum_{\ell=1}^{n_k}\left|\left(\tilde{\zeta}\Lambda^s S_\delta\tilde{\zeta}\frac{\partial}{\partial x^k_j}u, \tilde{\zeta}R^0_{s,\delta}\Lambda^{s-2}\left(\lambda_k a^k_{ij}\zeta_{x^k_i}\right)_{x^r_\ell}\frac{\partial}{\partial x^r_\ell}S_\delta\zeta u\right)\right| \\
&\quad +\left|\left(\tilde{\zeta}\Lambda^s S_\delta\tilde{\zeta}\frac{\partial}{\partial x^k_j}u, \tilde{\zeta}Q^{-0}_{s,\delta}\Lambda^{s-2}\zeta^1_0 S_\delta\zeta u\right)\right| \\
&\leq \sum_{r=1}^{m}\sum_{\ell=1}^{n_k}\left|\left(\tilde{\zeta}\Lambda^s S_\delta\tilde{\zeta}\frac{\partial}{\partial x^k_j}u, \tilde{\zeta}R^0_{s,\delta}\Lambda^{s-2}\left(\lambda_k a^k_{ij}\zeta_{x^k_i}\right)_{x^r_\ell}\frac{\partial}{\partial x^r_\ell}S_\delta\zeta u\right)\right| \qquad (37) \\
&\quad +C\left\{\left\|\tilde{\zeta}S_\delta\zeta u\right\|^2_s + \|\zeta' S_\delta\zeta' u\|^2_{s-1}\right\}.
\end{aligned}$$

We write the terms in the sum above as

$$\begin{aligned}
&\left(\tilde{\zeta}\Lambda^s S_\delta\tilde{\zeta}\frac{\partial}{\partial x^k_j}u, \tilde{\zeta}R^0_{s,\delta}\Lambda^{s-2}\left(\lambda_k a^k_{ij}\zeta_{x^k_i}\right)_{x^r_\ell}\frac{\partial}{\partial x^r_\ell}S_\delta\zeta u\right) \\
&= -\left(\frac{\partial}{\partial x^r_\ell}\left(\lambda_k a^k_{ij}\zeta_{x^k_i}\right)_{x^r_\ell}\Lambda^{-2}R^0_{s,\delta}\tilde{\zeta}\Lambda^s S_\delta\tilde{\zeta}\frac{\partial}{\partial x^k_j}u, \Lambda^s\tilde{\zeta}S_\delta\zeta u\right)
\end{aligned}$$

$$
\begin{aligned}
&+\left(\tilde{\zeta}\Lambda^s S_\delta \tilde{\zeta}\frac{\partial}{\partial x_j^k}u, R^0_{s,\delta}\left[\Lambda^s\tilde{\zeta}, \Lambda^{-2}\left(\lambda_k a_{ij}^k \zeta_{x_i^k}\right)_{x_\ell^r}\frac{\partial}{\partial x_\ell^r}\right]S_\delta\zeta u\right)\\
&+\left(\tilde{\zeta}\Lambda^s S_\delta \tilde{\zeta}\frac{\partial}{\partial x_j^k}u, \left[\tilde{\zeta}, R^0_{s,\delta}\Lambda^{s-2}\right]\Lambda^{-2}\left(\lambda_k a_{ij}^k \zeta_{x_i^k}\right)_{x_\ell^r}\frac{\partial}{\partial x_\ell^r}S_\delta\zeta u\right).
\end{aligned}
$$

Plugging this into (37), we obtain

$$
\begin{aligned}
\left|VI_6^{kij}\right| \quad \leq \quad & \sum_{r=1}^{m}\sum_{\ell=1}^{n_k}\left|\left(\frac{\partial}{\partial x_\ell^r}\left(\lambda_k a_{ij}^k\zeta_{x_i^k}\right)_{x_\ell^r}\Lambda^{-2}R^0_{s,\delta}\tilde{\zeta}\Lambda^s S_\delta\tilde{\zeta}\frac{\partial}{\partial x_j^k}u, \Lambda^s\tilde{\zeta}S_\delta\zeta u\right)\right|\\
&+C\left\{\left\|\tilde{\zeta}S_\delta\zeta u\right\|_s^2+\|\zeta' S_\delta\zeta' u\|_{s-1/2}^2\right\}.
\end{aligned}
$$

We estimate the first term on the right in the same way as we did (28-30). Since VI_7^{kij} can be estimated by the same procedure, we obtain

$$
\left|VI_6^{kij}\right|+\left|VI_7^{kij}\right| \leq C\left\{\left\|\tilde{\zeta}S_\delta\zeta u\right\|_s^2+\left\|\lambda_k\zeta_0^k S_\delta\zeta_0^k u\right\|_s^2+\|\zeta' S_\delta\zeta' u\|_{s-1/2}^2\right\}. \quad (38)
$$

Combining estimates (32), (33), (34), (36), and (38) yields

$$
\begin{aligned}
|VI| \quad &\leq \quad 2\sum_{k=1}^{m}\sum_{i,j=1}^{n_k}\left|VI^{kij}\right|\\
&\leq \quad C\left\{\left\|\tilde{\zeta}S_\delta\zeta u\right\|_s^2+\sum_{k=1}^{m}\left\|\lambda_k\zeta_0^k S_\delta\zeta_0^k u\right\|_s^2+\|\zeta' S_\delta\zeta' u\|_{s-1/2}^2\right\}. \quad (39)
\end{aligned}
$$

Applying (24), (25), (26), and (39) to the right of (21) we obtain the conclusion of the lemma. $\square$

Corollary 3.4. *Under the assumptions of Lemma 3.3 the following estimate holds*

$$
\begin{aligned}
\sum_{k=1}^{m}\left\|\sqrt{\lambda_k}\Lambda^\varepsilon_{x^k}\tilde{\zeta}\Lambda^s S_\delta\zeta u\right\|^2 \quad \leq \quad & C\left\{\left\|\tilde{\zeta}S_\delta\zeta u\right\|_s^2+\left|\left(\tilde{\zeta}\Lambda^s S_\delta\tilde{\zeta}Lu, \tilde{\zeta}\Lambda^s S_\delta\zeta u\right)\right|\right.\\
&\left.+\sum_{k=1}^{m}\left\|\lambda_k\zeta_0^k S_\delta\zeta_0^k u\right\|_s^2+\|\zeta' S_\delta\zeta' u\|_{s-1/2}^2\right\}.
\end{aligned}
$$

for all $u \in H^{s-1}(\mathbb{R}^n)$.

Proof. Since $u \in H^{s-1}(\mathbb{R}^n)$ we have that $\Lambda^s S_\delta \zeta u \in H^2(\mathbb{R}^n)$ so we can replace ζ by $\tilde{\zeta}$ and u by $\Lambda^s S_\delta \zeta u$ in Lemma 3.2 to obtain

$$\sum_{k=1}^{m} \left\| \sqrt{\lambda_k} \Lambda^{\varepsilon}_{x^k} \left(\tilde{\zeta} \Lambda^s S_\delta \zeta u \right) \right\|^2 \leq C \left\{ \left| \left(\tilde{\zeta} L \Lambda^s S_\delta \zeta u, \tilde{\zeta} \Lambda^s S_\delta \zeta u \right) \right| + \|\zeta' \Lambda^s S_\delta \zeta u\|^2 \right\}.$$

The corollary follows by commuting L with $\Lambda^s S_\delta \zeta$ and applying Lemma 3.3. $\square$

Lemma 3.5. *There exists $C > 0$ such that*

$$\sum_{k=1}^{m} \left\| \lambda_k \zeta_0^k S_\delta \zeta_0^k u \right\|_s^2 \leq C \left\{ \left| \left(\zeta' \Lambda^{s-\varepsilon} S_\delta \zeta' L u, \zeta' \Lambda^{s-\varepsilon} S_\delta \tilde{\zeta} u \right) \right| + \|\zeta'' S_\delta \zeta'' u\|_{s-\varepsilon}^2 \right\}$$

Proof. We have

$$\begin{aligned} \sum_{k=1}^{m} \left\| \lambda_k \zeta_0^k S_\delta \zeta_0^k u \right\|_s^2 &= \sum_{k=1}^{m} \left\| \Lambda^s \lambda_k \zeta_0^k S_\delta \zeta_0^k u \right\|^2 \\ &\leq \sum_{k=1}^{m} \left\| \Lambda^{\varepsilon} \lambda_k \zeta_0^k \Lambda^{s-\varepsilon} S_\delta \tilde{\zeta} u \right\|^2 + C \left\| \zeta' S_\delta \tilde{\zeta} u \right\|_{s-1}^2 \\ &\leq C \sum_{k,\ell=1}^{m} \left\| \Lambda^{\varepsilon}_{x^\ell} \lambda_k \zeta_0^k \Lambda^{s-\varepsilon} S_\delta \tilde{\zeta} u \right\|^2 + C \left\| \zeta' S_\delta \tilde{\zeta} u \right\|_{s-1}^2 \\ &\leq C \sum_{k,\ell=1}^{m} \left\| \lambda_k \zeta_0^k \Lambda^{\varepsilon}_{x^\ell} \tilde{\zeta} \Lambda^{s-\varepsilon} S_\delta \tilde{\zeta} u \right\|^2 + C \left\| \zeta' S_\delta \tilde{\zeta} u \right\|_{s-1}^2 \end{aligned} \tag{40}$$

Now we recall that by the hypotheses on λ_k, we have that if $1 \leq \ell \leq m$, and $\ell \neq k$ then $\lambda_\ell(x) > 0$ for all x in an open neighbourhood of the support of ζ_0^k. Hence

$$\delta_k = \min_{\substack{1 \leq \ell \leq m \\ \ell \neq k}} \inf_{x \in \text{support}\left(\zeta_0^k\right)} \lambda_\ell(x) > 0, \qquad 1 \leq \ell \leq m.$$

Consequently,

$$\lambda_k \zeta_0^k \leq \frac{\|\lambda_k\|_\infty}{\delta_k} \lambda_\ell(x) \zeta_0^k, \qquad 1 \leq \ell, k \leq m. \tag{41}$$

We apply (41) to the right side of (40) to obtain

$$\begin{aligned}\sum_{k=1}^{m}\left\|\lambda_k\zeta_0^k S_\delta\zeta_0^k u\right\|_s^2 &\le C\sum_{k=1}^{m}\left\|\lambda_k\zeta_0^k\Lambda_{x^k}^{\varepsilon}\tilde{\zeta}\Lambda^{s-\varepsilon}S_\delta\tilde{\zeta}u\right\|^2+C\left\|\zeta' S_\delta\tilde{\zeta}u\right\|_{s-1}^2\\ &\le C\sum_{k=1}^{m}\left\|\sqrt{\lambda_k}\Lambda_{x^k}^{\varepsilon}\tilde{\zeta}\Lambda^{s-\varepsilon}S_\delta\tilde{\zeta}u\right\|^2+C\left\|\zeta' S_\delta\tilde{\zeta}u\right\|_{s-1}^2.\end{aligned}$$

We apply Corollary 3.4 with $\tilde{\zeta}$ instead of ζ, ζ' instead of $\tilde{\zeta}$, etc., to the first term on the right to obtain

$$\begin{aligned}\sum_{k=1}^{m}\left\|\lambda_k\zeta_0^k S_\delta\zeta_0^k u\right\|_s^2 &\le C\sum_{k=1}^{m}\left\|\sqrt{\lambda_k}\Lambda_{x^k}^{\varepsilon}\tilde{\zeta}\Lambda^{s-\varepsilon}S_\delta\tilde{\zeta}u\right\|^2+C\left\|\zeta' S_\delta\tilde{\zeta}u\right\|_{s-1}^2\\ &\le C\left\{\left\|\zeta' S_\delta\tilde{\zeta}u\right\|_{s-\varepsilon}^2+\left|\left(\zeta'\Lambda^{s-\varepsilon}S_\delta\zeta' Lu,\zeta'\Lambda^{s-\varepsilon}S_\delta\tilde{\zeta}u\right)\right|\right.\\ &\quad\left.+\sum_{k=1}^{m}\left\|\lambda_k\tilde{\zeta}_0^k S_\delta\tilde{\zeta}_0^k u\right\|_{s-\varepsilon}^2+\left\|\zeta'' S_\delta\zeta'' u\right\|_{s-\varepsilon-1/2}^2+C\left\|\zeta' S_\delta\tilde{\zeta}u\right\|_{s-1}^2\right\}.\\ &\le C\left\{\left|\left(\zeta'\Lambda^{s-\varepsilon}S_\delta\zeta' Lu,\zeta'\Lambda^{s-\varepsilon}S_\delta\tilde{\zeta}u\right)\right|+\left\|\zeta'' S_\delta\zeta'' u\right\|_{s-\varepsilon}^2\right\}.\end{aligned}$$

□

The last auxiliary result we need is the following Poincaré-type inequality.

Lemma 3.6 (Poincaré-type inequality)**.** *For every* $0<\varepsilon<1$ *and cutoff* $\zeta=\prod_{k=1}^{m}\sigma_k\left(x^k\right)$ *as in Definition (2.7) such that* $d_0=\operatorname{diam}(\operatorname{support}\sigma_1)\le 1$, *there exists* $C>0$ *such that for any* $u\in H^{s+1}(\mathbb{R}^n)$

$$\left\|\tilde{\zeta}\Lambda^s\zeta u\right\|^2\le C d_0^{\frac{\varepsilon}{1+\varepsilon}}\left(\left\|\Lambda_{x^1}^{\varepsilon}\tilde{\zeta}\Lambda^s\zeta u\right\|^2+d_0^{-2}\left\|\zeta u\right\|_{s-1}^2\right).$$

Proof. We can write

$$\left\|\tilde{\zeta}\Lambda^s\zeta u\right\|^2=\left(\Lambda_{x^1}^{\varepsilon}\tilde{\zeta}\Lambda^s\zeta u,\Lambda_{x^1}^{-\varepsilon}\tilde{\zeta}\Lambda^s\zeta u\right)$$

Using Young's inequality, we obtain

$$\left\|\tilde{\zeta}\Lambda^s\zeta u\right\|^2\le a_0\left\|\Lambda_{x^1}^{\varepsilon}\tilde{\zeta}\Lambda^s\zeta u\right\|^2+\frac{1}{4a_0}\left\|\Lambda_{x^1}^{-\varepsilon}\tilde{\zeta}\Lambda^s\zeta u\right\|^2$$

Inductively, we have

$$\left\|\Lambda_{x^1}^{-\left(2^j-1\right)\varepsilon}\tilde{\zeta}\Lambda^s\zeta u\right\|^2 \leq a_j\left\|\Lambda_{x^1}^{\varepsilon}\tilde{\zeta}\Lambda^s\zeta u\right\|^2 + \frac{1}{4a_j}\left\|\Lambda_{x^1}^{-\left(2^{j+1}-1\right)\varepsilon}\tilde{\zeta}\Lambda^s\zeta u\right\|^2.$$

Taking $0 < a_0 < 1$, $a_j = 4^{2^j-1}a_0^{2^{(j+1)}-1}$, $j = 1, \ldots, N$ and combining the corresponding estimates yields

$$\left\|\tilde{\zeta}\Lambda^s\zeta u\right\|^2 \leq \frac{a_0}{1-a_0}\left\|\Lambda_{x^1}^{\varepsilon}\tilde{\zeta}\Lambda^s\zeta u\right\|^2 + \frac{1}{4^{2^N-1}\left(a_0\right)^{2^{N+1}-N-2}}\left\|\Lambda_{x^1}^{-\left(2^{N+1}-1\right)\varepsilon}\tilde{\zeta}\Lambda^s\zeta u\right\|^2.$$

We choose the integer N such that $\left(2^{N+1}-1\right)\varepsilon \geq 1 > \left(2^N-1\right)\varepsilon$. Using the monotonicity of the $||\cdot||_s$ norms, and Poincaré's inequality with respect to x^1 we have

$$\begin{aligned}
\left\|\Lambda_{x^1}^{-\left(2^{N+1}-1\right)\varepsilon}\tilde{\zeta}\Lambda^s\zeta u\right\|^2 &\leq \left\|\Lambda_{x^1}^{-1}\Lambda_{x^1}^{\varepsilon}\tilde{\zeta}\Lambda^s\zeta u\right\|^2 \leq \left\|\zeta'\Lambda_{x^1}^{-1}\Lambda_{x^1}^{\varepsilon}\tilde{\zeta}\Lambda^s\zeta u\right\|^2 + C\left\|\zeta u\right\|_{s-1}^2 \\
&\leq d_0^2\left\|\nabla_{x^1}\zeta'\Lambda_{x^1}^{-1}\Lambda_{x^1}^{\varepsilon}\tilde{\zeta}\Lambda^s\zeta u\right\|^2 + C\left\|\zeta u\right\|_{s-1}^2 \\
&\leq d_0^2\left\|\Lambda_{x^1}^{\varepsilon}\tilde{\zeta}\Lambda^s\zeta u\right\|^2 + C\left\|\zeta u\right\|_{s-1}^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left\|\tilde{\zeta}\Lambda^s\zeta u\right\|^2 &\leq \left(\frac{a_0}{1-a_0} + \frac{d_0^2}{4^{2^N-1}\left(a_0\right)^{2^{N+1}-N-2}}\right)\left\|\Lambda_{x^1}^{\varepsilon}\tilde{\zeta}\Lambda^s\zeta u\right\|^2 \\
&\quad + \frac{C}{4^{2^N-1}\left(a_0\right)^{2^{N+1}-N-2}}\left\|\zeta u\right\|_{s-1}^2
\end{aligned}$$

Taking $a_0 = \frac{d_0^2}{4^{2^N-1}(a_0)^{2^{N+1}-N-2}}$ and using that $2^{N+1} > 2\left(1+\frac{1}{\varepsilon}\right)$, we have

$$a_0 = \frac{d_0^{\frac{2}{2^{N+1}-N-1}}}{4^{\frac{2^N-1}{2^{N+1}-N-1}}} \leq \frac{d_0^{\frac{2}{2^{N+1}}}}{2} \leq \frac{d_0^{\frac{\varepsilon}{1+\varepsilon}}}{2} \leq \frac{1}{2}.$$

Hence,

$$\begin{aligned}
\left\|\tilde{\zeta}\Lambda^s\zeta u\right\|^2 &\leq a_0\left(\frac{2-a_0}{1-a_0}\right)\left\|\Lambda_{x^1}^{\varepsilon}\tilde{\zeta}\Lambda^s\zeta u\right\|^2 + \frac{C}{4^{2^N-1}\left(a_0\right)^{2^{N+1}-N-2}}\left\|\zeta u\right\|_{s-1}^2 \\
&\leq 2d_0^{\frac{\varepsilon}{1+\varepsilon}}\left\|\Lambda_{x^1}^{\varepsilon}\tilde{\zeta}\Lambda^s\zeta u\right\|^2 + Cd_0^{\frac{\varepsilon}{1+\varepsilon}-2}\left\|\zeta u\right\|_{s-1}^2.
\end{aligned}$$

$\square$

Lemma 3.7. *For each $0 < \varepsilon < 1$ there exist $C > 0$ such that for all $u \in H^{s-1}(\mathbb{R}^n)$ and all $0 < \delta \le 1$*

$$\begin{aligned} \|S_\delta \zeta u\|_s^2 &\le C d_0^{\varepsilon/2} \left| \left(\tilde{\zeta} \Lambda^s S_\delta \tilde{\zeta} L u, \tilde{\zeta} \Lambda^s S_\delta \zeta u \right) \right| \\ & \quad C d_0^{\varepsilon/2} \left| \left(\zeta' \Lambda^{s-\varepsilon} S_\delta \zeta' L u, \zeta' \Lambda^{s-\varepsilon} S_\delta \tilde{\zeta} u \right) \right| \\ & \quad + \left(C d_0^{-2(1-\varepsilon/4)} + 1 \right) \|S_\delta \zeta'' u\|_{s-\varepsilon}^2 , \end{aligned} \tag{42}$$

where d_0 is as in Lemma 3.6. We also have that

$$\|S_\delta \zeta u\|_s^2 \le C \left\{ d_0^{\varepsilon/2} \|S_\delta \zeta' L u\|_s^2 + \left(d_0^{-2(1-\varepsilon/4)} + 1 \right) \|S_\delta \zeta'' u\|_{s-\varepsilon}^2 \right\} . \tag{43}$$

Proof. We have

$$\begin{aligned} \|S_\delta \zeta u\|_s^2 &= \left\| \Lambda^s S_\delta \tilde{\zeta}^2 \zeta u \right\|^2 \le \left\| \tilde{\zeta} \Lambda^s S_\delta \tilde{\zeta} \zeta u \right\|^2 + C \|S_\delta \zeta u\|_{s-1}^2 \\ &\le \left\| \tilde{\zeta} \Lambda^s \zeta S_\delta \tilde{\zeta} u \right\|^2 + C \left\| S_\delta \tilde{\zeta} u \right\|_{s-1}^2 . \end{aligned}$$

By Property (iii) in Lemma 2.9 we have that $\tilde{\zeta} \Lambda^s \zeta S_\delta \tilde{\zeta} u \in H^{s+1}(\mathbb{R}^n)$; applying Poincaré's inequality, Lemma 3.6, we obtain

$$\begin{aligned} \|S_\delta \zeta u\|_s^2 &\le C d_0^{\varepsilon/2} \left\| \Lambda_{x^1}^\varepsilon \tilde{\zeta} \Lambda^s \zeta S_\delta \tilde{\zeta} u \right\|^2 + C \left(d_0^{\varepsilon/2-2} + 1 \right) \left\| S_\delta \tilde{\zeta} u \right\|_{s-1}^2 \\ &\le C d_0^{\varepsilon/2} \left\| \Lambda_{x^1}^\varepsilon \tilde{\zeta} \Lambda^s S_\delta \zeta u \right\|^2 + C \left(d_0^{\varepsilon/2-2} + 1 \right) \left\| S_\delta \tilde{\zeta} u \right\|_{s-1}^2 . \end{aligned}$$

Where we used that $0 < \varepsilon, d_0 \le 1$.

Since $\lambda_1 \equiv 1$, by Corollary 3.4 we then have

$$\begin{aligned} \|S_\delta \zeta u\|_s^2 &\le C d_0^{\varepsilon/2} \left\{ \|S_\delta \zeta u\|_s^2 + \left| \left(\tilde{\zeta} \Lambda^s S_\delta \tilde{\zeta} L u, \tilde{\zeta} \Lambda^s S_\delta \zeta u \right) \right| + \sum_{k=1}^m \left\| \lambda_k \zeta_0^k S_\delta \zeta_0^k u \right\|_s^2 \right\} \\ & \quad + C \left(d_0^{\varepsilon/2-2} + 1 \right) \|S_\delta \zeta' u\|_{s-1}^2 . \end{aligned}$$

Taking ε small enough, we absorb the first term on the right into the left, and apply Lemma 3.5 to the third term on the right. We get

$$\begin{aligned} \|S_\delta \zeta u\|_s^2 &\le C d_0^{\varepsilon/2} \left\{ \left| \left(\tilde{\zeta} \Lambda^s S_\delta \tilde{\zeta} L u, \tilde{\zeta} \Lambda^s S_\delta \zeta u \right) \right| + \left| \left(\zeta' \Lambda^{s-\varepsilon} S_\delta \zeta' L u, \zeta' \Lambda^{s-\varepsilon} S_\delta \tilde{\zeta} u \right) \right| \right\} \\ & \quad + C \left(d_0^{\varepsilon/2-2} + 1 \right) \|S_\delta \zeta'' u\|_{s-\varepsilon}^2 . \end{aligned}$$

This proves (42). The second inequality, (43), follows from (42) after another absorption into the left. □

4. Proof of Theorem 1.4

Suppose u is a distribution on $\mathbb{R}^n$ such that $u \in H^{s-1}(\mathbb{R}^n)$. If moreover $\zeta L u \in H^s(\mathbb{R}^n)$ for all $\zeta \in C_0^\infty(U)$, then, by (43), for all $0 < \delta \leq 1$ we have

$$\|S_\delta \zeta u\|_s \leq C d_0^{\varepsilon/2} \|S_\delta \zeta' L u\|_s^2 + C\left(d_0^{-2(1-\varepsilon/4)} + 1\right) \|S_\delta \zeta'' u\|_{s-\varepsilon}^2 ,$$

where the cutoff functions $\zeta \prec \zeta' \prec \zeta''$ are supported in U and the constants C, α, d_0 do not depend on δ. Letting $\delta \to 0^+$, by (iv) in Lemma 2.9 we obtain

$$\|\zeta u\|_s \leq C d_0^{\varepsilon/2} \|\zeta' L u\|_s^2 + C\left(d_0^{-2(1-\varepsilon/4)} + 1\right) \|\zeta'' u\|_{s-\varepsilon}^2 , \tag{44}$$

This inequality suffices to prove the hypoellipticity, without loss of weak derivatives, of L. Indeed, since $\zeta'' u$ is a compactly supported distribution there exists $s_0 \in \mathbb{R}$ such that $\|\varphi u\|_{s_0} \leq \infty$ for all $\varphi \in C_0^\infty(U)$. If $s_0 \geq s - \varepsilon$ then (44) implies that $\zeta u \in H^s(\mathbb{R}^n)$ for all $\zeta \in C_0^\infty(U)$. On the other hand, if $s_0 < s - \varepsilon$, let N be the positive integer such that $N\varepsilon \leq s - s_0 < (N+1)\varepsilon$. Let $\zeta = \zeta^0 \prec \zeta^1 \prec \zeta^2 \prec \cdots \prec \zeta^{2N}$ be a sequence of cutoff functions supported in U. Iterating (44)we obtain

$$\left\|\zeta^{2(j-1)} u\right\|_{s-(j-1)\varepsilon}^2 \leq C d_0^{\varepsilon/2} \left\|\zeta^{2j-1} L u\right\|_{s-(j-1)\varepsilon}^2 + C\left(d_0^{-2(1-\varepsilon/4)} + 1\right) \left\|\zeta^{2j} u\right\|_{s-j\varepsilon}^2 ,$$

for $j = 1, \ldots, N$. Assembling these estimates yields

$$\begin{aligned} \|\zeta u\|_s \leq\ & C d_0^{\varepsilon/2} \sum_{j=1}^{N} \left(d_0^{-2(1-\varepsilon/4)} + 1\right)^{j-1} \left\|\zeta^{2j-1} L u\right\|_{s-(2j-1)\varepsilon}^2 \\ & + C\left(d_0^{-2(1-\varepsilon/4)} + 1\right)^{2N} \left\|\zeta^{2N} u\right\|_{s-N\varepsilon}^2 . \end{aligned}$$

By the monotonicity of the H^s-norms, and since $s_0 \leq s - N\varepsilon$, it follows that

$$\begin{aligned} \|\zeta u\|_s \leq\ & C N d_0^{\varepsilon/2} \left(d_0^{-2(1-\varepsilon/4)} + 1\right)^{N-1} \left\|\zeta^{2N-1} L u\right\|_{s-\varepsilon}^2 \\ & + C\left(d_0^{-2(1-\varepsilon/4)} + 1\right)^{2N} \left\|\zeta^{2N} u\right\|_{s_0}^2 < \infty \end{aligned}$$

Hence $\zeta u \in H^s(\mathbb{R}^n)$ for all $\zeta \in C_0^\infty(U)$ and this finishes the proof. $\square$

[1] M. Christ, A remark on sums of squares of complex vector fields, (2004) *Author's web page.*

[2] V. S. Fediĭ, On a criterion for hypoellipticity, *Mat. Sb.* **14** (1971), 15–45.

[3] C. Fefferman, D. H. Phong, Subelliptic eigenvalue problems, *Conf. in Honor of A. Zygmund*, Wadsworth Math. Series 1981.

[4] L. Hörmander, Hypoelliptic second order differential equations, *Acta. Math.* **119** (1967), 141–171.

[5] L. Hörmander, The Analysis of Linear Partial Differential Operators I-IV, *Springer-Verlag, Berlin*, 1983.

[6] J. J. Kohn, Hypoellipticity of some degenerate subelliptic operators, *J. Funct. Anal.* **159** (1998), no. 1, 203–216.

[7] J. J. Kohn, Hypoellipticity and loss of derivatives, *Ann. Math.* **162** (2005), 943-986.

[8] J. J. Kohn, Loss of derivatives. *From Fourier analysis and number theory to radon transforms and geometry*, 353–369, Dev. Math., 28, Springer, New York, 2013.

[9] S. Kusuoka and D. Strook, Applications of the Mallavain calculus II, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **32** (1985), no. 1, 1–76.

[10] O. A. Ladyzhenskaya, N. N. Uraltseva, Linear and Quasilinear Elliptic Equations, *New York, Academic Press*, 1968.

[11] Morimoto, Y., On the hypoellipticity for infinitely degenerate semi-elliptic operators, Journal of the Mathematical Society of Japan, 1978, **30**, no.2, 327–358.

[12] Morimoto, Y., Hypoellipticity for infinitely degenerate elliptic operators, *Osaka J. Math.* **24** (1987), no. 1, 13–35.

[13] Morimoto, Y., A criterion for hypoellipticity of second order differential operators, *Osaka J. Math.* **24** (1987), no. 1, 661–675.

[14] Morimoto, Y. and Xu, C.J, Nonlinear hypoellipticity of infinite, *Funkcialaj Ekvacioj*, **50** (2007), no. 1, 33–65.

[15] Morimoto, Y. and Xu, C.J, Hypoellipticity for a class of kinetic equations, *Kyoto Journal of Mathematics*, 2007, 47, no.1, 129–152.

[16] Morioka, T, Hypoellipticity for some infinitely degenerate operators of second order, *J. Math. Kyoto Univ.* 32-2, 1992, 373-386.

[17] C. Parenti and A. Parmeggiani, On the hypoellipticity with a big loss of derivatives, *Kyushu J. Math.*, **59**, (2005), 155–230.

[18] C. Parenti and A. Parmeggiani, A note on Kohn's and Christ's examples. *Hyperbolic problems and regularity questions*, 151–158, Trends Math., Birkhäuser, Basel, (2007).

[19] C. Rios, E. T. Sawyer, R. L. Wheeden, A priori estimates for infinitely degenerate quasilinear equations, *Differential and Integral Equations-Athens*, 2008, 21, no.1, 131–200.

[20] C. Rios, E. Sawyer, R. Wheeden, Hypoellipticity for infinitely degenerate quasilinear equations and the Dirichlet problem, *To appear Journal d'Analyse Mathematique* (2012).

[21] E. Sawyer and R. Wheeden, A priori estimates for quasilinear equations related to the Monge-Ampère equation in two dimensions, *Journal d'Analyse Mathèmatique*, 2005, 97, no.1, 257–316.

[22] E. Sawyer and R. Wheeden, Regularity of degenerate Monge-Ampère and prescribed Gaussian curvature equations in two dimensions, *Potential Analysis*, **24** (2006), no. 3, 267–301.

[23] M. E. Taylor, Pseudodifferential operators and nonlinear PDE, Progress in Mathematics **100**. *Birkhäuser Boston, Inc.*, Boston, MA, 1991.

[24] Tri, N.M, Semilinear hypoelliptic differential operators with multiple characteristics, *Transactions of the American Mathematical Society*, 2008, **360**, no. 7, 3875–3907.

[25] Xu, C.J. and Zuily, C, Higher interior regularity for quasilinear subelliptic systems, *Calculus of Variations and Partial Differential Equations*, 1997, **5**, no.4, 323–343.