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A variation of Merton's corporate bond valuation model for firms with illiquid but observable assets

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We introduce a new model for pricing corporate bonds, which is a modification of the classical model of Merton. In this new model, we drop the liquidity assumption of the firm's asset value process, and assume that there is a liquidly tradeable asset in the market whose value is correlated with the firm's asset value, and all portfolios can be constructed using solely this asset and the money market account. We formulate the market price of the corporate bond as the product of the price of an optimal replicating portfolio and $\exp(-\kappa \times \text{replication error})$, where κ is a non-negative constant. The interpretation is that the representative investor uses the price of the optimal replicating portfolio as a benchmark and requests compensation for the non-hedgeable risk. We show that if the replication error is measured relative to the firm's value, the resulting formula is arbitrage free with mild restrictions on the parameters.

Keywords: Credit risk; Merton's model; Illiquidity; Mean variance hedging; Stochastic control

2000 Mathematics Subject Classifications: 60G55, 60G60

1. Introduction

In the classical model of Merton (1974), a corporate bond is a contingent claim on the assets of a firm. A geometric Brownian motion $(V_t)_{t\geq 0}$ models the value of the firm's assets,

$$\mathrm{d}V_t = \mu V_t\,\mathrm{d}t + \sigma V_t\,\mathrm{d}W_t, \tag{1}$$

where $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$, and $(W_t)_{t\geq 0}$ is a Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The market is endowed with a money market account accumulating interest at a constant rate r. In this model, a defaultable bond is defined as the pay-off $\min(V_T, D)$, where D is the face value and T is the maturity. The model assumes that the firm's assets are liquidly tradeable in the market, which is one of the critical assumptions enabling perfect replication; that is, the pay-off $\min(V_T, D)$ is hedgeable with a self financing portfolio made of a suitable number of the firms assets and the money market account. Hence, under Merton's model's assumptions, the arbitrage-free price, B_t, of the bond at time t is given by the formula (Merton 1974):

$$B_t = V_t\Phi(d_1) - D\exp(-r(T-t))\Phi(d_2), \tag{2}$$

where Φ is the standard normal distribution function, $d_1 = \frac{\ln(V_t/D)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = d_1 - \sigma\sqrt{T-t}$.

In this paper, we follow Merton's model, but assume that the firm's assets are not liquidly tradeable. Instead, there is a liquidly tradeable asset in the market, with the value S_t at time t, that is correlated with V_t, and all portfolios can be constructed using solely this asset and the money market account. Since the model is incomplete, there is no unique way to set the price of the claim $\min(V_T, D)$.

We model the price process $(B_t)_{0\leq t\leq T}$ of the claim $\min(V_T, D)$ as a stochastic process adapted to the natural filtration of S and V. We write B_t as the product of two components. One component is the price of an optimal replicating portfolio for $\min(V_T, D)$ determined at time t. We assume that the optimality criterion for replication is the expected square difference between the terminal value of the portfolio and the claim $\min(V_T, D)$. This optimization problem is referred to in the literature as the mean variance hedging (MVH) problem (Schweizer 1992). Note that we consider the dynamic version of the mean variance hedging problem; since at each time t, we re-solve the optimal replication problem for the period $[t, T]$ conditional on the information available

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up to time t. The other component of the price is a discount factor that takes into account the replication error. More specifically,

$$B_t = \text{The price of the time } t\text{-optimal replicating portfolio} \times e^{-\kappa \times \text{the time } t\text{-minimum replication error}}, \qquad (3)$$

where κ is a non-negative constant. The interpretation of the pricing formula (3) is that the representative agent using the price of the optimal replicating portfolio as a benchmark requests (or offers if selling) compensation for the additional risk involved in the claim $\min(V_T, D)$ (more on this later).

Our main contribution is that under certain conditions on the parameters, the price process $(B_t)_{0\le t\le T}$ in formula (3) allows no arbitrage in the sense of Definition 5.4.6 of Shreve (2008), if we measure the replication error relative to the value of the firm's assets, see Theorem 8 in section 4. We obtain explicit formulas for the value of the optimal mean variance replicating portfolio and the optimal replication error. As a by product, we provide a simple formula for the optimal mean variance replicating strategy for the claim $\min(V_T, D)$.

The idea of using the optimal mean variance replicating portfolio as an evaluation tool for a claim is well-known in the literature (see e.g. Schweizer 1996). What is new in (3) is the discount factor which we motivate as follows. For simplicity, let us assume that $t=0$. First, we show that the mean terminal wealth of the optimal replicating portfolio is the same as the mean of the claim $\min(V_T, D)$. Second, the variance of the terminal wealth of the optimal replicating portfolio is strictly less than the variance of the claim $\min(V_T, D)$. These properties simply follow from the fact that the optimal replicating portfolio is the L^2-projection of the claim $\min(V_T, D)$ onto the linear space of random variables of the form $x + \int_0^T \phi_t \, dS_t$, where $x \in \mathbb{R}$ and ϕ_t adapted to the filtration generated by V and S.

More precisely, let us denote the L^2-projection of the claim $\min(V_T, D)$ by $\Phi_T = x^* + \int_0^T \phi_t \, dS_t$. Then,

$$\begin{aligned} &\mathbb{E}\left[(\min(V_T, D) - \Phi_T)^2\right] \\ &\quad = \mathrm{Var}\left[\min(V_T, D)\right] + \mathrm{Var}\left[\Phi_T\right] \\ &\qquad - 2\mathrm{Cov}\left[\min(V_T, D), \Phi_T\right] \\ &\qquad + \left(\mathbb{E}\left[\min(V_T, D)\right] - \mathbb{E}\left[\Phi_T\right]\right)^2 . \end{aligned}$$

This decomposition implies that $\mathbb{E}[\min(V_T, D)] = \mathbb{E}[\Phi_T]$, since otherwise, we can make the left hand side smaller by adding a constant C to Φ_T, contradicting the optimality of Φ_T. Since $\min(V_T, D) - \Phi_T$ is orthogonal to $\Phi_T - \mathbb{E}(\min(V_T, D)) = \Phi_T - \mathbb{E}(\Phi_T)$ in L^2, we also have that

$$\begin{aligned} &\mathrm{Var}\left[\min(V_T, D)\right] \\ &\quad = \mathbb{E}\left[(\min(V_T, D) - \Phi_T)^2\right] \\ &\qquad + \mathbb{E}\left[(\min(V_T, D) - \Phi_T)(\Phi_T - \mathbb{E}(\min(V_T, D))\right] \\ &\qquad + \mathrm{Var}\left[\Phi_T\right] \\ &\quad = \mathbb{E}\left[(\min(V_T, D) - \Phi_T)^2\right] + \mathrm{Var}\left[\Phi_T\right] . \end{aligned} \qquad (4)$$

Hence, $\mathrm{Var}[\min(V_T, D)] > \mathrm{Var}[\Phi_T]$.

The existence of a discount factor for the price of the claim $\min(V_T, D)$ is consistent with the Markowitz portfolio choice theory: the price of the claim $\min(V_T, D)$ must be less than the price of the optimal replicating portfolio, since $\mathbb{E}[\min(V_T, D)] = \mathbb{E}[\Phi_T]$, while the variance of $\min(V_T, D)$ is higher than the variance of Φ_T. It is intuitive that the discount factor is a function of the replication error, since equation (4) shows that the replication error is exactly the difference between the variance of the claim $\min(V_T, D)$ and the variance of Φ_T. Replacing the expectations with conditional expectations, similar arguments hold for an arbitrary time t, as well. The choice of the exponential function is for convenience: we need the discount factor to be non-negative, and less than 1, and converge to 1 as $t \to T$. Any other function satisfying these properties can be used as well, but we prefer the exponential function for mathematical convenience.

We interpret (3) as the market price of the claim $\min(V_T, D)$ determined by a representative agent (representing the aggregate behavior of the investors in the economy). The discount factor represents the market's risk aversion towards a product that is more risky, where the riskiness of the product is measured by the standard deviation of its return as in Markowitz portfolio choice theory. The key question for us is whether this pricing model is arbitrage free.

The representative agent approach is well known in asset pricing theory, see for example Magill and Quinzii (2002). In particular, in complete markets, the state-price density is interpreted as the marginal utility of wealth of a representative agent. With the addition of a tradeable asset with the price process $(B_t)_{0\le t\le T}$ in the market, we find that our model is also complete, hence it makes sense to talk about a representative agent, although we do not explicitly construct his/her utility function.

Further merit of the pricing formula (3) is that it takes into account both the drift μ and the volatility σ of the firm's asset value, unlike Merton's formula, which ignores μ. The drift μ is a key parameter determining the firm's default probability, therefore should effect the price of the bond. Even though in the literature there are different findings of how much of the credit yield spreads are due to default risk (see e.g. Longstaff *et al.* 2005, Huang and Huang 2012), credit yield spreads increase as the default risk increases at a large scale, for instance, as credit ratings deteriorate. According to Merton's model, however, two firms can have the same yield spread even if their default probabilities are substantially different, for example if $\sigma_A = \sigma_B$ and $\mu_B >> \mu_A$. This is not the case for the pricing formula (3), since it uses the value of the optimal mean variance replicating portfolio as a basis, which increases as the drift of the firm increases, see section 5.

A limiting assumption in our approach is that the firm's asset value is perfectly observable even though its assets are not liquid. For example, V can be realized as the book value of the firm. The book values of companies are provided on a quarterly basis which can be used to estimate the parameters of V . There is the question of how truthful the published book values are as a measure of a company's true value, and this is a weakness of our approach. The assumption of perfect observability of the firm's true value makes our short term spreads resemble the short term spreads of Merton's model, (see section 5.5). However, a more powerful methodology can be obtained by introducing accounting noise as in Duffie and Lando (2001). Our current work can be considered as a preparation for such a work.

Another limitation of our approach is that it allows only non-negative values of κ, since a negative value of κ may make the price of the bond exceed D, which is prohibited. With the restriction $\kappa \geq 0$, the discount factor adds an extra cost for the seller of the bond who wants to hedge his/her position using the optimal mean variance replicating portfolio. The seller's expected terminal wealth is negative with respect to the historical measure $\mathbb{P}$, but this is not a contradiction, because there is still a chance that the seller can make a profit in case the optimal mean variance replicating portfolio exceeds the value of $\min(V_T, D)$ by a sufficient amount. The pricing formula (3) implies that one unit of currency is valued more in those states of the world where the optimal mean variance replicating portfolio super-replicates than those where it under-replicates. More precisely, let π_T be the state price density corresponding to (3). Then $\mathbb{E}[\pi_T \min(V_T, D)] = B_0$. Since Φ_T is replicable, we also have that $\mathbb{E}(\pi_T \Phi_T) = b_0$, where b_0 denotes the value of the time 0 optimal mean variance replicating portfolio. Since $B_0 < b_0$, it follows that $\mathbb{E}[\pi_T(\min(V_T, D) - \Phi_T)] < 0$. Thus, $\mathbb{E}[\pi_T(\min(V_T, D) - \Phi_T)^-] > \mathbb{E}[\pi_T(\min(V_T, D) - \Phi_T)^+]$, since $\min(V_T, D) - \Phi_T = (\min(V_T, D) - \Phi_T)^+ - (\min(V_T, D) - \Phi_T)^-$.

Our methodology is to formulate the dynamic MVH as a stochastic control problem as in Bertsimas *et al.* (2001), Bobrovnytska and Schweizer (2004), Jeanblanc *et al.* (2012) and Kohlmann and Zhou (2006). We solve the corresponding stochastic control problem using the Hamilton Jacobi Belman equation since the underlying processes are Itô processes (as in Bertsimas *et al.* 2001). To prove that the formula (3) is arbitrage free, we use Novikov's condition and the first fundamental theorem of asset pricing, (see e.g. Shreve 2008 Theorem 5.4.7).

We show that there exists a unique probability measure $\mathbb{Q}$ which makes $(S_t)_{t\in[0,T]}$ and $(B_t)_{t\in[0,T]}$ both martingales with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$, $\mathcal{F}_t = \sigma(V_u, S_u, 0 \leq u \leq t)$), (see Theorem 2). By the second fundamental theorem of asset pricing (see e.g. Shreve 2008, Theorem 5.4.9), the market is now complete, hence all contingent claims Y can now be priced by the formula $\mathrm{e}^{-rT}\mathbb{E}^{\mathbb{Q}}(Y)$. Consequently, the price of equity is uniquely implied. An efficient numerical procedure can be potentially found to compute the pricing formula for the equity, with the benefit of allowing equity prices to be used in the estimation of the parameters; however, we do not explore this in the current paper.

Our pricing framework can be extended to more general credit risk derivatives and/or to multifactor models due to the existence of a vast literature on the mean variance hedging problem and its solution in terms of backward stochastic differential equations (BSDE), (Bobrovnytska and Schweizer 2004, Kohlmann and Zhou 2006, Jeanblanc *et al.* 2012). In particular, both the price of the optimal replicating portfolio and the replication error can be represented as solutions of BSDEs, see e.g Bobrovnytska and Schweizer (2004). These representations can be explored to answer the no arbitrage property in a more general framework.

Within the current literature, this paper stands in the intersection of incomplete markets and credit risk pricing. The idea of using a correlated asset for hedging and valuation of pay-offs contingent on illiquid assets has been considered by various authors in the literature. Our approach is closest to mean variance hedging (Schweizer 1992, 1996, Bertsimas *et al.* 2001, Bobrovnytska and Schweizer 2004, Kohlmann and Zhou 2006, Jeanblanc *et al.* 2012). The cost of the optimal mean variance hedging portfolio, herein denoted as $(b_t)_{t\in[0,T]}$, under certain assumptions on the underlying filtration, can be represented as a conditional expectation of the claim with respect to the variance optimal martingale measure equivalent to $\mathbb{P}$ (Bobrovnytska and Schweizer 2004). This ensures that $(b_t)_{t\in[0,T]}$ is nonnegative and forms an arbitrage free price process for the corresponding claim. Note that this can be considered as a special case of our result that the pricing formula (3) is arbitrage free, specifically for the case $\kappa = 0$.

We mention here some other approaches: Davis (1997) uses the expected utility maximization framework, and defines the fair price of an option as the marginal price—the price at which diverting a little money into a derivative has neutral effect on the expected terminal utility. Karatzas and Kou (1996) show that the marginal price is independent of the utility function and the corresponding pricing measure is the minimal martingale measure. Another price considered in the utility maximization framework is the indifference price; see Henderson (2002), Halperin and Itkin (2014), Henderson and Hobson (2008), Leung *et al.* (2008) and Bielecki and Jeanblanc (2008). Indifference price p of a claim contingent on an illiquid asset is the price defined for an initial endowment v that makes the buyer indifferent between investing the v dollars in the tradeable assets versus buying the claim and investing the remaining $v-p$ dollars in the tradeable assets. Unlike the marginal price, the indifference price depends on the choice of the utility function, Henderson (2002). The indifference price is interpreted as an individual's own valuation of the claim, not as a market price, see e.g. Bielecki and Jeanblanc (2008). Thus, the question of arbitrage has not been a central question in indifference pricing.

Finally, Okhrati *et al.* (2014) considers a modification of Merton's model similar to ours and uses local risk minimization for the valuation of the claim $\min(V_T, D)$, another well-known hedging and valuation tool used in incomplete markets.

In section 2, we define our setting and review the MVH problem for a general contingent claim and the corresponding stochastic control problem. In particular, we review Bertsimas *et al.* (2001)'s characterization of the solution of the stochastic control problem in terms of a certain boundary value problem for a system of partial differential equations (PDEs). In section 3, we introduce our pricing model for a general contingent claim and apply the results of Bertsimas *et al.* (2001) to formulate the price in terms of the solutions of a certain system of PDEs. In section 4, we focus on the pricing of the contingent claim $\min(D, V_T)$ and obtain an analytical formula for the price by solving the corresponding system of PDEs. We then use these explicit formulas to analyze the qualitative properties of the price, in particular, whether it is arbitrage free. In section 5 we perform numerical experiments to see how the price and yield spreads react to changes in the parameters.

2. MVH as a stochastic control problem

In this section, we give an overview of the stochastic control approach of Bertsimas *et al.* (2001) to solve the MVH. Let V_t be as defined in equation (1). From now on, we denote the (percentage) drift and (percentage) volatility of V as μ_1 and σ_1 and the Brownian motion driving V as $W^{(1)}$. Let $(S_t)_{t\geq 0}$ be another geometric Brownian motion:

$$\mathrm{d}S_t = \mu_2 S_t \,\mathrm{d}t + \sigma_2 S_t \,\mathrm{d}W_t^{(2)}, \tag{5}$$

where $W^{(2)}$ is another Brownian motion with $\mathrm{d}\langle W^{(1)}, W^{(2)}\rangle_t = \rho\,\mathrm{d}t$, where $\rho \in (-1, 1)$. We assume both σ_1 and σ_2 are positive.

We assume a trading horizon $[0, T]$ with $T > 0$, and that initially the market contains only a tradeable asset with the price process $(S_t)_{0\leq t\leq T}$ and a money market account which accumulates interest at a constant deterministic rate $r > 0$. In order not to introduce new notation, we assume that both V and S are already discounted. We consider a contingent claim with discounted pay-off $F(S_T, V_T)$, where $F : \mathbb{R}^2 \to \mathbb{R}^+$ is a nonnegative deterministic function.

The dynamic for the discounted wealth of a self-financing portfolio is

$$\mathrm{d}P_t = \theta_t \,\mathrm{d}S_t = \theta_t \mu_2 S_t \,\mathrm{d}t + \theta_t \sigma_2 S_t \,\mathrm{d}W_t^{(2)}, \quad 0 \leq t \leq T,$$

where $(\theta_t)_{0\leq t\leq T}$ is a predictable S-integrable process with respect to $(\mathcal{F}_t)_{t\in[0,T]}$, where

$$\mathcal{F}_t = \sigma(V_u,\ S_u,\ 0 \leq u \leq t).$$

Since there are more random factors than the tradeable assets in the market, the market is incomplete. Therefore, not every contingent claim $F(S_T, V_T)$ can be perfectly hedged. MVH is an approach in which one finds an optimal trading strategy that best approximates the payoff $F(S_T, V_T)$; that is,

$$\text{minimize } \mathbb{E}[(P_T - F(S_T, V_T))^2] \text{ over all pairs } (p, \theta), \tag{6}$$

where the pair $[p, (\theta_t)_{t\in[0,T]}]$ describes a dynamic self financing trading strategy which starts at time 0 with initial capital p and holds θ_t shares of the tradeable asset at time $t \in [0, T]$, leading to a discounted wealth of P_t at time t. This problem can be treated as a stochastic optimal control problem as follows:

$$\text{minimize } \mathbb{E}[(P_T - F(S_T, V_T))^2] \text{ over all } \theta \in \Theta, \tag{7}$$

with the dynamics

$$\begin{aligned} \mathrm{d}V_t &= \mu_1 V_t \,\mathrm{d}t + \sigma_1 V_t \,\mathrm{d}W_t^{(1)}, \\ \mathrm{d}S_t &= \mu_2 S_t \,\mathrm{d}t + \sigma_2 S_t \,\mathrm{d}W_t^{(2)}, \\ \mathrm{d}P_t &= \theta_t \,\mathrm{d}S_t = \theta_t \mu_2 S_t \,\mathrm{d}t + \theta_t \sigma_2 S_t \,\mathrm{d}W_t^{(2)}, \end{aligned}$$

$$P_0 = p, \quad S_0 = s, \quad V_0 = v,$$

where $0 \leq t \leq T$, Θ is the set of all $\mathbb{R}$-valued $(\mathcal{F}_t)_{t\in[0,T]}$-predictable S-integrable processes such that $\int_0^T \theta_t \,\mathrm{d}S_t$ is well-defined and square integrable. In this context, the 3-dimensional process $(P_t, S_t, V_t)_{t\in[0,T]}$ is the state process, the process $(\theta_t)_{t\in[0,T]}$ is the control process, and Θ is the constraint set.

Note that in the formulation given in (7), the initial cost of the portfolio p is taken as fixed. Once the problem (7) is solved for any p, then one can optimize over p. To make this clearer, we first solve the optimal trading strategy $\theta^*(p, s, v)$ for the initial conditions $P_0 = p,\ S_0 = s,\ V_0 = v$. That is, for given initial values p, s and v, $\theta^*(p, s, v) = (\theta_t^*(p, s, v))_{0\leq t\leq T}$ is the predictable S-integrable process with respect to $(\mathcal{F}_t)_{t\in[0,T]}$, solving (7). Let $\mathbb{E}_{p,s,v}(\cdot)$ be the conditional expectation operator $\mathbb{E}(\cdot \mid P_0 = p,\ S_0 = s,\ V_0 = v)$. Then we define:

$$\varepsilon(s, v) := \sqrt{\min_p \mathbb{E}_{p,s,v}\left[(P_T - F(S_T, V_T))^2\right]}.$$

Let $p^*(s, v)$ be the optimal solution of the problem $\min_p \mathbb{E}_{p,s,v}[(P_T - F(S_T, V_T))^2]$. The uniqueness of $p^*(s, v)$ is well known; in fact, $\mathbb{E}_{p,s,v}[(P_T - F(S_T, V_T))^2]$ is a quadratic function of p, Bertsimas *et al.* (2001). Then, the optimal solution of (6) is $(p^*(s, v), \theta^*(s, v))$ where $\theta^*(s, v) = \theta(p^*(s, v), s, v)$ and the minimal replication error is $\varepsilon(s, v)$, as in Bertsimas *et al.* (2001). Note that we did not require Θ to contain only admissible processes (i.e. θ such that $P_t \geq 0$ for all t). However, it turns out that the optimal replicating strategy $\theta^*(s, v)$ is always admissible. This follows from the fact that S is continuous and therefore the value process of the optimal replicating portfolio can be represented as a conditional expectation of the final pay-off with respect to the variance optimal martingale measure (see e.g. Bobrovnytska and Schweizer 2004).

We fix $t \in [0, T]$ and let $\mathbb{E}_{t,p,s,v}(\cdot)$ be the conditional expectation operator $\mathbb{E}(\cdot \mid P_t = p,\ S_t = s,\ V_t = v)$. Let $\Theta_t = \{(\theta_u)_{u\in[t,T]} : \int_t^T \theta_u \,\mathrm{d}S_u$ is well defined and square integrable$\}$. We can define a dynamic version of (7) by considering the problem:

$$\text{minimize } \mathbb{E}_{t,p,s,v}[(P_T - F(S_T, V_T))^2] \text{ over all } \theta \in \Theta_t. \tag{8}$$

Let $J : [0, T] \times \mathbb{R}^+ \times (0, \infty) \times (0, \infty) \to \mathbb{R}^+,\ (t, p, s, v) \mapsto J(t, p, s, v)$, be the optimal value function of this control problem.

Using the Markov property of (S, V), Bertsimas *et al.* (2001) shows that, regardless of the value of t, p, s and v, the solution of (8), denoted by $\theta^*(t, p, s, v) = (\theta_u^*(t, p, s, v))_{u\in[t,T]}$, is of the form $(H(u, P_u, S_u, V_u))_{u\in[t,T]}$, where H refers to a deterministic function of (u, p, s, v).

It is well known that J is characterized as the solution of the Hamilton Jacobi and Bellman equation,(see e.g. Björk 2004, p. 279):

$$\frac{\partial J}{\partial t} + \inf_{\{h\}} \left\{ \left[\mu_2 s \frac{\partial}{\partial s} + \mu_1 v \frac{\partial}{\partial v} + h\mu_2 s \frac{\partial}{\partial p} + \frac{1}{2}\sigma_2^2 s^2 \frac{\partial^2}{\partial s^2} \right.\right.$$
$$\left. + \frac{1}{2}\sigma_1^2 v^2 \frac{\partial^2}{\partial v^2} \right.$$

$$+\sigma_2\sigma_1 sv\rho\frac{\partial}{\partial s\partial v}+\frac{1}{2}h^2\sigma_2^2 s^2\frac{\partial^2}{\partial p^2}+\sigma_2^2 s^2 h\frac{\partial^2}{\partial s\partial p}$$
$$+\sigma_2\sigma_1 svh\rho\frac{\partial^2}{\partial v\partial p}\Big]J\Big\}=0 \qquad (9)$$

for $t\in(0,T)$, $p\in\mathbb{R}^+$, $s\in(0,\infty)$, $v\in(0,\infty)$, and with the following boundary condition at $t=T$:

$$J(T,p,s,v)=[p-F(s,v)]^2 \quad \text{for } (s,v)\in(0,\infty)\times(0,\infty).$$

The inf in equation (9) is achieved at $h=H^*(t,v,s,p)$ which is explicitly found as:

$$H^*(t,p,s,v)=\frac{\begin{array}{c}-\mu_2 s\frac{\partial J}{\partial p}(t,p,s,v)-\sigma_2^2 s^2\frac{\partial^2 J}{\partial s\partial p}(t,p,s,v)\\ -\rho\sigma_1\sigma_2 vs\frac{\partial^2 J}{\partial v\partial p}(t,p,s,v)\end{array}}{\sigma_2^2 s^2\frac{\partial^2 J}{\partial p^2}(t,p,s,v)}.$$

Substituting the expression for h into the PDE (9) gives us a PDE for the only unknown function J.

Solving the resulting nonlinear PDE for J is facilitated by the observation that the value process of the MVH problem has a quadratic structure. More specifically, Bertsimas *et al.* (2001) proves the following theorem:

THEOREM 1 Bertsimas *et al.* (2001) *The value function $J:(t,p,s,v)\mapsto J(t,p,s,v)$ is quadratic in p, the initial wealth to start the replicating strategy. Moreover, there are continuous functions $a:[0,T]\times(0,\infty)\times(0,\infty)\to\mathbb{R}^+$, $b:[0,T]\times(0,\infty)\times(0,\infty)\to\mathbb{R}^+$ and $c:[0,T]\times(0,\infty)\times(0,\infty)\to\mathbb{R}^+$ such that*

$$J(t,p,s,v)=a(t,s,v)\cdot[p-b(t,s,v)]^2$$
$$+c(t,s,v),\quad 0\le t\le T, \qquad (10)$$

where a, b and c are first order differentiable in t and second order differentiable in (s,v) on the domain $[0,T)\times(0,\infty)\times(0,\infty)$ and satisfy

$$\frac{\partial a}{\partial t}=\left(\frac{\mu_2}{\sigma_2}\right)^2 a+\mu_2 s\frac{\partial a}{\partial s}+\left[\frac{2\sigma_1\rho v\mu_2}{\sigma_2}-\mu_1 v\right]\frac{\partial a}{\partial v}$$
$$-\frac{1}{2}v^2 s^2\frac{\partial^2 a}{\partial s^2}-\frac{1}{2}\sigma_1^2 v^2\frac{\partial^2 a}{\partial v^2}$$
$$-v^2\sigma_1 s\rho\frac{\partial^2 a}{\partial s\partial v}+\frac{1}{a}v^2 s^2\left(\frac{\partial a}{\partial s}\right)^2$$
$$+\frac{1}{a}\rho^2\sigma_1^2 v^2\left(\frac{\partial a}{\partial v}\right)^2+2v^2\sigma_1 s\rho\frac{\partial a}{\partial s}\frac{\partial a}{\partial v}, \qquad (11)$$

$$\frac{\partial b}{\partial t}=\left[\frac{\sigma_1}{\sigma_2}v\rho\mu_2-\mu_1 v\right]\frac{\partial b}{\partial v}-\frac{1}{2}\sigma_1^2 v^2\frac{\partial^2 b}{\partial v^2}$$
$$-\frac{1}{2}\sigma_2^2 s^2\frac{\partial^2 b}{\partial s^2}-\sigma_2\sigma_1 sv\rho\frac{\partial^2 b}{\partial v\partial s}$$
$$+\frac{\sigma_1^2 v^2}{a}(\rho^2-1)\frac{\partial a}{\partial v}\frac{\partial b}{\partial v}, \qquad (12)$$

$$\frac{\partial c}{\partial t}=-\mu_1 v\frac{\partial c}{\partial v}-\mu_2 s\frac{\partial c}{\partial s}-\frac{1}{2}\sigma_1^2 v^2\frac{\partial^2 c}{\partial v^2}$$
$$-\sigma_2\sigma_1 sv\rho\frac{\partial^2 c}{\partial s\partial v}-\frac{1}{2}\sigma_2^2 s^2\frac{\partial^2 c}{\partial s^2}$$
$$+a\sigma_1^2 v^2(\rho^2-1)\left(\frac{\partial b}{\partial v}\right)^2, \qquad (13)$$

with boundary conditions

$$a(T,s,v)=1,\quad \textit{for } (s,v)\in(0,\infty)\times(0,\infty);$$
$$b(T,s,v)=F(s,v),\quad \textit{for } (s,v)\in(0,\infty)\times(0,\infty);$$
$$c(T,s,v)=0,\quad \textit{for } (s,v)\in(0,\infty)\times(0,\infty).$$

Bertsimas *et al.* (2001) shows that $a(t,s,v)$ is positive, hence the initial wealth $p^*(t,s,v)$ that minimizes the quadratic function $J:(t,p,s,v)\mapsto J(t,p,s,v)$ is exactly $b(t,s,v)$ and the time t-optimal-replicating strategy $\theta^*(t,s,v)=(\theta_u^*(t,s,v))_{u\in[t,T]}$ and the corresponding portfolio process $(P_u^*(t,s,v))_{u\ge t}$ are given by the system of equations:

$$P_t^*(t,s,v)=b(t,S_t,V_t),$$
$$\theta_u^*(t,s,v)=A_u P_u^*(t,s,v)-b(u,S_u,V_u))+C_u,\quad u\ge t,$$
$$\mathrm{d}P_u^*(t,s,v)=\theta_u^*(t,s,v)\,\mathrm{d}S_u,\quad u\ge t,$$
$$A_u=-\frac{\begin{array}{c}\mu_2+\sigma_2^2 S_u\frac{\partial}{\partial s}a(u,S_u,V_u)+\\ \rho\sigma_1\sigma_2 V_u\frac{\partial}{\partial v}a(u,S_u,V_u)\end{array}}{\sigma_2^2 S_u a(u,S_u,V_u)},\quad u\ge t,$$
$$C_u=\frac{\partial b}{\partial s}(u,S_u,V_u)+\rho\frac{\sigma_1}{\sigma_2}\frac{V_u}{S_u}\frac{\partial b}{\partial v}(u,S_u,V_u),\quad u\ge t. \qquad (14)$$

Moreover, equation (10) implies that the replication error of the strategy $\theta^*(t,s,v)$ in the mean square error sense is $c(t,s,v)$ under the conditions $S_t=s$ and $V_t=v$.

3. A pricing model

In this section, we propose a pricing model to price a contingent claim of the form $F(V_T)$, where $F:\mathbb{R}^+\to\mathbb{R}^+$ is a non-negative deterministic function. Because the pay-off is only a function of V_T but not S_T, it turns out that the functions a, b and c given in Theorem 1 are only functions of t and v but not s. (See Appendix A.1, in particular, the derivation of b, for a justification.) Hence, in what follows, we have $a,b,c:[0,T]\times(0,\infty)\to\mathbb{R}^+$.

We propose the following formula to calculate the price of a contingent claim $F(V_T)$:

$$B_t=b(t,V_t)\cdot \mathrm{e}^{-\kappa\cdot\frac{c(t,V_t)}{d(t,V_t)}},\quad t\in[0,T], \qquad (15)$$

where $b(t,V_t)$ is the cost of building an optimal-replicating portfolio for the claim $F(V_T)$ at time t, and $c(t,V_t)$ is the expected squared error of the replication, $\kappa\ge 0$, and $d(t,V_t)$ is a suitable normalization factor. We interpret the term $b(t,V_t)$ as a benchmark price. Indeed, in the mean square error sense, it is the price of the closest tradeable instrument available in the market. However, the variance of the pay-off of this benchmark instrument is less than the variance of the contingent claim, therefore it makes sense for the investor to ask for a discount from the benchmark price; which is the rationale for the term $e^{-\kappa\cdot\frac{c(t,V_t)}{d(t,v)}}$. Here $\kappa \geq 0$ is a preference parameter, measuring the sensitivity of the investor to the difference in the variances of the pay-off of the optimal mean variance replicating portfolio and the contingent claim.

Here, let us comment on the main features of equation (15). The price of the contingent claim B_t converges to the payoff $F(V_T)$ as t approaches T. This is due to the boundary condition of $b(t,v)$ and assumption (16). Recall from Theorem 1 that $b(t,V_t) \to F(V_T)$ as $t \to T$. We assume that $d : [0,T] \times (0,\infty) \to (0,\infty)$ is first order differentiable in t and second order differentiable in v in the domain $[0,T) \times (0,\infty)$, and

$$\lim_{t\to T} \frac{c(t,v)}{d(t,v)} = 0. \tag{16}$$

Because $c(t,v) \to 0$ as $t \to T$, assumption (16) will be satisfied for a wide range of choices for d. Normalizing the mean squared replication error $c(t,V_t)$ by $d(t,V_t)$ allows the investor to measure the replication error in relative terms. A possible choice for $d(t,v)$ is the conditional variance of $F(V_T)$ given $V_t = v$. Simpler normalization factors can be chosen depending on the type of the contingent claim. For example, when $F(V_T) = \min(D, V_T)$, it makes sense to choose $d(t,v) = v^2$. That is, the replication error is measured as a percentage of the value of the firm, and thus κ is a unitless constant. In general, we need normalization for technical reasons, in particular, to show that the pricing formula is arbitrage free.

In what follows, all references to martingales, local martingales, semimartingales are made with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$, $\mathcal{F}_t = \sigma(V_s,\ S_s,\ 0 \leq s \leq t)$, and the probability measure $\mathbb{P}$, unless stated explicitly otherwise.

To show that the proposed pricing model is arbitrage free, we first find two processes $(\lambda_t^{(1)})_{t\in[0,T]}$ and $(\lambda_t^{(2)})_{t\in[0,T]}$ such that if $\tilde{W}_t^{(i)} = \int_0^t \lambda_u^{(i)}\, du + W_t^{(i)}$, then for $t \in [0,T]$,

$$B_t = B_0 + \int_0^t N_u\, d\tilde{W}_u^{(1)}, \tag{17}$$

$$S_t = S_0 + \int_0^t L_u\, d\tilde{W}_u^{(2)}, \tag{18}$$

for some continuous and adapted processes N and L. The following theorem identifies the processes $\lambda^{(1)}$ and $\lambda^{(2)}$ and gives sufficient conditions when there is a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that $\tilde{W}^{(1)}$ and $\tilde{W}^{(2)}$ are $\mathbb{Q}$-martingales.

THEOREM 2 *Let* $\tilde{c}(t,v) = \frac{c(t,v)}{d(t,v)}$. *Assume that* $\frac{\partial b}{\partial v} - \kappa b \frac{\partial \tilde{c}}{\partial v} > 0$ *for all* $(t,v) \in (0,T) \times (0,\infty)$. *Let* $L_t = S_t\sigma_2$, *and* $(M_t)_{t\in[0,T]}$ *and* $(N_t)_{t\in[0,T]}$ *be as defined below:*

$$\begin{aligned}
M_t = {} & e^{-\kappa\tilde{c}(t,V_t)} \left[\frac{\partial b}{\partial t}(t,V_t) + \frac{\partial b}{\partial v}(t,V_t)\mu_1 V_t \right.\\
& \left. + \frac{1}{2}\frac{\partial^2 b}{\partial v^2}(t,V_t)\sigma_1^2 V_t^2\right]\\
& - \kappa b(t,V_t)\, e^{-\kappa\tilde{c}(t,V_t)} \left[\frac{\partial \tilde{c}}{\partial t}(t,V_t) + \frac{\partial \tilde{c}}{\partial v}(t,V_t)\mu_1 V_t \right.\\
& \left. + \frac{1}{2}\frac{\partial^2 \tilde{c}}{\partial v^2}(t,V_t)\sigma_1^2 V_t^2\right]\\
& + \frac{1}{2}\kappa^2 b(t,V_t)\, e^{-\kappa\tilde{c}(t,V_t)} \frac{\partial \tilde{c}}{\partial v}(t,V_t)^2\sigma_1^2 V_t^2\\
& - \kappa\, e^{-\kappa\tilde{c}(t,V_t)} \frac{\partial \tilde{c}}{\partial v}(t,V_t)\frac{\partial b}{\partial v}(t,V_t)\sigma_1^2 V_t^2,\\
N_t = {} & e^{-\kappa\tilde{c}(t,V_t)} V_t\sigma_1 \left[\frac{\partial b}{\partial v}(t,V_t) - \kappa b_t \frac{\partial \tilde{c}}{\partial v}(t,V_t)\right].
\end{aligned} \tag{19}$$

Then $N_t > 0$ *for all* $t \in [0,T)$, *and equations* (17) *and* (18) *hold with* $\tilde{W}_t^{(i)} = \int_0^t \lambda_u^{(i)}\, du + W_t^{(i)}$, *where* $\lambda_t^{(1)} = \frac{M_t}{N_t}$ *and* $\lambda_t^{(2)} = \frac{\mu_2}{\sigma_2}$.

Moreover, if

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2(1-\rho^2)}\int_0^{\mathrm{T}}\left[\frac{1}{2}(\frac{M_t}{N_t})^2 + \rho\frac{M_t\mu_2}{N_t\sigma_2}\right] dt\right)\right] < \infty, \tag{20}$$

then there exists a probability measure $\mathbb{Q}$ *equivalent to* $\mathbb{P}$ *such that* $(\tilde{W}_t^{(1)}, \tilde{W}_t^{(2)})_{0\leq t\leq T}$ *is a two dimensional Brownian motion with correlation* ρ *and this is the unique probability measure equivalent to* $\mathbb{P}$ *rendering* $(B_t, S_t)_{t\in[0,T]}$ *a local martingale.*

Proof We first show that equations (17) and (18) are true. Note that

$$\begin{aligned}
S_0 + \int_0^t L_u\, d\tilde{W}_u^{(2)} &= S_0 + \int_0^t S_u\sigma_2\frac{\mu_2}{\sigma_2} du + \int_0^t S_u\sigma_2\, dW_u^{(2)}\\
&= S_t.
\end{aligned}$$

Similarly,

$$\begin{aligned}
B_0 + \int_0^t N_u\, d\tilde{W}_u^{(1)} &= B_0 + \int_0^t \frac{M_u}{N_u}N_u\, du + \int_0^t N_u\, dW_u^{(1)}\\
&= B_0 + \int_0^t M_u\, du + \int N_u\, dW_u^{(1)}\\
&= B_0 + \int_0^t \left(\frac{\partial}{\partial t}(b\, e^{-\kappa\tilde{c}})(u,V_u)\right.\\
&\quad \left. + \frac{1}{2}\frac{\partial^2}{\partial v^2}(b\, e^{-\kappa\tilde{c}})(u,V_u)\sigma_1^2 V_u^2\right) du\\
&\quad + \int_0^t \frac{\partial}{\partial v}(b\, e^{-\kappa\tilde{c}})(u,V_u)\, dV_u\\
&= b(t,V_t)\, e^{-\kappa\tilde{c}(t,V_t)},
\end{aligned}$$

where the last line follows from Itô's formula. Note that we are able to apply Itô's formula because we know from Theorem 1 that both b and c are first order differentiable in t and second order differentiable in v in $[0,T) \times (0,\infty) \times (0,\infty)$, and continuous at $\{T\} \times (0,\infty) \times (0,\infty)$, the same regularity conditions apply for $b e^{-\kappa\tilde{c}}$ in $[0,T) \times (0,\infty) \times (0,\infty)$ under the assumption (16).

Note that because $N_t > 0$, and both M and N are continuous, the process

$$X_t = -\int_0^t \frac{\lambda_u^{(1)} - \rho\lambda_u^{(2)}}{1-\rho^2} dW_u^{(1)} - \int_0^t \frac{\lambda_u^{(2)} - \rho\lambda_u^{(1)}}{1-\rho^2} dW_u^{(2)}$$

is well defined and a continuous local martingale. Let $(Z_t)_{t\in[0,T]}$ be the stochastic exponential of X. By the well known Novikov's condition (see e.g. Karatsaz and Shreve 2004), $(Z_t)_{t\in[0,T]}$ is a strictly positive martingale if $\mathbb{E}(\exp[\frac{1}{2}\langle X,X\rangle_T]) < \infty$. An elementary calculation gives

$$\langle X,X\rangle_t = \frac{1}{(1-\rho^2)}\int_0^t \left[(\lambda_u^{(1)})^2 + (\lambda_u^{(2)})^2 - 2\rho\lambda_u^{(1)}\lambda_u^{(2)}\right] \mathrm{d}u.$$

Hence, we have

$$\exp\left(\frac{1}{2}\langle X,X\rangle_T\right) = \exp\left(\frac{1}{2(1-\rho^2)}\int_0^{\mathrm{T}}\left[\left(\frac{M_t}{N_t}\right)^2 + \left(\frac{\mu_2}{\sigma_2}\right)^2 - 2\rho\frac{M_t\mu_2}{N_t\sigma_2}\right]\mathrm{d}t\right).$$

Hence, Hypothesis (20) implies that Novikov's condition is satisfied. Therefore, $(Z_t)_{t\in[0,T]}$ is a martingale with $Z_0 = 1$. We define the probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ as $\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}(Z_T 1_A)$. Since $\mathrm{d}Z_t = Z_t(-\frac{\lambda_t^{(1)}-\rho\lambda_t^{(2)}}{1-\rho^2}\mathrm{d}W_t^{(1)} - \frac{\lambda_t^{(2)}-\rho\lambda_t^{(1)}}{1-\rho^2}\mathrm{d}W_t^{(2)})$, for $i=1,2$

$$\begin{aligned}
\mathrm{d}(Z_t\tilde{W}_t^{(i)}) &= Z_t\,\mathrm{d}\tilde{W}_t^{(i)} + \tilde{W}_t^{(i)}\,\mathrm{d}Z_t + \mathrm{d}\left\langle Z,\tilde{W}^{(i)}\right\rangle_t\\
&= Z_t\,\mathrm{d}W_t^{(i)} + \tilde{W}_t^{(i)}\,\mathrm{d}Z_t + Z_t\lambda_t^{(i)}\,\mathrm{d}t\\
&\quad + Z_t\left(\frac{\lambda_t^{(i)}-\rho\lambda_t^{(i^*)}}{1-\rho^2} + \rho\frac{\lambda_t^{(i^*)}-\rho\lambda_t^{(i)}}{1-\rho^2}\right)\mathrm{d}t\\
&= Z_t\,\mathrm{d}W_t^{(i)} + \tilde{W}_t^{(i)}\,\mathrm{d}Z_t,
\end{aligned}$$

where $i^*=2$ if $i=1$, and $i^*=1$ if $i=2$. Hence, both $\tilde{W}^{(1)}$ and $\tilde{W}^{(2)}$ are $\mathbb{Q}$-local martingales which implies that $(\tilde{W}_t^{(1)},\tilde{W}_t^{(2)})_{t\in[0,T]}$ is a two dimensional Brownian motion with correlation ρ by Lévy's characterization theorem, (see e.g. Shreve 2008, p. 169). Hence $(B_t,S_t)_{t\in[0,T]}$ is a $\mathbb{Q}$-local martingale. Let $\tilde{\mathbb{Q}}$ be another probability measure equivalent to $\mathbb{P}$ such that $(B_t,S_t)_{t\in[0,T]}$ is a $\tilde{\mathbb{Q}}$-local martingale. Then, there are processes $\tilde{\lambda}^{(1)}$ and $\tilde{\lambda}^{(2)}$ such that $\frac{\mathrm{d}\tilde{\mathbb{Q}}_t}{\mathrm{d}\mathbb{P}_t}$ is the stochastic exponential of

$$\tilde{X}_t = -\int_0^t \frac{\tilde{\lambda}_u^{(1)}-\rho\tilde{\lambda}_u^{(2)}}{1-\rho^2}\mathrm{d}W_u^{(1)} - \int_0^t\frac{\tilde{\lambda}_u^{(2)}-\rho\tilde{\lambda}_u^{(1)}}{1-\rho^2}\mathrm{d}W_u^2.$$

Then, we have $(W_t + \int_0^t\tilde{\lambda}_u\,\mathrm{d}u)_{t\in[0,T]}$ is a $\tilde{\mathbb{Q}}$-Brownian motion with correlation ρ, and the drift terms of (B,S) with respect to $\tilde{\mathbb{Q}}$ are respectively $\int_0^t N_u\lambda_u^{(1)}\,\mathrm{d}u - \int_0^t N_u\tilde{\lambda}_u^{(1)}\,\mathrm{d}u$ and $\int_0^t L_u\lambda_u^{(2)}\,\mathrm{d}u - \int_0^t L_u\tilde{\lambda}_u^{(2)}\,\mathrm{d}u$ both of which must be identically zero. Since $N_t > 0$ and $L_t \neq 0$, $\tilde{\lambda} = \lambda$ and $\tilde{\mathbb{Q}} = \mathbb{Q}$. ■

Remark 3 The result of Dalbean and Schachermayer Delbaen and Schachermayer (1994) states that if the asset prices are continuous semimartingales then there is no arbitrage in the sense of *no free lunch with vanishing risk (NFLVR)* if and only if there exists an equivalent probability measure $\mathbb{Q}$ rendering the price processes local martingales. Note that for a claim $F(V_T)$ with a continuous function F, $(B_t)_{t\in[0,T]}$ is a continuous semimartingale. $(S_t)_{t\in[0,T]}$ is also a continuous semimartingale hence the conditions of Theorem 2 imply that the price processes $(B_t,S_t)_{t\in[0,T]}$ satisfy no arbitrage in the sense of NFLVR. If F is bounded as in the case of $\min(V_T,D)$, we can use the first fundamental theorem of asset pricing (see e.g. Shreve 2008, Theorem 5.4.7) to deduce that $(B_t,S_t)_{t\in[0,T]}$ allows no arbitrage in the elementary sense as in Definition 5.4.6 of Shreve (2008).

4. Application to credit risk pricing

In this section, we apply the results of the previous section to the following credit risk pricing model:

$$B_t = b(t,S_t,V_t)\cdot \mathrm{e}^{-K\cdot\frac{c(t,S_t,V_t)}{V_t^2}},$$

where B_t represents the price of the defaultable bond $\min(V_T,D)$ at time t, and the functions a, b and c are the solutions to the system of PDEs given by (11), (12), (13), with boundary conditions, for $(s,v)\in(0,\infty)\times(0,\infty)$,

$$a(T,s,v) = 1, \tag{21}$$
$$b(T,s,v) = \min(v,D), \tag{22}$$
$$c(T,s,v) = 0. \tag{23}$$

It is possible to solve the system of PDEs given by (11), (12), and (13) analytically, and derive explicit formulas for the functions a, b and c, (see Appendix A.1, in particular equations (A1), (A8), (A10)).

Theorem 4 *The PDE system given by* (11), (12), (13) *with boundary conditions* (21), (22), (23) *have unique solutions a, b and c, such that $a(t,s,v)$ is independent of s and v, and $b(t,s,v)$ and $c(t,s,v)$ are both independent of s. More precisely, for $0\le t<T$ and $v\in(0,\infty)$,*

(i) $a(t,s,v) = a(t) = \mathrm{e}^{-(\frac{\mu_2}{\sigma_2})^2(T-t)}$;

(ii) $b(t,s,v) = b(t,v) = v\,\mathrm{e}^{(\mu_1-\frac{\mu_2\rho\sigma_1}{\sigma_2})(T-t)}\Phi(d_1) + D(1-\Phi(d_2))$,*where*

$$d_1(t,v) = \frac{\ln\frac{D}{v} - \sigma_1^2\left[\frac{1}{2} + \frac{1}{\sigma_1^2}(\mu_1 - \frac{\sigma_1\rho\mu_2}{\sigma_2})\right](T-t)}{\sigma_1\sqrt{T-t}}, \tag{24}$$

$$d_2(t,v) = d_1 + \sigma_1\sqrt{T-t}; \tag{25}$$

(iii) $c(t,s,v) = c(t,v) = \sigma^2(1-\rho^2)\mathbb{E}(\int_t^{\mathrm{T}} a(u)V_u^2(\frac{\partial b}{\partial v}(u, V_u))^2\,\mathrm{d}u \mid V_t = v)$.

Proof See Appendix A.1 for the analytical derivations of the formulas for a, b and c. Below, we only prove the representation in part (iii).

Applying Itô's formula to $(c(t,V_t))_{t\in[0,T]}$, we get that

$$c(T,V_T) - c(t,V_t) = \int_t^{\mathrm{T}}\frac{\partial c}{\partial t}(u,V_u)\,\mathrm{d}u + \frac{1}{2}\frac{\partial^2 c}{\partial v^2}(u,V_u)d\langle V,V\rangle_u$$

$$
\begin{aligned}
&+\int_t^{\mathrm{T}} \frac{\partial c}{\partial v}(u, V_u)\,\mathrm{d}V_u \\
&= \int_t^{\mathrm{T}} \left(\frac{\partial c}{\partial t}(u, V_u) + \mu_1 V_u \frac{\partial c}{\partial v}(u, V_u) \right. \\
&\quad \left. + \frac{1}{2}\sigma_1^2 V_u^2 \frac{\partial^2 c}{\partial v^2}(u, V_u) \right) \mathrm{d}u \\
&\quad + \int_t^{\mathrm{T}} \sigma_1 \frac{\partial c}{\partial v}(u, V_u) V_u \,\mathrm{d}W_u^{(1)}.
\end{aligned}
$$

Since

$$
\frac{\partial c}{\partial t} = -\mu_1 v \frac{\partial c}{\partial v} - \frac{1}{2}\sigma_1^2 v^2 \frac{\partial^2 c}{\partial v^2} + a(t)\sigma_1^2 v^2(\rho^2 - 1)\left(\frac{\partial b}{\partial v}\right)^2,
$$

we get that

$$
\begin{aligned}
&c(T, V_T) - c(t, V_t) \\
&= \int_t^{\mathrm{T}} a(u)\sigma_1^2 V_u^2(\rho^2 - 1)\left(\frac{\partial b}{\partial v}(u, V_u)\right)^2 \mathrm{d}u \\
&\quad + \int_t^{\mathrm{T}} \sigma_1 \frac{\partial c}{\partial v}(u, V_u) V_u \,\mathrm{d}W_u^{(1)}.
\end{aligned}
$$

Note that $c(T, V_T) = 0$. Hence,

$$
c(t, V_t) = \mathbb{E}\left[\int_t^{\mathrm{T}} a(u)\sigma_1^2 V_u^2(1 - \rho^2)\left(\frac{\partial b}{\partial v}(u, V_u)\right)^2 \mathrm{d}u \,\middle|\, \mathcal{F}_t \right].
$$

Now, the representation in part (iii) follows from the Markov property of V. ■

Corollary 5 *Let $t \in [0, T)$. Under the conditions $S_t = s$ and $V_t = v$, the time t-optimal-replicating strategy $(\theta_u^*(t, s, v))_{u\geq t}$ and the corresponding portfolio process $(P_u^*(t, s, v))_{u\geq t}$ are given by:*

$$
\begin{aligned}
\mathrm{d}P_t^*(t, s, v) &= b(t, V_t), \\
\theta_u^*(t, s, v) &= \frac{-\mu_2}{\sigma_2^2 S_u a(u)}[P_u^*(t, s, v) - b(u, V_u)] \\
&\quad + \rho \frac{\sigma_1}{\sigma_2}\frac{V_u}{S_u}\frac{\partial b}{\partial v}(u, V_u), \quad u \geq t, \\
\mathrm{d}P_u^*(s, v) &= \theta_u^*(t, s, v)\,\mathrm{d}S_u, \quad u \geq t.
\end{aligned} \tag{26}
$$

Proof This is a consequence of equation (14) and the formulas in Theorem 4. ■

4.1. Analysis of the function $c : (t, v) \mapsto c(t, v)$

We derive a more explicit formula for the function c and analyze its qualitative properties:

Lemma 6 *The function $(t, v) \mapsto \frac{c(t,v)}{v^2}$ is monotone decreasing in v and $\lim_{t\to T} \frac{c(t,v)}{v^2} = 0$. In particular,*

$$
\begin{aligned}
c(t, v) &= \sigma_1^2(1 - \rho^2)v^2\,\mathrm{e}^{(2\mu_1+\sigma_1^2)(T-t)} \\
&\quad \times \int_t^{\mathrm{T}} \mathrm{e}^{\left(-(\frac{\mu_2}{\sigma_2})^2 - \sigma_1^2 - 2\frac{\mu_2}{\sigma_2}\rho\sigma_1\right)(T-u)}\,\mathbb{E}(\Phi(\mathbf{d})^2)\,\mathrm{d}u,
\end{aligned} \tag{27}
$$

where $\mathbf{d}$ is normally distributed with variance $\frac{u-t}{T-u}$ and mean

$$
\mu(t, u, v) = \frac{\begin{array}{c}\ln D - \ln v - (\mu_1 + \frac{3}{2}\sigma_1^2)(u - t) \\ -\sigma_1^2(\frac{1}{2} + \frac{\alpha}{\sigma_1^2})(T - u)\end{array}}{\sigma_1\sqrt{T - u}}, \tag{28}
$$

where $\alpha = \mu_1 - \rho\frac{\mu_2\sigma_1}{\sigma_2}$.

Proof It follows from equation (A8) in Appendix A.1 that

$$
\frac{\partial b}{\partial v}(u, v) = \mathrm{e}^{\left(\mu_1 - \frac{\mu_2\rho\sigma_1}{\sigma_2}\right)(T-u)}\Phi(d_1(u, v)), \tag{29}
$$

where

$$
d_1(u, v) = \frac{\ln\frac{D}{v} - \sigma_1^2(\frac{1}{2} + \frac{\alpha}{\sigma_1^2})(T - u)}{\sigma_1\sqrt{T - u}}.
$$

Let

$$
Z_u = \frac{V_u^2}{V_0^2}\mathrm{e}^{[-2\mu_1-\sigma_1^2]u}.
$$

Note that

$$
\begin{aligned}
Z_u &= \frac{V_u^2}{V_0^2}\mathrm{e}^{[-2\mu_1-\sigma_1^2]u} \\
&= \mathrm{e}^{[2\mu_1-\sigma_1^2]u+2\sigma_1 W_u}\,\mathrm{e}^{[-2\mu_1-\sigma_1^2]u} \\
&= \mathrm{e}^{-2\sigma_1^2 u+2\sigma_1 W_u},
\end{aligned}
$$

hence $(Z_u)_{u\in[0,T]}$ is a Girsanov density. Let $\tilde{\mathbb{Q}}$ be the probability measure on $(\Omega, \mathcal{F}_T)$ such that $\frac{\mathrm{d}\tilde{\mathbb{Q}}_u}{\mathrm{d}\mathbb{P}_u} = Z_u$, for $u \in [0, T]$, where $\tilde{\mathbb{Q}}_u$ and $\mathbb{P}_u$ respectively denote the restrictions of $\tilde{\mathbb{Q}}$ and $\mathbb{P}$ to $\mathcal{F}_u$. Now, we have that

$$
\begin{aligned}
c(t, v) &= \sigma_1^2(1 - \rho^2)v^2\,\mathrm{e}^{[-2\mu_1-\sigma_1^2]t}\mathbb{E}_{\tilde{\mathbb{Q}}} \\
&\quad \times \left(\int_t^{\mathrm{T}} a(u, V_u)\,\mathrm{e}^{[2\mu_1+\sigma^2]u}\left(\frac{\partial b}{\partial v}(u, V_u)\right)^2 \mathrm{d}u \,\middle|\, V_t = v \right).
\end{aligned}
$$

Substituting the formulas for a and $\frac{\partial b}{\partial v}$ we get that

$$
\begin{aligned}
c(t, v) &= \sigma_1^2(1 - \rho^2)v^2\,\mathrm{e}^{[-2\mu_1-\sigma^2]t} \\
&\quad \times \mathbb{E}_{\tilde{\mathbb{Q}}}\left(\int_t^{\mathrm{T}} \mathrm{e}^{-\left(\frac{\mu_2}{\sigma_2}\right)^2 T-u}\mathrm{e}^{[2\mu_1+\sigma^2]u}\mathrm{e}^{\left(2\mu_1 - 2\frac{\mu_2\rho\sigma_1}{\sigma_2}\right)(T-u)} \right. \\
&\quad \left. \times\ \Phi(d_1(u, V_u))^2\,\mathrm{d}u \,\middle|\, V_t = v \right) \\
&= \sigma_1^2(1 - \rho^2)v^2\,\mathrm{e}^{[2\mu_1+\sigma^2](T-t)} \\
&\quad \times \mathbb{E}_{\tilde{\mathbb{Q}}}\left(\int_t^{\mathrm{T}} \mathrm{e}^{\left[-\left(\frac{\mu_2}{\sigma_2}\right)^2 - \sigma^2 - 2\frac{\mu_2\rho\sigma_1}{\sigma_2}\right](T-u)} \right. \\
&\quad \left. \times\ \Phi(d_1(u, V_u))^2\,\mathrm{d}u \,\middle|\, V_t = v \right).
\end{aligned}
$$

Because W has a drift equal to $2\sigma_1$ with respect to $\tilde{\mathbb{Q}}$, the $\tilde{\mathbb{Q}}$-conditional distribution of $\log V_u = \log V_t + (\mu_1 - \frac{1}{2}\sigma_1^2)(u - t)) + \sigma_1(W_u - W_t)$ given $V_t = v$ is normal with mean $\log v +$

$(\mu_1 - \frac{\sigma_1^2}{2} + 2\sigma_1^2)(u-t)$ and standard deviation $\sigma_1\sqrt{u-t}$. Therefore $d_1(u, V_u)$ also has normal distribution, with mean given by equation (28) and standard deviation $\frac{\sqrt{u-t}}{\sqrt{T-u}}$. This proves equation (27). To complete the proof it is sufficient to show that the expected value of $\Phi(\mathbf{d})^2$ is monotone decreasing in v where

$$\mathbf{d} = \mu(t,u,v) + \sqrt{\frac{u-t}{T-u}}Z,$$

where Z is a standard normal random variable. Note that

$$\begin{aligned}\frac{\partial \mathbb{E}(\Phi(\mathbf{d})^2)}{\partial v} &= \mathbb{E}\left[\frac{\partial(\Phi(\mathbf{d})^2)}{\partial v}\right] \\ &= \mathbb{E}\left[2\Phi(\mathbf{d})\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{\mathbf{d}^2}{2}}\frac{\partial \mu(t,u,v)}{\partial v}\right].\end{aligned}$$

Since $\frac{\partial \mu(t,u,v)}{\partial v} < 0$, and all the other terms inside the expectation are always non-negative, $\frac{\partial \mathbb{E}(\Phi(\mathbf{d})^2)}{\partial v}$ is negative as desired. Equation (27) also implies that $\lim_{t\to T}\frac{c(t,v)}{v^2} = 0$. ■

The following proposition is needed to prove that our model meets Hypothesis (20) of Theorem 2.

PROPOSITION 7 *Assume that* $\frac{1}{2} + \frac{\alpha}{\sigma_1^2} > 0$. *Then there is a function* $O : [0,T) \times (0,\infty) \to \mathbb{R}$ *satisfying* $\sup_{t\in[0,T],v>0} |O(t,v)| < \infty$ *such that* $\frac{M_t}{N_t} = O(t, V_t)$.

The proof of Proposition 7 is deferred to Appendix A.2.

THEOREM 8 *Assume that* $\frac{1}{2} + \frac{\alpha}{\sigma_1^2} > 0$. *Let* $\tilde{c}(t,v) = c(t,v)/v^2$. *For any* $\kappa \geq 0$, $B_t = b(t,V_t)\,\mathrm{e}^{-\kappa\tilde{c}(t,V_t)}$, $t \in [0,T)$ *gives an arbitrage free price for the defaultable bond* $\min(V_T, D)$.

Proof We first observe that with $d(t,v) = v^2$, condition (16) is satisfied. Because $\frac{\partial b}{\partial v} > 0$ and $\frac{\partial \tilde{c}}{\partial v} < 0$, $\frac{\partial b}{\partial v} - \kappa\frac{\partial \tilde{c}}{\partial v} > 0$. Also Proposition 7 implies that Hypothesis (20) of Theorem 2 is satisfied. Hence, by Theorem 2, there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that $(S_t)_{t\in[0,T]}$ and $(B_t)_{t\in[0,T]}$ are $\mathbb{Q}$-martingales, hence (S, B) does not allow arbitrage in the sense of Definition 5.4.6 of Shreve (2008). ■

REMARK 9 It is possible that no arbitrage property still holds without the assumption $\frac{1}{2} + \frac{\alpha}{\sigma_1^2} > 0$, however one may need to use a criterion other than Novikov's criterion. A possible approach is to use the absolute continuity of the laws processes of the diffusion type (see e.g. Chapter 7 of Lipster and Shiryaev 2001). This would require the formulation of $(B_t, S_t)_{t\in[0,T]}$ as a solution of a stochastic differential equation in the sense of Lipster and Shiryaev (in particular, see section 7.6.4 of Lipster and Shiryaev 2001). To keep the presentation simple, we will not pursue this approach in this paper.

5. The effects of the parameters on the price

In this section, we fix $T=10$, $D=100$ and look at how various parameters affect the price of the defaultable bond $\min(V_T, D)$. Examining the formulas we see that there are five parameters affecting the price: μ_1, $\theta := \frac{\mu_2}{\sigma_2}$, ρ, σ_1 and κ.

5.1. *The effect of μ_1*

We observe that μ_1 affects b only through the parameter $\alpha = \mu_1 - \frac{\mu_2\rho\sigma_1}{\sigma_2}$. Hence we can calculate $\frac{\partial b}{\partial \alpha}$ and deduce $\frac{\partial b}{\partial \mu_1}$ from this. An elementary calculation gives

$$\frac{\partial b}{\partial \alpha}(t,v) = v(T-t)\,\mathrm{e}^{\alpha(T-t)}\Phi(d_1),$$

which is always positive. Since $\frac{\partial \alpha}{\partial \mu_1} = 1$, we find that b is strictly increasing with μ_1. Note that in our pricing model, we may interpret b as the price of $\min(V_t, D)$ when $\kappa = 0$. Hence

$$y(t) = -\log\frac{D}{b(t,V_t)}$$

is the yield of the bond for the case $\kappa = 0$. The fact that b is increasing with μ_1 means that the growth rate of the underlying firm, μ_1, affects the yield of the bond negatively which makes sense because the bond becomes less risky.

We also observe that when $\alpha = 0$, b is the price of the bond in the Merton model. Hence, in this model, depending on the sign of α, the price of the bond can be higher (when $\alpha > 0$) or lower ($\alpha < 0$) than its price in the Merton model.

The relationship between μ_1 and $\tilde{c}$ is not monotone, even when α remains constant.

5.2. *The effect of θ*

This parameter affects b only through α as well. Since $\frac{\partial \alpha}{\partial \theta} = -\rho\sigma_1$, b is decreasing with θ if $\rho > 0$ and increasing with θ if $\rho < 0$. If $\rho = 0$, θ has no effect on b. One can interpret θ as a measure of systemic risk in the underlying security. The condition $\rho > 0$ implies that the optimal replicating portfolio is also positively correlated with the same systemic risk, therefore it is intuitive that this portfolio gives a premium that increases with ρ. Also, the greater σ_1, the greater the exposure to the systemic risk, hence it is again intuitive that the premium increases with σ_1.

θ has a monotone decreasing relationship with $\tilde{c}$ provided that α remains constant. But since α is a function of θ, in general there is no monotone relationship between θ and $\tilde{c}$.

5.3. *The effect of σ_1*

We explicitly calculate $\frac{\partial b}{\partial \sigma_1}$ as

$$\frac{\partial b}{\partial \sigma_1} = v(T-t)\,\mathrm{e}^{\alpha(T-t)}\Phi(d_1)\frac{\partial \alpha}{\partial \sigma_1} - D\frac{\mathrm{e}^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}}\sqrt{T-t}.$$

Note that if $\rho\mu_2 \geq 0$, then $\frac{\partial \alpha}{\partial \sigma_1} < 0$, hence, in this case b is monotone decreasing as σ_1 increases. If $\rho\mu_2 < 0$, it is not clear if the effect of σ_1 on b would be monotone.

In figure (1) we plot $\tilde{c}$ as σ_1 varies from 0.05 to 0.25 for each of the cases $\rho = 0.6$, $v=66$ and $\rho = 0.6$, $v=133$, and $\rho = -0.6$, $v=66$ and $\rho = -0.6$, $v=133$, while setting $\mu_1 = 0.02$, $\theta = 0.4$. We see that in all the cases $\tilde{c}$ increases as σ_1 increases, suggesting that there is a monotone increasing relationship between $\tilde{c}$ and σ_1. However, we have not confirmed this theoretically.

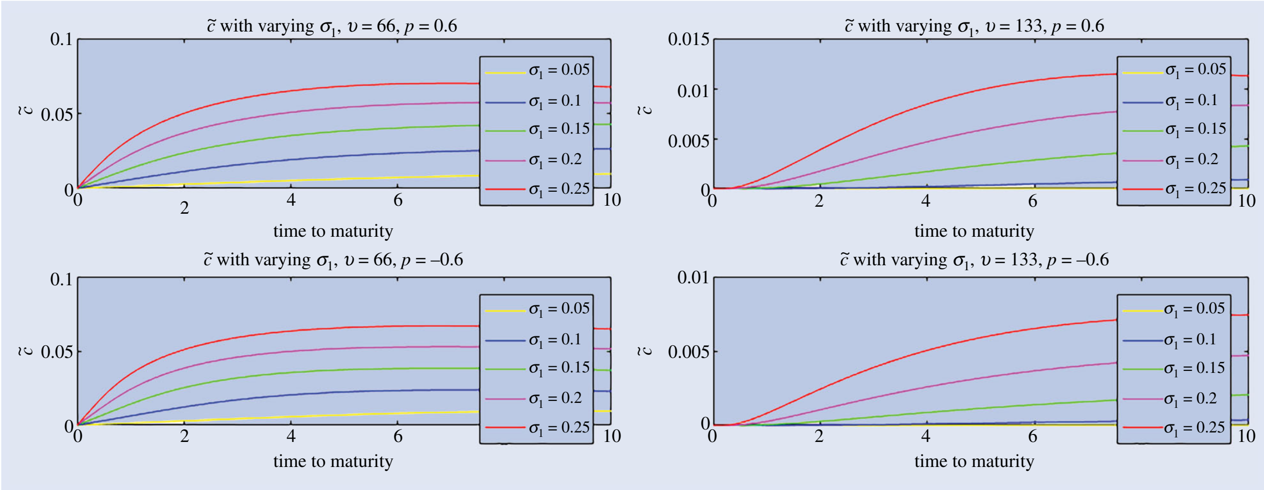

Figure 1. $\tilde{c}$ with varying σ_1, $\mu_1 = 0.02, \theta = 0.4$.

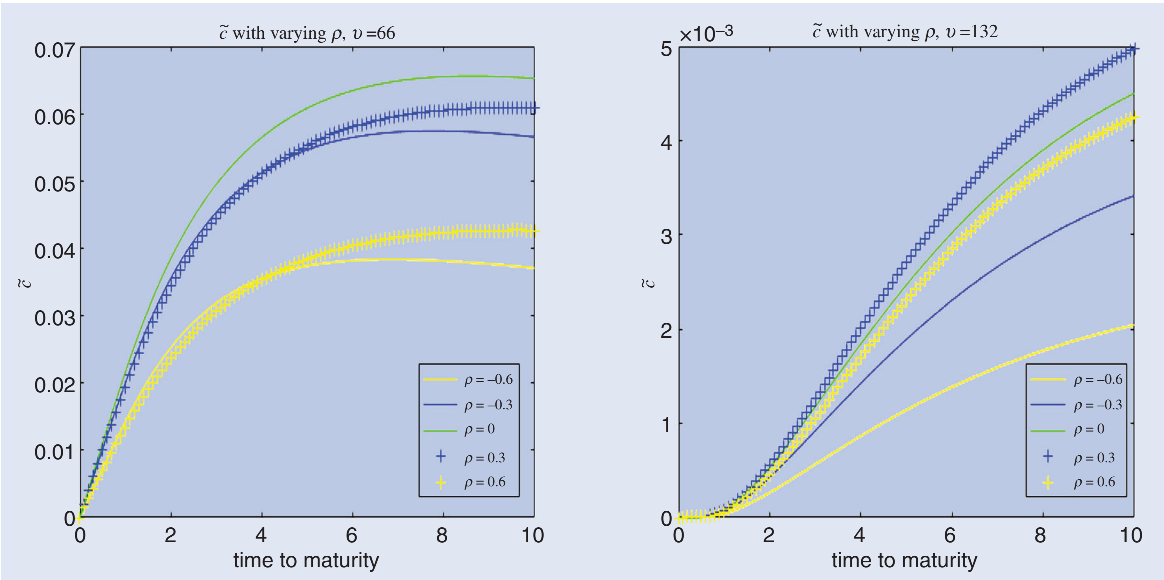

Figure 2. $\tilde{c}$ with varying ρ, $\mu_1 = 0.02, \sigma_1 = 0.15, \theta = 0.4$.

5.4. *The effect of ρ*

The correlation parameter ρ affects both the price of the replicating portfolio and the replication error. Examining the formula for b, we see that ρ appears only in the term α. Since α is monotone in ρ and b is monotone in α, b is monotone in ρ. In particular, b is monotone decreasing (resp. increasing) as ρ increases if $\mu_2 > 0$ (resp. $\mu_2 < 0$). If $\mu_2 = 0$ then b is constant in ρ.

The replication error c is a product of $1 - \rho^2$ and another term which also depends on ρ, but only through α. Hence if α remains constant, $\tilde{c}$ will decrease as ρ^2 increases, which is intuitive. But, surprisingly, there is no monotone relationship between $\tilde{c}$ and ρ^2 in general. In figure (2), we plotted $\tilde{c}$ as ρ varies from -0.6 to 0.6 setting $\mu_1 = 0.02, \sigma_1 = 0.15, \theta = 0.4$ for each of the cases $v = 66$ and $v = 132$. For example, when $v = 132$, $\tilde{c}$ does not decrease if ρ is changed from 0 to 0.3. We also observe that the replication error is smaller for negative values of ρ when ρ^2 is the same.

5.5. *The yield spreads and the effect of κ*

The yield at date t of the defaultable bond $\min(V_T, D)$ with maturity T is defined as the function

$$y(t,T) = \frac{\log(D) - \log(B_t)}{T - t}.$$

Since we assume that B_t is already discounted, $y(t,T)$ is also the yield spread. Recall that given the parameters, the price of the bond at time t is a function of $T - t$ and the firm value V_t only, hence we can represent $y(t,T) = y(V_t, T - t)$. In this section, we plot the function $y : (v, T) \mapsto y(v, T)$ as a function of T (time to maturity), for $v = 66$ and $v = 132$. We interpret $y(v, T)$ as the time 0 yield of the bond $\min(V_T, D)$, when the firm value at time 0 is equal to v. The values $v = 66$ and $v = 132$ represent two distinct scenarios where the firm's value is respectively less and greater than the face value of the debt $(D = 100)$.

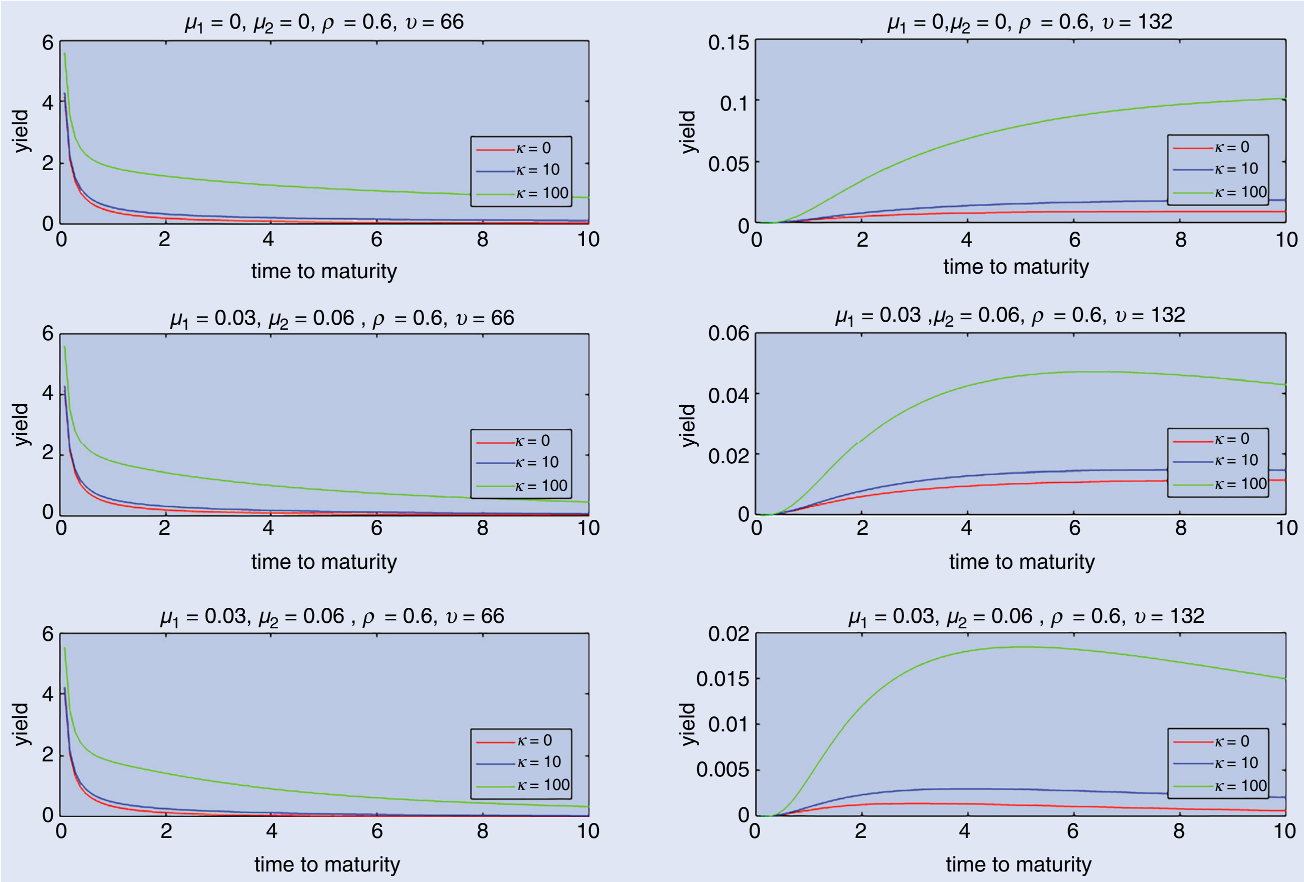

Figure 3. yield with varying κ for various configurations of the parameters.

We observe that in all the plots, the yields go to 0 when the firm value is greater than D, as in Merton's model. Hence, our model is subject to the same criticism as Merton's model that it underestimates the short spreads for firms that are not in danger of imminent default. Also, as in Merton's model, the yields go to infinity when the firm value is less than D, since in this case, the default is imminent for short maturities and the price of the bond simply reflects its recovery value, (see also Lando 2004 for a discussion of yield spreads in Merton's model).

In the first set of plots in figure 3 we set $\sigma_1 = 0.15$, $\rho = 0.6$, and $\mu_1 = \mu_2 = 0$. Note that this gives $\alpha = \mu_1 - \rho\theta\sigma_1 = 0$. Since the price of the replicating portfolio is the Merton model's price when $\alpha = 0$, we obtain the same yield curves as the Merton model when $\kappa = 0$. The yield curve is monotone decreasing when the firm's value is 66 and monotone increasing when $v = 133$. As we vary κ, the yield spreads increase, however the shape of the curves remain the same.

In the second set of plots in figure 3 we set $\mu_1 = 0.03$ and $\theta = 0.4$, while keeping $\sigma_1 = 0.15$ and $\rho = 0.6$ as before. Note that in this configuration $\alpha = -0.006$, which causes an increase in the yield spread at the baseline case, $\kappa = 0$. This is because the price of the replicating portfolio increases with α. The parameter κ increases the yield spreads, and when $v = 132$, and we see a hump shape pattern emerging when κ increases to 100.

In the third set of plots in figure 3, we set $\mu_1 = 0.03$ and $\theta = 0.4$, $\sigma_1 = 0.15$ and $\rho = -0.6$. In this configuration note that $\alpha = 0.009$, hence the yields decrease in the baseline case, $\kappa = 0$. The yields increase with κ as before, and the hump shape of the yield spread is more noticeable when $\kappa = 100$ and $v = 132$.

6. Conclusion

To summarize our results, we propose a pricing formula for a contingent claim on the assets of a firm whose assets are not liquidly tradeable. The price has two components, one is the price of the optimal mean variance replicating portfolio constructed by a correlated asset that is liquidly tradeable in the market. The second component is a discount factor which is determined by a parameter κ and the error of the replication. In the specific application to the pricing of the defaultable bond $\min(V_T, D)$, we are able to show that the pricing formula is arbitrage free without any restrictions on the value of κ, other than non-negativity. This gives a wide range of prices for the bond $\min(V_T, D)$, and the value of κ can be interpreted as the representative investor's risk aversion towards the non-hedgeable risk.

We are able to describe meaningful relationships between the price of the optimal mean variance replicating portfolio and (μ_1, θ, ρ) via both theoretical and numerical means. We are not able to qualitatively describe the relationship between the parameters and the replication error by theoretical means. However, numerical simulations suggest that σ_1 has a monotone relationship with the replication error. The numerical simulations indicate non-monotone relationships between the replication error and the other parameters.

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Appendix

A.1. Solutions of the PDEs

In this section we show the analytical derivation of the solutions of PDEs (11), (12), (13) with boundary conditions (21), (22), (23).

As the boundary condition for a is independent of s and v, it is natural to guess that the solution to PDE (11) is independent of s and v. It is easy to verify that the function a defined by

$$a(t) = \mathrm{e}^{\left(-\frac{\mu_2}{\sigma_2}\right)^2 (T-t)} \tag{A1}$$

is a solution to PDE (11). Moreover, (A1) is the unique solution according to Theorem 8.1 in Ladyzhenskaya *et al.* (1968).

To treat PDE (12), we use the change of variables:

$$\tau = T - t, \quad \tau \in [0, T]; \tag{A2}$$

$$u = \ln v, \quad u \in (-\infty, \infty); \tag{A3}$$

$$w = \ln s, \quad w \in (-\infty, \infty). \tag{A4}$$

Then PDE (12) becomes

$$\frac{\partial b}{\partial \tau} = \frac{1}{2}\sigma_1^2 \frac{\partial^2 b}{\partial u^2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 b}{\partial w^2} + \sigma_1\sigma_2\rho \frac{\partial^2 b}{\partial u \partial w} - \frac{1}{2}\sigma_1^2 \frac{\partial b}{\partial u} - \frac{1}{2}\sigma_2^2 \frac{\partial b}{\partial w} - \left(\frac{\sigma_1 \rho \mu_2}{\sigma_2} - \mu_1\right)\frac{\partial b}{\partial u} + \frac{\sigma_1^2}{a}(\rho^2 - 1)\frac{\partial a}{\partial u}\frac{\partial b}{\partial u}, \tag{A5}$$

with initial condition at $\tau = 0$:

$$b(0, w, u) = \min(\mathrm{e}^u, D).$$

Note that the initial condition as well as the coefficients in equation (A5) are independent of s. We thus may assume that the solution to (A5) is independent of s. In order to put the equation into a canonical form, we can make the following transformation

$$b(\tau, u) = \tilde{b}(\tau, u)\, \mathrm{e}^{\eta u + \beta \tau}, \tag{A6}$$

where η and β are chosen so that the lower order terms disappear. One can check that by choosing

$$\eta = \frac{1}{2} + \frac{1}{\sigma_1^2}\left[\frac{\sigma_1 \rho \mu_2}{\sigma_2} - \mu_1\right],$$

$$\beta = -\left[\frac{\sigma_1^2}{2} + \left(\frac{\sigma_1\rho\mu_2}{\sigma_2} - \mu_1\right)\right]\eta + \frac{\sigma_1^2}{2}\eta^2 = -\frac{\sigma_1^2}{2}\eta^2,$$

PDE (A5) reduces to

$$\frac{\partial\tilde{b}}{\partial\tau} = \frac{1}{2}\sigma_1^2\frac{\partial^2\tilde{b}}{\partial u^2}, \tag{A7}$$

with initial condition

$$\tilde{b}(0,u) = \mathrm{e}^{-\eta u}\min(\mathrm{e}^u, D).$$

It is known Evans (1998) that the solution to the Cauchy problem given by (A7) is unique and is given by

$$\tilde{b}(\tau,u) = \frac{1}{\sigma_1\sqrt{2\pi\tau}}\int_{-\infty}^{+\infty}\tilde{b}(0,y)\,\mathrm{e}^{-\frac{(u-y)^2}{2\sigma_1^2\tau}}\,\mathrm{d}y.$$

Substituting $\tilde{b}(0,y) = \mathrm{e}^{-\eta y}\min(\mathrm{e}^y, D)$, we obtain

$$\begin{aligned}\tilde{b}(\tau,u) &= \frac{1}{\sigma_1\sqrt{2\pi\tau}}\int_{-\infty}^{\ln D}\mathrm{e}^{(1-\eta)y}\,\mathrm{e}^{-\frac{(u-y)^2}{2\sigma_1^2\tau}}\,\mathrm{d}y\\ &\quad+\frac{1}{\sigma_1\sqrt{2\pi\tau}}\int_{\ln D}^{+\infty}D\,\mathrm{e}^{-\eta y}\,\mathrm{e}^{-\frac{(u-y)^2}{2\sigma_1^2\tau}}\,\mathrm{d}y\\ &= \mathrm{e}^{u(1-\eta)+\frac{\sigma_1^2\tau(1-\eta)^2}{2}}\,\Phi\left(\frac{\ln D - u - \sigma_1^2\tau(1-\eta)}{\sigma_1\sqrt{\tau}}\right)\\ &\quad- D\,\mathrm{e}^{-u\eta+\frac{\sigma_1^2\tau\eta^2}{2}}\,\Phi\left(\frac{\ln D - u - \sigma_1^2\tau\eta}{\sigma_1\sqrt{\tau}}\right)\\ &\quad+ D\,\mathrm{e}^{-u\eta+\frac{\sigma_1^2\tau\eta^2}{2}}.\end{aligned}$$

Substituting (A2), (A3) and (A6), we obtain the unique solution to PDE (12):

$$\begin{aligned}&b(t,v) = v\,\mathrm{e}^{\left(\mu_1-\frac{\mu_2\rho\sigma_1}{\sigma_2}\right)(T-t)}\Phi(d_1(t,v)) + D\left(1-\Phi(d_2(t,v))\right),\\ &b(T,v) = \min(v,D),\end{aligned} \tag{A8}$$

where Φ is the standard normal distribution function and

$$\begin{aligned}d_1(t,v) &= \frac{\ln\frac{D}{v} - \sigma_1^2(1-\eta)(T-t)}{\sigma_1\sqrt{T-t}},\\ d_2(t,v) &= d_1 + \sigma_1\sqrt{T-t},\end{aligned}$$

where we have $\eta = \frac{1}{2} + \frac{1}{\sigma_1^2}[\frac{\sigma_1\rho\mu_2}{\sigma_2} - \mu_1]$. Note that $1-\eta = \frac{1}{2} + \frac{\alpha}{\sigma_1^2}$ where $\alpha = \mu_1 - \frac{\mu_2\rho\sigma_1}{\sigma_2}$. In regarding to the PDE for c, by the change of variables (A2), (A3) and (A4), equation (13) becomes

$$\begin{aligned}\frac{\partial c}{\partial\tau} &= \mu_1\frac{\partial c}{\partial u} + \mu_2\frac{\partial c}{\partial w} + \frac{1}{2}\sigma_1^2\left(\frac{\partial^2 c}{\partial u^2} - \frac{\partial c}{\partial u}\right) + \frac{1}{2}\sigma_2^2\left(\frac{\partial^2 c}{\partial w^2} - \frac{\partial c}{\partial w}\right)\\ &\quad- a\sigma_1^2(\rho^2-1)\left(\frac{\partial b}{\partial u}\right)^2,\end{aligned} \tag{A9}$$

with initial condition at $\tau = 0$:

$$c(0,w,u) = 0.$$

Note that the initial condition as well as the coefficients in equation (A9) are independent of s. We hence may assume that the solution to (A9) is independent of w. Similar to what we have done for b, we make the following transformation

$$c(\tau,u) = \tilde{c}(\tau,u)\,\mathrm{e}^{\alpha^{(1)}u+\beta^{(1)}\tau}, \tag{A10}$$

with

$$\alpha^{(1)} = \frac{1}{2} - \frac{1}{\sigma_1^2}\mu_1,$$

$$\beta^{(1)} = \left(\mu_1 - \frac{1}{2}\sigma_1^2\right)\alpha^{(1)} + \frac{\sigma_1^2}{2}(\alpha^{(1)})^2 = -\frac{\sigma_1^2}{2}(\alpha^{(1)})^2.$$

Therefore, PDE (A9) reduces to

$$\frac{\partial\tilde{c}}{\partial\tau} = \frac{1}{2}\sigma_1^2\frac{\partial^2\tilde{c}}{\partial u^2} + \mathrm{e}^{-\alpha^{(1)}u-\beta^{(1)}\tau}a\sigma_1^2(1-\rho^2)\left(\frac{\partial b}{\partial u}\right)^2, \tag{A11}$$

with initial condition

$$\tilde{c}(0,u) = 0.$$

In order to solve PDE (A11), we need the value of $\frac{\partial b}{\partial u}$. We differentiate equation (A8) to obtain

$$\frac{\partial b}{\partial u} = \mathrm{e}^{u+\left(\mu_1-\frac{\mu_2\rho\sigma_1}{\sigma_2}\right)\tau}\Phi(d_1). \tag{A12}$$

By substituting (A12) into (A11), we obtain the unique solution to PDE (A11) in the form

$$\begin{aligned}&\tilde{c}(\tau,u) = \int_0^\tau\frac{1}{\sigma_1\sqrt{2\pi(\tau-s)}}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{y^2}{2\sigma_1^2(\tau-s)}}f(s,u-y)\,\mathrm{d}y\,\mathrm{d}s,\\ &\tilde{c}(0,u) = 0,\end{aligned} \tag{A13}$$

where we have the substitutions

$$\begin{aligned}u &= \ln v,\\ f(\tau,u) &= \mathrm{e}^{-\alpha^{(1)}u-\beta^{(1)}\tau}a\sigma_1^2(1-\rho^2)\left(\frac{\partial b}{\partial u}\right)^2,\\ c(\tau,u) &= \tilde{c}(\tau,u)\,\mathrm{e}^{\alpha^{(1)}u+\beta^{(1)}\tau},\\ \alpha^{(1)} &= \frac{1}{2} - \frac{1}{\sigma_1^2}\mu_1,\\ \beta^{(1)} &= -\frac{\sigma_1^2}{2}(\alpha^{(1)})^2.\end{aligned}$$

A.2. *Proof of Proposition* 7

Proof Using equation (19) and the PDEs for b and c, we obtain the following formula for M_t:

$$M_t = M(t, V_t),$$

where

$$\begin{aligned}M(t,v) &= \mathrm{e}^{-\kappa\tilde{c}}\left[\frac{\sigma_1}{\sigma_2}\rho\mu_2 v\frac{\partial b}{\partial v}\right] - \kappa b\,\mathrm{e}^{-\kappa\tilde{c}}a\sigma_1^2(\rho^2-1)\left(\frac{\partial b}{\partial v}\right)^2\\ &\quad+ 2\kappa b\,\mathrm{e}^{-\kappa\tilde{c}}\frac{\partial\tilde{c}}{\partial v}v\sigma_1^2\\ &\quad+ (2\mu_1+\sigma_1^2)\kappa b\,\mathrm{e}^{-\kappa\tilde{c}}\tilde{c}\sigma_1^2 + \frac{1}{2}\kappa^2 b\,\mathrm{e}^{-\kappa\tilde{c}}\left(\frac{\partial\tilde{c}}{\partial v}\right)^2\sigma_1^2 v^2\\ &\quad- \kappa\,\mathrm{e}^{-\kappa\tilde{c}}\frac{\partial\tilde{c}}{\partial v}\frac{\partial b}{\partial v}\sigma_1^2 v^2.\end{aligned}$$

We also have that $N_t = N(t,v)$, where

$$N(t,v) = \mathrm{e}^{-\kappa\tilde{c}}v\sigma_1\left[\frac{\partial b}{\partial v} - \kappa b\frac{\partial\tilde{c}}{\partial v}\right].$$

Hence, we have that $\frac{M_t}{N_t} = O(t,V_t)$, where $O(t,v) = \frac{M(t,v)}{N(t,v)}$. Because

$$\left[\frac{\partial b}{\partial v} - \kappa b\frac{\partial\tilde{c}}{\partial v}\right] \geq \max\left(\frac{\partial b}{\partial v}, \kappa b\left|\frac{\partial\tilde{c}}{\partial v}\right|\right),$$

and noting that a, b, $\mathrm{e}^{-\kappa\tilde{c}}$, $\frac{\partial b}{\partial v}$ and $\tilde{c}$ are bounded functions, we have that $\sup_{t\in[0,T],v>0}|O(t,v)| < \infty$, if we show the following:

(i) $\frac{b}{v}$ is bounded when $t \in [0,T]$, $v > 0$,
(ii) $\frac{\partial \tilde{c}}{\partial v}.v$ is bounded when $t \in [0,T)$, $v > 0$,
(iii) $\frac{b}{\frac{\partial b}{\partial v}v}$ is bounded when $t \in [0,T)$, $0 < v \le D$,
(iv) $\frac{\tilde{c}}{\frac{\partial \tilde{c}}{\partial v}v}$ is bounded when $t \in [0,T)$, $v > D$.

We estimate each quantity as follows:

(i) We note that $(\frac{b}{v})' = -\frac{D(1-\Phi(d_2))}{v} < 0$, hence $\sup_v \frac{b}{v} = \lim_{v\to 0}\frac{b}{v} = \lim_{v\to 0}\frac{\partial b}{\partial v} = \mathrm{e}^{(\mu_1 - \frac{\mu_2\rho\sigma_1}{\sigma_2})(T-t)}$, which is bounded.
(ii) Let $\mathbf{d}$ be normally distributed with mean $\mu(t,u,v)$ and variance $\frac{u-t}{T-u}$. We observe that

$$\begin{aligned}\left|\frac{\partial E(\Phi(\mathbf{d})^2)}{\partial v}\right| &= \left|\frac{\partial}{\partial v}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{z^2}{2}}\right.\\ &\quad \left.\times\ \Phi\left(\mu(t,u,v)+\sqrt{\frac{u-t}{T-u}}z\right)^2 \mathrm{d}z\right|\\ &= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{z^2}{2}}2\Phi\left(\mu(t,u,v)+\sqrt{\frac{u-t}{T-u}}z\right)\\ &\quad\times\frac{\mathrm{e}^{-\frac{(\mu(t,u,v)+\sqrt{\frac{u-t}{T-u}}z)^2}{2}}}{\sqrt{2\pi}}\frac{1}{v\sigma_1\sqrt{T-u}}\mathrm{d}z\\ &\le \frac{2}{v\sqrt{2\pi}\sigma_1\sqrt{T-u}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\left|\frac{\partial\tilde{c}}{\partial v}\right|.v &\le \sigma_1^2(1-\rho^2)\,\mathrm{e}^{[(2\mu_1+\sigma_1^2)](T-t)}\\ &\quad\times\int_t^{\mathrm{T}}\mathrm{e}^{\left(-(\frac{\mu_2}{\sigma_2})^2-\sigma_1^2-2\frac{\mu_2}{\sigma_2}\rho\sigma_1\right)(T-u)}\\ &\quad\times\frac{2}{\sqrt{2\pi}\sigma_1\sqrt{T-u}}\mathrm{d}u\\ &\le \tilde{K}\int_t^{\mathrm{T}}\frac{1}{\sigma_1\sqrt{T-u}}\mathrm{d}u\\ &\le 2\frac{\tilde{K}}{\sigma_1}\sqrt{T-t},\end{aligned}$$

for some constant $\tilde{K}$. Hence $\frac{\partial\tilde{c}}{\partial v}.v$ is bounded as desired.
(iii) Note that

$$\begin{aligned}\frac{b(t,v)}{\frac{\partial b}{\partial v}(t,v)v} &= \frac{v\,\mathrm{e}^{\alpha(T-t)}\Phi(d_1(t,v))+D[1-\Phi(d_2(t,v))]}{v\,\mathrm{e}^{\alpha(T-t)}\Phi(d_1(t,v))}\\ &= 1+\frac{D[1-\Phi(d_2(t,v))]}{\Phi(d_1(t,v))}.\end{aligned}\tag{A14}$$

Next, observe that $\frac{b(t,v)}{\frac{\partial b}{\partial v}(t,v)v}$ is bounded in the domain $[0,T)\times[D/2,D]$, because $d_1(t,v)$ is bounded below by $K = -|\sigma||\frac{1}{2}+\frac{\alpha}{\sigma_1^2}|T$ and

$$\frac{b(t,v)}{\frac{\partial b}{\partial v}(t,v)v} \le 1+\frac{D}{v\Phi(K)} \le 1+\frac{2}{\Phi(K)},$$
$$\text{for } (t,v)\in[0,T)\times[D/2,D].$$

To show that $\frac{b(t,v)}{\frac{\partial b}{\partial v}(t,v)v}$ is also bounded in $[0,T)\times[0,D/2)$, due to continuity, it is sufficient that $\lim_{n\to\infty}\frac{b(t_n,v_n)}{\frac{\partial b}{\partial v}(t_n,v_n)v_n}$ exists and is finite, whenever $(t_n,v_n)\to(t,0)$, $t\in[0,T]$ or $(t_n,v_n)\to(T,v)$, $0<v\le D/2$: Indeed, let $(t_n,v_n)\to(t,0)$, $t\in[0,T]$. Then $\lim_{n\to\infty}d_1(t_n,v_n) = \lim_{n\to\infty}d_2(t_n,v_n) = \infty$. Hence, for sufficiently large n,

$$\frac{1-\Phi(d_2(t_n,v_n))}{v_n\Phi(d_1(t_n,v_n))} \le \frac{\mathrm{e}^{-\frac{(d_2(t_n,v_n))^2}{2}}\mathrm{e}^{-\ln(v_n)}}{\sqrt{2\pi}},$$

where we use the upper tail inequality for the standard normal distribution:

$$1-\Phi(x) \le \frac{\mathrm{e}^{-\frac{x^2}{2}}}{x\sqrt{2\pi}},\quad \text{for } x>0.\tag{A15}$$

Since $d_2(t_n,v_n)^2$ is a quadratic polynomial in $\ln(v_n)$ with a positive coefficient of $(\ln(v_n))^2$, we have that $\lim_{n\to\infty}-\frac{d_2(t_n,v_n))^2}{2}-\ln(v_n) = -\infty$, hence, by equation (A14),

$$\lim_{n\to\infty}\frac{b(t_n,v_n)}{\frac{\partial b}{\partial v}(t_n,v_n)v_n} = 0,\quad \text{as } (t_n,v_n)\to(t,0).$$

Next, let $(t_n,v_n)\to(T,v)$, $0<v\le D/2$. Then,

$$\lim_{n\to\infty}d_1(t_n,v_n) = \lim_{n\to\infty}d_2(t_n,v_n) = \infty.$$

Since $v_n\Phi(d_1(t_n,v_n))\to v$ and $D(1-\Phi(d_2(t_n,v_n))\to 0$, by equation (A14),

$$\lim_{n\to\infty}\frac{b(t_n,v_n)}{\frac{\partial b}{\partial v}(t_n,v_n)v_n} = 0\quad \text{as } (t_n,v_n)\to(T,v),$$

as well.
We conclude that $\sup_{(t,v)\in[0,T)\times(0,D]}\frac{b(t,v)}{\frac{\partial b}{\partial v}(t,v)v} < \infty$.
(iv) Let us assume $t\in[0,T)$, $v>D$. Let $K_1 = -(\frac{\mu_2}{\sigma_2})^2-\sigma_1^2-2\frac{\mu_2}{\sigma_2}\rho\sigma_1$. We observe that

$$\frac{\tilde{c}(t,v)}{\frac{\partial\tilde{c}}{\partial v}(t,v)v} = K_2\frac{\int_t^{\mathrm{T}}\mathrm{e}^{K_1(T-u)}\mathbb{E}[\Phi(\mathbf{d})^2]\,\mathrm{d}u}{\int_t^{\mathrm{T}}\frac{\mathrm{e}^{K_1(T-u)}}{\sigma_1\sqrt{T-u}}\mathbb{E}[\Phi(\mathbf{d})\,\mathrm{e}^{-\frac{\mathbf{d}^2}{2}}]\,\mathrm{d}u},$$

where $K_2>0$ is a constant independent of t and v. For $u\in[t,T]$, we define

$$z(t,u,v) = \frac{-\mu(t,u,v)\sqrt{T-u}}{\sqrt{u-t}},$$

where $\mu(t,u,v)$ is defined by equation (28). Note that $\mathbf{d} = \frac{\sqrt{u-t}}{\sqrt{T-u}}Z+\mu(t,u,v)$, and $\mathbf{d}>0$ if and only if $Z>z(t,u,v)$. Next, we decompose $\int_t^{\mathrm{T}}\mathrm{e}^{K_1(T-u)}\mathbb{E}[\Phi(\mathbf{d})^2]du$ into 3 parts:

$$\begin{aligned}&\int_t^{\mathrm{T}}\mathrm{e}^{K_1(T-u)}\mathbb{E}(\Phi(\mathbf{d})^2)\,\mathrm{d}u\\ &\quad= \int_t^{\mathrm{T}}\mathrm{e}^{K_1(T-u)}\mathbb{E}[\Phi(\mathbf{d})^2 1_{\{\mathbf{d}<-1\}}]\,\mathrm{d}u\\ &\qquad+\int_t^{\mathrm{T}}\mathrm{e}^{K_1(T-u)}\mathbb{E}[\Phi(\mathbf{d})^2 1_{\{\mathbf{d}\ge-1\}}]1_{\{z(t,u,v)\le 1\}}\,\mathrm{d}u\\ &\qquad+\int_t^{\mathrm{T}}\mathrm{e}^{K_1(T-u)}\mathbb{E}[\Phi(\mathbf{d})^2 1_{\{\mathbf{d}\ge-1\}}]1_{\{z(t,u,v)>1\}}\,\mathrm{d}u.\end{aligned}\tag{A16}$$

Let *I*, *II*, and *III* be the integrals on the right hand side of equation (A16).

Due to (A15), $\Phi(\mathbf{d})^2 1_{\{\mathbf{d}<-1\}} \le \Phi(\mathbf{d}) \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{\mathbf{d}^2}{2}} 1_{\{\mathbf{d}<-1\}}$, hence

$$\mathbb{E}(\Phi(\mathbf{d})^2 1_{\{\mathbf{d}<-1\}}) \le \frac{1}{\sqrt{2\pi}} \mathbb{E}(\Phi(\mathbf{d})\,\mathrm{e}^{-\frac{\mathbf{d}^2}{2}}) \le K_3 \frac{\mathbb{E}(\Phi(\mathbf{d})\,\mathrm{e}^{-\frac{\mathbf{d}^2}{2}})}{\sigma_1\sqrt{T-u}},$$

where $K_3 = \frac{1}{\sqrt{2\pi}}\sigma_1\sqrt{T}$. Hence

$$\frac{I}{\int_t^{\mathrm{T}} \frac{\mathrm{e}^{K_1(T-u)}}{\sigma_1\sqrt{T-u}} \mathbb{E}[\Phi(\mathbf{d})\,\mathrm{e}^{-\frac{\mathbf{d}^2}{2}}]\,\mathrm{d}u} \le K_3.$$

The condition $z(t,u,v) - \sqrt{\frac{T-u}{u-t}} \le Z \le z(t,u,v)$ is equivalent to $-1 \le \mathbf{d} \le 0$. Since $\Phi(\mathbf{d})\,\mathrm{e}^{-\frac{\mathbf{d}^2}{2}}$ is bounded below by some $\gamma > 0$ when $-1 \le \mathbf{d} \le 0$, we obtain the bound

$$\mathbb{E}[\Phi(\mathbf{d})\,\mathrm{e}^{-\frac{\mathbf{d}^2}{2}}] \ge K_4 \mathbb{P}(z(t,u,v) - \sqrt{\frac{T-u}{u-t}} < Z < z(t,u,v)), \tag{A17}$$

where $K_4 = 1/\gamma$.

To bound the term $\frac{II}{\int_t^{\mathrm{T}} \frac{\mathrm{e}^{K_1(T-u)}}{\sigma_1\sqrt{T-u}} \mathbb{E}[\Phi(\mathbf{d})\,\mathrm{e}^{-\frac{\mathbf{d}^2}{2}}]\,\mathrm{d}u}$, we first argue that if $z(t,u,v) \le 1$ then

$$\mathbb{E}[\Phi(\mathbf{d})\,\mathrm{e}^{-\frac{\mathbf{d}^2}{2}}] > \epsilon \min\left(1, \frac{\sqrt{T-u}}{\sqrt{t-u}}\right),$$

where ϵ is a positive number independent of u,t and v: Let $\tilde{\alpha} = \mu_1 + \frac{3}{2}\sigma_1^2$. Because of the assumptions $v > D$ and $\frac{1}{2} + \frac{\alpha}{\sigma_1^2} > 0$, and

$$z(t,u,v) = -\frac{\ln(D/v)}{\sigma_1\sqrt{u-t}} + \frac{\tilde{\alpha}\sqrt{u-t}}{\sigma_1} + \sigma_1(\frac{1}{2} + \frac{\alpha}{\sigma_1^2})\frac{T-u}{\sqrt{u-t}},$$

$z(t,u,v) > \frac{-|\tilde{\alpha}|\sqrt{T}}{\sigma_1}$ for all u,v. If $\sqrt{\frac{T-u}{u-t}} > 1$, then $\mathbb{P}(z(t,u,v) - \sqrt{\frac{T-u}{u-t}} < Z < z(t,u,v)) \ge \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}\max(z(t,u,v)^2,(z(t,u,v)-1)^2)}$, which is bounded below by $\epsilon > 0$ independent of u,v or t, since $1 \ge z(t,u,v) > -\frac{\tilde{\alpha}\sqrt{T}}{\sigma_1}$. If $\sqrt{\frac{T-t}{u-t}} \le 1$, then $\mathbb{P}(z(t,u,v) - \sqrt{\frac{T-u}{u-t}} < Z < z(t,u,v)) \ge \sqrt{\frac{T-u}{u-t}}\frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\max(z(t,u,v)^2,(z(t,u,v)-1)^2)}$, which proves the claim. Hence,

$$\begin{aligned}
&\frac{II}{\int_t^{\mathrm{T}} \frac{\mathrm{e}^{K_1(T-u)}}{\sigma_1\sqrt{T-u}} \mathbb{E}[\Phi(\mathbf{d})\,\mathrm{e}^{-\frac{\mathbf{d}^2}{2}}]\,\mathrm{d}u} \\
&\le \frac{\int_t^{\mathrm{T}} \mathrm{e}^{K_1(T-u)} 1_{\{z(t,u,v)\le 1\}}\,\mathrm{d}u}{\epsilon \int_t^{\mathrm{T}} \mathrm{e}^{K_1(T-u)} K_4 \frac{\min(1,\frac{\sqrt{T-u}}{\sqrt{t-u}})}{\sigma_1\sqrt{T-u}} 1_{\{z(t,u,v)\le 1\}}\,\mathrm{d}u} \\
&\le K_5 \frac{\int_t^{\mathrm{T}} \mathrm{e}^{K_1(T-u)} 1_{\{z(t,u,v)\le 1\}}\,\mathrm{d}u}{\int_t^{\mathrm{T}} \mathrm{e}^{K_1(T-u)} 1_{\{z(t,u,v)\le 1\}}\,\mathrm{d}u} \\
&= K_5,
\end{aligned}$$

where $K_5 = \frac{\sigma_1\sqrt{T}}{\epsilon K_4}$.

To bound the term $\frac{III}{\int_t^{\mathrm{T}} \frac{\mathrm{e}^{K_1(T-u)}}{\sigma_1\sqrt{T-u}} \mathbb{E}[\Phi(\mathbf{d})\,\mathrm{e}^{-\frac{\mathbf{d}^2}{2}}]\,\mathrm{d}u}$, we observe that $\mathbb{E}(\Phi(\mathbf{d})^2 1_{\{\mathbf{d}\ge -1\}}) \le \mathbb{P}(Z > z(t,u,v) - \sqrt{\frac{T-u}{t-u}})$ and

$$\begin{aligned}
&\frac{III}{\int_t^{\mathrm{T}} \frac{\mathrm{e}^{K_1(T-u)}}{\sigma_1\sqrt{T-u}} \mathbb{E}[\Phi(\mathbf{d})\,\mathrm{e}^{-\frac{\mathbf{d}^2}{2}}]\,\mathrm{d}u} \\
&\le \frac{\int_t^{\mathrm{T}} \mathrm{e}^{K_1(T-u)} \mathbb{P}(z(t,u,v) - \sqrt{\frac{T-u}{u-t}} \le Z \le z(t,u,v)) 1_{\{z(t,u,v)>1\}}\,\mathrm{d}u}{\int_t^{\mathrm{T}} \frac{\mathrm{e}^{K_1(T-u)}}{\sigma_1\sqrt{T-u}} K_4 \mathbb{P}(z(t,u,v) - \sqrt{\frac{T-u}{u-t}} < Z < z(t,u,v)) 1_{\{z(t,u,v)>1\}}\,\mathrm{d}u} \\
&\quad + \frac{\int_t^{\mathrm{T}} \mathrm{e}^{K_1(T-u)} \mathbb{P}(Z > z(t,u,v)) 1_{\{z(t,u,v)>1\}}\,\mathrm{d}u}{\int_t^{\mathrm{T}} \frac{\mathrm{e}^{K_1(T-u)}}{\sigma_1\sqrt{T-u}} K_4 \mathbb{P}(z(t,u,v) - \sqrt{\frac{T-u}{u-t}} < Z < z(t,u,v)) 1_{\{z(t,u,v)>1\}}\,\mathrm{d}u} \\
&\le K_6 + \frac{\int_t^{\mathrm{T}} \mathrm{e}^{K_1(T-u)} \mathbb{P}(Z > z(t,u,v)) 1_{\{z(t,u,v)>1\}}\,\mathrm{d}u}{\int_t^{\mathrm{T}} \frac{\mathrm{e}^{K_1(T-u)}}{\sigma_1\sqrt{T-u}} K_4 \mathbb{P}(z(t,u,v) - \sqrt{\frac{T-u}{u-t}} < Z < z(t,u,v)) 1_{\{z(t,u,v)>1\}}\,\mathrm{d}u},
\end{aligned}$$

where $K_6 = \frac{\sigma_1\sqrt{T}}{K_4}$. Since $\mathbb{P}(Z > z(t,u,v)) \le \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{z(t,u,v)^2}{2}}$ when $z(t,u,v) > 1$, (due to (A15)), and, when $z(t,u,v) > 1$,

$$\begin{aligned}
&\mathbb{P}(z(t,u,v) - \sqrt{\frac{T-u}{u-t}} < Z < z(t,u,v)) \\
&\ge \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}\max(z(t,u,v)^2,(z(t,u,v)-1)^2)} \min(1, \frac{\sqrt{T-u}}{\sqrt{u-t}}) \\
&\ge \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}z(t,u,v)^2} \min\left(1, \frac{\sqrt{T-u}}{\sqrt{u-t}}\right),
\end{aligned}$$

we get that

$$\begin{aligned}
&\frac{III}{\int_t^{\mathrm{T}} \frac{\mathrm{e}^{K_1(T-u)}}{\sigma_1\sqrt{T-u}} \mathbb{E}[\Phi(\mathbf{d})\,\mathrm{e}^{-\frac{\mathbf{d}^2}{2}}]\,\mathrm{d}u} \\
&\le K_6 + \frac{\int_t^{\mathrm{T}} \mathrm{e}^{K_1(T-u)} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{z(t,u,v)^2}{2}} 1_{\{z(t,u,v)>1\}}\,\mathrm{d}u}{\int_t^{\mathrm{T}} \mathrm{e}^{K_1(T-u)} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}z(t,u,v)^2} \frac{\min(1,\frac{\sqrt{T-u}}{\sqrt{u-t}})}{\sigma_1\sqrt{T-u}} 1_{\{z(t,u,v)>1\}}\,\mathrm{d}u}. \\
&= K_6 + \frac{\sigma_1\sqrt{T}}{K_4}.
\end{aligned}$$

We conclude that

$$\frac{\tilde{c}(t,v)}{\frac{\partial\tilde{c}}{\partial v}(t,v)v} \le K_2(K_3 + K_5 + K_6 + \frac{\sigma_1\sqrt{T}}{K_4}), \quad t \in [0,T), v > D. \blacksquare$$