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ON PERMANENCE AND STABILITY OF A LOGISTIC MODEL WITH HARVESTING AND A CARRYING CAPACITY DEPENDENT DIFFUSION

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Abstract. For the reaction-diffusion equation describing the harvested population with the logistic type of growth and diffusion stipulated by the carrying capacity K

$$\frac{\partial u(t,x)}{\partial t} = D\Delta\left(\frac{u(t,x)}{K(x)}\right) + r(x)u(t,x)\left(1-\frac{u(t,x)}{K(x)}\right) - E(x)u(x,t)$$

existence, positivity, persistence and stability of solutions is investigated. In the logistic model the introduction of the standard diffusion term Δu (incorporated with the zero Neumann boundary conditions) leads to the situation when the population tends to be equally distributed over the space available, even if the carrying capacity $K(x)$ varies significantly with location. A K-driven diffusion was introduced to account for this effect. Harvesting with a prescribed revenue is also considered, which can lead to more than one possible solution for the harvesting effort.
Keywords. logistic equation, diffusion, harvesting, revenue, positive periodic solution, global attractivity, rate of convergence.

1 Introduction

Population biology is one of most important and dynamic areas of mathematical biology, while sustainable harvesting and optimal resources management are among the most important problems of population ecology [9]. If the model describing population growth encounters spatial distribution and population diffusion, in addition to harvesting, the analysis of such systems becomes more complicated. However, models with diffusion more adequately describe management of spatially distributed resources, which is the reason why they recently attracted a lot of attention. Among other issues, optimal harvesting and stability of solutions were investigated [1, 2, 3] and how stability properties are influenced by diffusion [16].

Let us note that the solution of the diffusive logistic model

$$\frac{\partial u(t,x)}{\partial t} = D\Delta u(t,x) + r(x)u(t,x)\left(1-\frac{u(t,x)}{K(x)}\right), \quad t>0, \quad x\in\Omega, \tag{1.1}$$

where $u(t,x)$ is the population density, $r(x)$ is the intrinsic growth rate, $K(x)$ is the carrying capacity of the environment (generally, both can also be time-dependent), with the zero Neumann boundary condition and a high diffusion rate D tends to be uniformly distributed. This assumption is reasonable for spatially uniform carrying capacity (K is not x-dependent), but for nonuniform resources distribution model (1.1) suggests that species may move to the regions with lower per capita available resources which doesn't seem to be biologically feasible. Diffusive systems of type (1.1) were considered in many papers, see, for example, [2, 3, 4, 6, 8, 12, 13, 15, 19, 20] and references therein.

We assume an alternative type of diffusion when not u but u/K diffuses, which is described by the equation

$$\frac{\partial u(t,x)}{\partial t} = D\Delta\left(\frac{u(t,x)}{K(x)}\right) + r(x)u(t,x)\left(1-\frac{u(t,x)}{K(x)}\right), \quad t>0,\ x\in\Omega, \tag{1.2}$$

with the Neumann boundary condition

$$\frac{\partial\left(\frac{u}{K}\right)}{\partial n} = 0,\ x\in\partial\Omega,\ t\in(0,\infty).$$

This means that for very high diffusion the population tends to have not a uniform distribution over the domain but the uniform per capita available resources (u/K is constant).

This model was first introduced in [5], its stability properties (also in the case of time-dependent K and r) were studied in [14]. In [5], the optimal harvesting of (1.3) was also investigated, but not existence and stability issues for the harvested model.

In the present paper we study the generalization of model (1.3) to the case of harvested populations when the term of type $qE(x)u(t,x)$ is subtracted from the right-hand side. Here $E(x)$ is the harvesting effort, $q(x)$ characterizes either catchability ($0<q\le 1$) per unit effort or both by-catch mortality and harvesting incorporated in the harvesting event ($q\ge 1$). We will assume that in the former case per capita catchability never exceeds the effort applied, while in the latter case the effect of harvesting (which can include disturbance of grazing and mating habits, noise, pollution, as well as direct by-catch mortality) is included in q, overall, this can even lead to $qE>1$.

In Sections 2-4 we combine the product qE into one coefficient which is denoted as $E(x)$ and study positivity of solutions, persistence and stability of the initial value problem with the Neumann boundary conditions for equation (1.3) with harvesting.

The paper is organized as follows. Section 2 includes the results on existence, uniqueness and positivity of solutions. Section 3 is concerned with extinction and persistence. Section 4 contains stability analysis, including the convergence rate to the unique positive equilibrium. Section 5 considers harvesting with a prescribed revenue. In Section 6 summary of the results and open problems are presented. Finally, Appendix A involves all auxiliary theorems used in the proofs of our results.

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2 Existence and Uniqueness

Consider the following initial-boundary value problem for the logistic equation with diffusion and harvesting

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} = \quad & D\Delta\left(\tfrac{u(t,x)}{K(x)}\right) + r(x)u(t,x)\left(1-\tfrac{u(t,x)}{K(x)}\right) \\ & -E(x)u(t,x), \quad t>0, \quad x\in\Omega, \end{aligned} \tag{2.1}$$

with the Neumann boundary condition

$$\frac{\partial\left(\frac{u}{K}\right)}{\partial n} = 0,\ x\in\partial\Omega,\ t\in(0,\infty) \tag{2.2}$$

and the initial condition

$$u(0,x)=u_0(x),\ x\in\Omega. \tag{2.3}$$

Denote

$$Q_T=(0,T]\times\Omega, \quad \overline{Q}_T=[0,T]\times\overline{\Omega},$$
$$\partial Q_T=(0,T]\times\partial\Omega \text{ for any } T>0.$$

Everywhere in this and the following sections we assume that $K(t,x)$ is in class $C^{1+\alpha}(\overline{\Omega})$ in x and is a function which is continuously differentiable in t, $K(t,x)>0$ for any $(t,x)\in\overline{Q}_T$, $r(t,x)>0$ is continuous in t and C^α in x in $\overline{Q}_T$. Ω is a an open nonempty bounded domain with $\partial\Omega\in C^{2+\alpha}$, $0<\alpha<1$, $u_0(x)\in C(\Omega)$, $E(x)\in C^\alpha(\Omega)$. For detailed definitions of Hölder spaces see e.g. [11].

We will formulate the existence and uniqueness theorem for the general case of a time-dependent carrying capacity $K(t,x)$ and a growth function $r(t,x)$:

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} = \quad & D\Delta\left(\tfrac{u(t,x)}{K(t,x)}\right) + r(t,x)u(t,x)\left(1-\tfrac{u(t,x)}{K(t,x)}\right) \\ & -E(x)u(t,x), \quad (t,x)\in Q_T \end{aligned} \tag{2.4}$$

with (2.2) and (2.3) as the boundary and the initial condition, respectively. The result will obviously hold for time independent functions $K(x)$ and $r(x)$. Since $u(t,x)$ describes the population density, we assume nonnegative values of $u(t,x)$ at $t=0$.

Theorem 2.1. *Let $u_0(x)\ge 0$ in Ω and $u_0(x)>0$ in some open bounded nonempty domain $\Omega_1\subset\Omega$. Then there exists a unique solution $u(t,x)$ of problem (2.5),(2.2),(2.3) and it is positive.*

Proof. The proof follows the main steps of the proof of Theorem 3.1 in [14], so we omit some details here.

We make the following substitution

$$v(t,x)=\frac{u(t,x)}{K(t,x)}.$$

Since the function K is positive and bounded from below by a positive number and from above in $\overline{Q_T}$, function $v(t,x)$ is well defined; it satisfies the initial-boundary value problem

$$\begin{aligned} \frac{\partial v}{\partial t}+\frac{v}{K}\frac{\partial K}{\partial t}-\frac{D}{K}\Delta v = \quad & r(t,x)v(t,x)(1-v(t,x)) \\ & -E(x)v(t,x), \quad (t,x)\in Q_T, \end{aligned} \tag{2.5}$$

$$\begin{aligned} & \frac{\partial v}{\partial n}=0 \ \text{ on } \partial Q_T, \\ & v(0,x)=u_0(x)/K(0,x) \ \text{ in } \Omega. \end{aligned} \tag{2.6}$$

We will use the method of upper and lower solutions (see e.g. [17]). Define $a(t,x)$ as

$$a(t,x)=\frac{1}{K}\frac{\partial K}{\partial t}+E$$

and consider the function

$$\overline{v}(t,x)=\max\left\{\sup_{x\in\overline{\Omega}}(v(0,x)),\ \sup_{(t,x)\in\overline{Q_T}}\frac{r(t,x)-a(t,x)}{r(t,x)}\right\}>0.$$

Then

$$\frac{\partial\overline{v}}{\partial t}-\frac{D}{K}\Delta\overline{v}=0\ge\overline{v}\left(r-a-r\overline{v}\right),$$
$$\frac{\partial\overline{v}}{\partial n}=0$$

and by definition $\overline{v}\ge v(0,x)$, therefore $\overline{v}$ is an upper solution of (2.7). The function $\underline{v}(t,x)\equiv 0$ is obviously a lower solution. Therefore, there exists a unique solution of (2.7) $0\le v(t,x)\le\overline{v}$.

To show that the solution is positive for any nonnegative initial function $u_0(x)$ positive in some open nonempty subdomain of Ω let us consider the function

$$Z(t,x)=v(t,x)e^{At}=\frac{u(t,x)}{K(t,x)}\,e^{At},$$

where

$$A=\overline{v}\sup_{(t,x)\in\overline{Q_T}}(r(t,x))+\sup_{(t,x)\in\overline{Q_T}}|a(t,x)|.$$

Substituting $v(t,x)=Z(t,x)e^{-At}$ into (2.7) we have

$$\begin{aligned} & \frac{\partial Z(t,x)}{\partial t}-\frac{D}{K(t,x)}\Delta Z(t,x)=[r\,v(1-v)-av+Av]e^{At}\ge 0, \\ & \qquad (t,x)\in Q_T, \\ & \frac{\partial Z}{\partial n}=0,\ (t,x)\in\partial Q_T, \\ & Z(0,x)=v(0,x)\ge 0,\ x\in\Omega. \end{aligned} \tag{2.7}$$

By the maximum principle [17] we have $Z(t,x)>0$ in $\overline{Q_T}$, and therefore $v(t,x)>0$ for any $t>0$, $x\in\Omega$, which completes the proof. □

3 Persistence and Extinction

In the previous section we demonstrated that all solutions of the problem with a nonnegative initial function (positive in a nonempty open domain) are positive. However, this cannot exclude the case when a solution extincts, i.e. tends to zero as $t\to\infty$. We will say that the solution is *persistent* if there exists a function $\varepsilon(x)>0$ for $x\in\Omega$ such that

$$\liminf_{t\to\infty}u(t,x)\ge\varepsilon(x)$$

for any $x\in\Omega$.

For the rest of the paper, we assume that the carrying capacity $K(x)$ and the intrinsic growth rate $r(x)$ are time-independent.

The next two theorems establish sufficient conditions for extinction and persistence of the population subject to continuous harvesting.

Theorem 3.1. *Let $E(x)>r(x)$ for all $x\in\Omega$. Then for any nonnegative $u_0(x)$ we have $u(t,x)\to 0$ as $t\to\infty$.*

Proof. Let us follow the proof of Theorem 2.1 and once again introduce the substitution $v(t,x)=u(t,x)/K(x)$. Substituting $u(t,x)=v(t,x)K(x)$ in equations (2.2)-(2.3) and dividing the first equation by $K(x)$ we obtain

$$\begin{aligned} \frac{\partial v}{\partial t} = \quad & \tfrac{D}{K}\Delta v+r(x)v(t,x)(1-v(t,x)) \\ & -E(x)v(t,x), \quad (t,x)\in Q_T, \\ & \frac{\partial v}{\partial n}=0 \ \text{ on } \partial Q_T, \\ & v(0,x)=u_0(x)/K(0,x) \ \text{ in } \Omega. \end{aligned} \tag{3.1}$$

We will consider problem (3.1) and apply Theorem 7.1 from Appendix A. Obviously, if $v(t,x)\to 0$ then $u(t,x)\to 0$ as well. First, we present (3.1) in the form (7.1). Since

$$\frac{D}{K(x)}\Delta v=\nabla\cdot\frac{D}{K(x)}\nabla v-\nabla\frac{D}{K(x)}\cdot\nabla v,$$

the function v satisfies (7.1) with

$$d(x)=D/K(x),$$

$$\vec{b}(x) = -\nabla(D/K(x)) = D/K(x)\nabla \ln K(x),$$
$$\beta(x) = 0$$

and

$$f(x,v) = r(x)v(1-v) - E(x)u.$$

Moreover,

$$f(x,v) \le (r(x) - E(x))v := g_0(x)v,$$

and since $\vec{b}/(D/K)$ is a gradient all the differentiability conditions on the parameters of the system of Theorem 7.1 are satisfied. Thus, to apply Theorem 7.1 to system (2.7) we only need to show that the principal eigenvalue of the problem

$$\nabla \cdot \frac{D}{K}\nabla\psi - \nabla\frac{D}{K}\cdot\nabla\psi + (r-E)\psi = \sigma\psi \tag{3.2}$$

$$\frac{\partial\psi}{\partial n} = 0 \tag{3.3}$$

is negative. First, we multiply (3.2) by K and note that

$$K\nabla\cdot\frac{D}{K(x)}\nabla\psi - K\nabla\frac{D}{K(x)}\cdot\nabla\psi = D\Delta\psi,$$

thus

$$D\Delta\psi + K(r-E)\psi = K\sigma\psi. \tag{3.4}$$

Next, since the eigenfunction ψ_1 corresponding to σ_1 can be chosen positive we obtain after integrating (3.4) and using boundary condition (3.3) :

$$\sigma_1 = \frac{\int\limits_\Omega (r-E)K\psi_1 dx}{\int\limits_\Omega K\psi_1 dx} \tag{3.5}$$

Therefore $\sigma_1 < 0$ if $r(x) < E(x)$ which concludes the proof. □

Theorem 3.2. *Let $E(x) < r(x)$ for all $x \in \Omega$. Then there exists a unique positive equilibrium solution $u^*(x)$ of (2.2), (2.2) such that for any $u_0(x) \ge 0$ in Ω and $u_0(x) > 0$ in some open bounded nonempty domain $u(t,x) \to u^*(x)$ as $t \to \infty$.*

Proof. Proceeding as in Theorem 3.1 and using Theorem 7.2 of Appendix A with

$$g(x,v) = r(x) - E(x) - r(x)v$$

we obtain

$$\sigma_1 = \frac{\int\limits_\Omega (r-E)K\psi_1 dx}{\int\limits_\Omega K\psi_1 dx},$$

which is positive if $E(x) < r(x)$, so there exists a positive attracting equilibrium solution.

To prove the uniqueness we need to show that the function $f(x,u)$ satisfies the conditions of Theorem 7.3 from Appendix A. It is easy to see that this is true with

$$M = \sup_{x\in\Omega}(r-E)K/r,$$

which completes the proof. □

Remark 3.1. In the case of harvesting effort proportional to the growth rate $E(x) = \alpha r(x)$ the equilibrium solution is given by

$$u^*(x) = (1-\alpha)K(x).$$

4 Stability analysis

In this section we conduct a stability analysis of the positive equilibrium solution.

Theorem 4.1. *Suppose that there exists a unique positive equilibrium solution $u^*(x)$ of problem (2.2)-(2.2). Then for any nonnegative initial function $u_0(x)$, positive on a subdomain of positive measure, the solution $u(t,x)$ of (2.2)-(2.3) converges to $u^*(x)$ uniformly on $\overline{\Omega}$ as $t \to \infty$. Moreover, there exist positive numbers ρ and α such that for large t*

$$|u(t,x) - u^*(x)| \le \rho e^{-\alpha t}\psi(x),$$

where $\psi(x)$ is the principal eigenfunction of the problem

$$\begin{aligned} \nabla\cdot\frac{D}{K}\nabla\psi + \nabla\frac{D}{K}\cdot\nabla\psi + \psi\Delta\frac{D}{K} & \\ +r\psi\left(1-2\frac{u^*}{K}\right) - E\psi &= \lambda\psi,\ \ x\in\Omega \\ \frac{\partial(\psi/K)}{n} &= 0,\ \ x\in\partial\Omega \end{aligned} \tag{4.1}$$

Proof. Let us demonstrate that the hypotheses of Theorems 7.5 and 7.6 from Appendix A are satisfied. First, we note that $\underline{u}(x) = \underline{v}(x)K(x)$ and $\overline{u}(x) = \overline{v}(x)K(x)$, where $\underline{v}$ and $\overline{v}$ are the functions constructed in the proof of Theorem 2.1, are lower and upper solutions of problem (2.2)-(2.3). Next, the function $f = ru(1-u/K) - Eu$ is obviously C^1 in u, therefore the conditions of Theorem 7.5 from Appendix A are satisfied. Moreover, since according to Theorem 3.2 we have $\underline{u}^*(x) = \overline{u}^*(x) = u^*(x)$, the first statement of the theorem follows.

Next, we will show that the principal eigenvalue of the linearized problem corresponding to (2.2), (2.2) is negative. Consider the right-hand side of (2.2), we can write it in the form

$$\begin{aligned} D\Delta\frac{u}{K} + ru\left(1-\frac{u}{K}\right) - Eu = \quad & \nabla\cdot\tfrac{D}{K}\nabla u + \nabla\tfrac{D}{K}\cdot\nabla u \\ & +u\Delta\tfrac{D}{K} + ru\left(1-\tfrac{u}{K}\right) - Eu \end{aligned}$$

similar to (7.1). Therefore, the linearized eigenvalue problem corresponding to (2.2),(2.2) becomes (4.1).

Next, we note that the function u^* is positive and satisfies

$$D\Delta\frac{u^*}{K} + ru^*\left(1-\frac{u^*}{K}\right) - Eu^* = 0,$$

therefore according to Theorem 7.4 from Appendix A it is the principal eigenfunction of the problem

$$\begin{aligned} \nabla\cdot\frac{D}{K}\nabla\phi + \nabla\frac{D}{K}\cdot\nabla\phi + \phi\Delta\frac{D}{K} & \\ +r\phi\left(1-\frac{u^*}{K}\right) - E\phi &= \mu\phi,\ \ x\in\Omega \qquad (4.2) \\ & \qquad\qquad (4.3) \\ \frac{\partial\phi/K}{n} &= 0,\ \ x\in\partial\Omega \end{aligned}$$

with the principal eigenvalue zero. Moreover, since

$$r\left(1-2\frac{u^*}{K}\right) < r\left(1-\frac{u^*}{K}\right)$$

on a subdomain of a positive measure, it follows from Corollary 7.1 of Appendix A that $\lambda_1 < \mu_1$ where λ_1 and μ_1 are the principal eigenvalues of problems (4.1) and (4.2), respectively. Thus, we obtain $\lambda_1 < 0$ and we can apply Theorem 7.6 from Appendix A, which concludes the proof. □

5 Harvesting with a Prescribed Revenue

In this section we will distinguish between the harvesting effort and the effect of harvesting on population density to obtain the effort required to attain the desired revenue. We consider the following equation with harvesting assuming as previously that K, r and harvesting effort E are time-independent, the density of harvested population is Eu, and harvesting effect q is a constant ($q \ge 1$):

$$\frac{\partial u(t,x)}{\partial t} = \quad D\Delta\left(\tfrac{u(t,x)}{K(x)}\right) + r(x)u(t,x)\left(1-\tfrac{u(t,x)}{K(x)}\right) \tag{5.1}$$

$$-qE(x)u(t,x),\ t>0,\ x\in\Omega. \tag{5.2}$$

In order to consider the problem of a constant revenue m over domain Ω let us introduce a unit price per harvested population $p > 0$ and a cost of harvest effort $c > 0$. Then $\int_\Omega E(x)(pu(t,x) - c)dx$ is the total revenue at time t. The assumption that the total revenue is to be kept at a constant level m leads to the equation

$$\int_\Omega E(x)(pu(t,x)-c)dx = m. \tag{5.3}$$

Further, suppose that

$$qE(x) < r(x), \quad x \in \Omega$$

and the harvesting effort at any point of domain Ω is proportional to the growth rate

$$E(x) = \alpha r(x), \quad 0 \le \alpha < \frac{1}{q}. \tag{5.4}$$

Then equation (5.2) has the stationary solution

$$u(x) = K(x)\left(1 - \frac{qE(x)}{r(x)}\right) = (1-q\alpha)K(x). \tag{5.5}$$

For which values of m will α exist such that the prescribed revenue is attained at any moment of time? After substituting (5.4) and (5.5) into (5.3) we obtain

$$\alpha \int_\Omega r(x)\left[pK(x)(1-q\alpha) - c\right]dx = m, \tag{5.6}$$

which is a quadratic equation in α:

$$\alpha^2 \int_\Omega qpr(x)K(x)dx - \alpha \int_\Omega r(x)(pK(x)-c)dx + m = 0. \tag{5.7}$$

Denote

$$A = \int_\Omega qpr(x)K(x)dx, \quad B = \int_\Omega r(x)(pK(x)-c)dx. \tag{5.8}$$

Equation (5.7) has two distinct real solutions if

$$\begin{aligned} B^2 &= \left[\int_\Omega r(x)(pK(x)-c)dx\right]^2 \\ &\quad > 4Am \\ &= 4m \int_\Omega qpr(x)K(x)dx. \end{aligned} \tag{5.9}$$

Since $A > qB$,

$$0 \le \alpha_1 = \frac{B - \sqrt{B^2-4Am}}{2A} < \frac{1}{2q}, \tag{5.10}$$

$$0 \le \alpha_2 = \frac{B + \sqrt{B^2-4Am}}{2A} < \frac{1}{q}, \tag{5.11}$$

thus both α_1 and α_2 can be used in the model. Let us note that the prescribed revenue m can be attained if

$$m \le \left[\int_\Omega r(x)(pK(x)-c)dx\right]^2 \Big/ \left(4\int_\Omega qpr(x)K(x)dx\right). \tag{5.12}$$

In the case when there is an equality in (5.12), there is only one value of α (and thus of the harvesting effort) which can lead to the desired revenue m. If there is a strict inequality in (5.12), then α_1 corresponds to the case when the harvesting effort is smaller but the population level is higher, while α_2 describes more intensive harvesting of a lower population.

The purpose of the optimal harvesting is to keep the system production at the highest level and persistently extract all species which decrease production due to competition for resources. Thus, the tendency to connect the harvesting effort with the area fertility and thus the growth rate is quite natural, and can be easily observed in nature (the choice of grazing and hunting areas is stipulated by both the carrying capacity and the time it takes for resources to restore). If some planning for harvesting is required (for example, to issue area-dependent licences), then some analysis of the population history is required. For example, for populations with low r when the population size is not easily restored, the judicious policy would be to abstain from harvesting.

6 Discussion and Open Problems

In this paper we have obtained the following results for equation (2.2) with a carrying capacity driven diffusion and harvesting and the Neumann boundary condition.

1. For any initial conditions positive in an open nonempty domain the solution of the initial-boundary value problem exists, is unique and positive for $t > 0$.
2. If the harvesting rate exceeds the intrinsic growth rate at every point of the domain, then the population extincts.
3. If the intrinsic growth rate exceeds the harvesting rate, while both rates are time-independent, then all solutions with positive initial values in the domain are persistent; moreover, all solutions are bounded and converge to the positive equilibrium.
4. For positive solutions of equation (2.2) in the case when it has a positive equilibrium, an estimation of the rate of convergence to the equilibrium is obtained.
5. If the harvesting effort is stipulated by economic considerations, relevant analysis of the harvesting policy is presented. In most cases when the desired revenue is attainable, there are two possible policies to get the desired revenue: high harvesting level of relatively low population sizes and less intensive harvesting for a population of a higher size. We recall that the optimal harvesting policy for (2.2) was obtained in [5].

Finally, let us formulate some open problems.

1. So far we have studied the cases when either $E(x) > r(x)$ or $E(x) < r(x)$ for any $x \in \Omega$. Consider the case when $E(x) \le r(x)$ but the equality is attained in some closed nonempty subdomain of Ω.
2. Consider the prescribed revenue model as in Section 5 when space-dependent catchability rather than by-catch mortality is incorporated in the system

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} = & \quad D\Delta\left(\frac{u(t,x)}{K(x)}\right) \\ & + r(x)u(t,x)\left(1 - \frac{u(t,x)}{K(x)}\right) - q(x)E(x)u(t,x), \end{aligned}$$

   where $0 \le q(x) \le 1$,

$$m = \int_\Omega r(x)E(x)(pq(x)u(t,x) - c)\,dx$$

   is the prescribed revenue value. When will this revenue be attainable? What harvesting effort(s) will it correspond?
3. Consider the equation

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} = & D\Delta\left(\frac{u(t,x)}{K(x)}\right) \\ & + r(x)u(t,x)\left(1 - \frac{u(t,x)}{K(x)}\right), \ t > 0, \ x \in \Omega, \end{aligned} \tag{6.1}$$

   with the impulsive harvesting

$$u(t_k^+, x) - u(t_k, x) = E_k(x)u(t_k, x), \tag{6.2}$$

   where t_k are fixed moment of time, E_k are coefficients combining catchability and harvesting effort at time t_k. What are necessary conditions for extinction and persistence of impulsive equation (6.1),(6.2)? What are stability properties of (6.1),(6.2)?
4. Investigate optimal impulsive harvesting for (6.1),(6.2). In the case when the time between impulses is constant, find the value of $E(x)$ leading to a prescribed revenue. Let us note that some aspects of impulsive harvesting (6.2) of equation (6.1) were studied in [5], which included only the limit cases $D = 0$ and $D \to \infty$.

5. Most results of the present paper were obtained under the assumption that both the intrinsic growth rate $r(t)$ and the deduction rate $E(x)$ are time-independent. Consider the time-dependent case, for instance, when the parameters of the equation (r,K and E) are T-periodic.
6. Is it possible to make some conclusion on persistence and extinction of solutions in the case when $E(x) > r(x)$ and $E(x) < r(x)$ are satisfied in two nonempty open subdomains of Ω? This model can correspond to the familiar situation in fisheries when closer to the shore area is more favorable for population growth but also involves much higher harvesting dangers than the open ocean (or any other refuge from harvesting which has scarcer food sources), see [9, 10, 18] and references therein for the description of the models including inshore and offshore fishering.
7. Consider all the above questions for models with a growth law which is not logistic.
8. Everywhere above, by the optimal harvesting we meant maximal system production or maximal revenue. However, it is possible to consider models where the best outcome is population extinction (pest control). Study stability of the zero equilibrium.
9. In this paper, harvesting with a prescribed revenue was studied in the case when $E = \alpha r$. Consider the case when $E \neq \alpha r$. Without the revenue considerations, it was demonstrated in [5] that for time-independent carrying capacity the optimal harvesting in attained when E is proportional to r.

7 Appendix A

Consider a general model of the form

$$\frac{\partial u}{\partial t} = \nabla \cdot d(x)\nabla u + \vec{b}(x)\cdot\nabla u + f(x,u) \qquad \text{in } \Omega\times(0,\infty),$$
$$d(x)\frac{\partial u}{\partial n} + \beta(x)u = 0 \qquad \text{on } \partial\Omega\times(0,\infty). \tag{7.1}$$

The two following results are concerned with equilibrium solutions.

Theorem 7.1. *[7, Proposition 3.1] Suppose that*

$$f(x,u) \le g_0(x)u \qquad \text{for } x\in\Omega,$$

where $g_0(x)$ is a bounded measurable function if $\vec{b}/d$ is a gradient and $g_0(x)\in C^\alpha(\overline{\Omega})$ if $\vec{b}/d$ is not a gradient. If the principal eigenvalue σ_1 of

$$\nabla\cdot d(x)\nabla\psi + \vec{b}(x)\cdot\nabla\psi + g_0(x)\psi = \sigma\psi \qquad \text{in } \Omega\times(0,\infty),$$
$$d(x)\frac{\partial\psi}{\partial n} + \beta(x)\psi = 0 \qquad \text{on } \partial\Omega\times(0,\infty) \tag{7.2}$$

is negative, then (7.1) has no positive equilibria and all nonnegative solutions decay exponentially to zero as $t\to\infty$.

Theorem 7.2. *[7, Proposition 3.2] Suppose that $f(x,u) = g(x,u)u$ with $g(x,u)$ of class C^2 in u and C^α in x, and there exists $M>0$ such that $g(x,u)<0$ for $u>M$. If the principal eigenvalue σ_1 is positive in the problem*

$$\nabla\cdot d(x)\nabla\psi + \vec{b}(x)\cdot\nabla\psi + g(x,0)\psi = \sigma\psi \qquad \text{in } \Omega\times(0,\infty)$$
$$d(x)\frac{\partial\psi}{\partial n} + \beta(x)\psi = 0 \qquad \text{on } \partial\Omega\times(0,\infty),$$

then (7.1) has a minimal positive equilibrium u^*, and all solutions of (7.1) which are initially positive on an open subset of Ω are eventually bounded below by orbits which increase toward u^* as $t\to\infty$.

Sufficient conditions for the uniqueness of the positive equilibrium are given by the following theorem.

Theorem 7.3. *[7, Proposition 3.3] Let all the conditions of Theorem 7.2 be satisfied and suppose that $f(x,u) = g(x,u)u$ with $g(x,u)$ strictly decreasing in u for $u\ge 0$. Then the minimal positive equilibrium u^* is the only positive equilibrium of (7.1).*

Existence and properties of the principal eigenvalues are given by the following variational characterization [7].

Theorem 7.4. *[7] Suppose that $\partial\Omega$ is piecewise of class $C^{2+\alpha}$ and that Ω satisfies the interior cone condition. Suppose that $d(x)\in C^{1+\alpha}(\overline{\Omega})$ with $d(x)\ge d_0>0$; that $g_0(x)\in L^\infty(\Omega)$, and $\beta(x)\in L^\infty(\Omega)$ with $\beta(x)\ge 0$. Also, suppose that there exists a function $B(x)$ such that*

$$\vec{b}(x) = -d(x)\nabla B(x).$$

The principal eigenvalue of (7.2) is given by

$$\sigma_1 = \sup_{\psi\neq 0,\psi\in W^{1,2}} \sigma(\psi),$$
$$\sigma(\psi) = \frac{-\int_\Omega de^{-B}|\nabla\psi|^2dx + \int_\Omega g_0e^{-B}\psi^2dx - \int_{\partial\Omega}\beta e^{-B}\psi^2dS}{\int_\Omega e^{-B}\psi^2dx}. \tag{7.3}$$

The eigenfunction ψ_1 corresponding to σ_1 can be chosen so that $\psi_1>0$ on Ω ($\psi_1>0$ on $\overline{\Omega}$ except for Dirichlet boundary conditions) and σ_1 is the only eigenvalue admitting a positive eigenfunction.

Corollary 7.1. *Denote the principal eigenvalue of (7.2) as $\sigma_1(g_0)$, then $\sigma_1(g_0)$ is increasing with respect to g_0 in the sense that if $g_0^1\ge g_0^2$ then $\sigma_1(g_0^1)\ge\sigma_1(g_0^2)$, and if $g_0^1>g_0^2$ on a subset of positive measure then $\sigma_1(g_0^1)>\sigma_1(g_0^2)$.*

The following two theorems deal with stability of the steady states [17].

Theorem 7.5. *[17] Let $\hat{u}(x)$, $\tilde{u}(x)$ be ordered lower and upper solutions of (7.1) and let $\underline{U}(t,x)$, $\overline{U}(t,x)$ be the solutions of (7.1) with initial conditions $\underline{U}(0,x)=\hat{u}(x)$ and $\overline{U}(0,x)=\tilde{u}(x)$ respectively. Assume that f is a C^1-function in $\langle\hat{u},\tilde{u}\rangle$. If $u(t,x)$ is a solution of (7.1) with $u_0(x)\in\langle\hat{u},\tilde{u}\rangle$, then $\underline{U}(t,x)\le u(t,x)\le\overline{U}(t,x)$ in Q. Moreover,*

$$\lim_{t\to\infty}\underline{U}(t,x) = \underline{u}^*(x), \quad \lim_{t\to\infty}\overline{U}(t,x) = \overline{u}^*(x)$$

monotonically in t, where $\underline{u}^*(x)$ and $\overline{u}^*(x)$ are the minimal and maximal equilibrium solutions of (7.1).

Theorem 7.6. *[17, Theorem 5.3.3] Let $u(t,x)$ be the solution of (7.1) and $u^*(x)$ be the steady-state of the same problem. If $\lambda_1<0$, then there exist positive numbers ρ and β such that*

$$|u(t,x)-u^*(x)| \le \rho e^{-\beta t}\psi(x) \tag{7.4}$$

whenever this inequality holds at time $t=0$. Here λ_1 and $\psi(x)$ are the principal eigenvalue and the principal eigenfunction of the problem

$$\nabla\cdot d(x)\nabla\psi + \vec{b}(x)\cdot\nabla\psi + f_u(x,u^*)\psi = \lambda\psi \qquad \text{in } \Omega\times(0,\infty),$$
$$d(x)\frac{\partial\psi}{\partial n} + \beta(x)\psi = 0 \qquad \text{on } \partial\Omega\times(0,\infty),$$

respectively.

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