REGULARITY OF SOLUTIONS TO QUASILINEAR INFINITELY DEGENERATE SECOND ORDER EQUATIONS

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Abstract. The main result of the paper is on the continuity of weak solutions of infinitely degenerate quasilinear second order equations. Namely, we show that every weak solution to a certain class of degenerate quasilinear equations is continuous. More precisely, we show that it is Hölder continuous with respect to a certain metric associated to the operator. One of the essential features of this metric is that the metric balls are non doubling with respect to Lebesgue measure. The proof of the continuity together with a recent result by Rios et al. [10] completes the result on hypoellipticity of a class of second order infinitely degenerate elliptic operators.

1. Introduction

In this paper we consider a second order divergence form quasilinear operator of the form

$$L = \nabla^T A(x, \cdot)\nabla$$

where $x \in \Omega$ with Ω a bounded domain of $\mathbb{R}^n$, and the matrix $A(x, \cdot)$ is nonnegative semidefinite, and study regularity of weak solutions to the corresponding second order equations. We do not assume that A is elliptic, moreover, it can degenerate to infinite order i.e. its eigenvalues are allowed to vanish together with all of their derivatives. Our assumptions on the degeneracy of the operator will be stated in terms of the associated metric balls.

Our main interest is the regularity of weak solutions. One of the ways to describe the regularity is in terms of the operator being subelliptic or hypoelliptic. The criteria of subellipticity for linear operators have been given by Hormander [5] for operators given in terms of vector fields; and by Fefferman and Phong [3] in terms of subunit metric balls associated to the operator. In particular, it follows that an infinitely degenerate operator cannot be subelliptic. Hypoellipticity of a certain class of such operators, however, have been established by Fediĭ [2] and later for a more general class by Kohn [6] and many others [8, 9, 1]. For the infinitely degenerate quasilinear operators hypoellipticity has been recently shown by Rios et al [10] in the assumption that the coefficients can only vanish on hyperplanes. This result, however, relies on the a priori assumption that the solution is continuous.

Our main result implies continuity of weak solutions of certain degenerate quasilinear second order equations. Namely, we show that every weak solution to a certain class of degenerate quasilinear equations is Hölder continuous with respect

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to the induced metric. This in term implies continuity, thus completing the result on hypoellipticity.

We mainly use the approach of E. Sawyer and R. Wheeden [11], namely we consider metric balls associated to the operator. The main difficulty is that the balls associated to infinitely degenerate quadratic forms do not satisfy a doubling condition

$$|B(x, 2r)| \leq C|B(x, r)|$$

where $x \in \Omega$, and $0 < r < \infty$. This prevents us from using the theory of homogeneous spaces and the vast body of results established for this type of spaces. We also do not assume the containment condition

$$E(x, r) \subset B(x, Cr^{\varepsilon})$$

where $E(x, r)$ is the Eucledian ball. It is known [3] that the above containment condition is necessary for subellipticity, and in the case it is not satisfied the highest regularity result one can hope to establish is hypoellipticity.

The paper is organized as follows. In Section 2 we give the basic definitions, state our main assumtions and formulate the main result. In Section 3 we introduce classes of weak solutions needed to implement Moser iterations. Section 4 is dedicated to the proof of our main result — "metric" Hölder continuity of weak solutions. First, we prove a Harnack inequality by means of performing Moser iterations. Then in the usual way from Harnack inequality we derive Hölder inequality with respect to the metric $d(x, y)$ associated to the operator. This estimate together with a weak containment condition will then imply continuity of solutions. Finally, in Section 5 we consider subunit metric associated to the operator and establish some of its properties.

2. Degenerate Sobolev spaces and weak solutions

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain. Consider the following second order quasilinear equation

$$Lu = \nabla^T A(x, u(x))\nabla u = f \tag{2.1}$$

in Ω where the matrix $A = \{a_{ij}\}$ is nonnegative semidefinite and can degenerate to infinite order. We will assume that $A(x, z)$ satisfies the following structural condition for a.e. $x \in \Omega$ and all $z \in \mathbb{R}$, $\xi \in \mathbb{R}^n$

$$k\,\xi^T Q(x)\xi \leq \xi^T A(x, z)\xi \leq K\,\xi^T Q(x)\xi \tag{2.2}$$

for some $K \geq k > 0$ and nonnegative semidefinite locally integrable matrix $Q(x)$. Note that if we denote $\tilde{A}(x) = A(x, u(x))$ for a particular solution $u(x)$ then (2.2) obviously holds with $\tilde{A}(x)$ in place of $A(x, z)$. To define solutions to (2.1) in the weakest sense possible we adopt the notion of strong degenerate Sobolev spaces [12]. Let $\mathcal{Q}(x, \xi) = \xi^T Q(x)\xi$ be a locally integrable nonnegative semidefinite quadratic form

Definition 1. *A form-weighted vector-valued L^2 space $\mathcal{L}^2(\Omega, Q)$ is a space consisting of all measurable $\mathbb{R}^n$-valued functions $\boldsymbol{f}(x)$, $x \in \Omega$, satisfying*

$$||\boldsymbol{f}||_{\mathcal{L}^2(\Omega,Q)} = \left\{\int_\Omega \boldsymbol{f}(x)^T Q(x)\boldsymbol{f}(x)dx\right\}^{\frac{1}{2}} = \left\{\int_\Omega [\boldsymbol{f}(x)]_Q^2 dx\right\}^{\frac{1}{2}} < \infty$$

where we denote $[\boldsymbol{U}(x)]_Q^2 = \boldsymbol{U}(x)^T Q(x) \boldsymbol{U}(x)$ *for any vector-valued function* $\boldsymbol{U}(x)$.

Definition 2. *We define a space* $W_Q^{1,2}(\Omega)$ *to be the completion of* $Lip(\Omega)$ *under the norm*

$$||w||_{W_Q^{1,2}(\Omega)} = \left\{ \int_\Omega (|w|^2 + [\nabla w]_Q^2) \right\}^{\frac{1}{2}} \approx ||w||_{L^2(\Omega)} + ||\nabla w||_{\mathcal{L}^2(\Omega,Q)}$$

By $\left(W_Q^{1,2}\right)_0(\Omega)$ *we mean the closure of* $Lip_c(\Omega)$ *in* $W_Q^{1,2}(\Omega)$ *where* $Lip_c(\Omega)$ *is the space of Lipschitz functions with common compact support.*

As shown in [12] if $u \in W_Q^{1,2}(\Omega)$ the gradient ∇u is not uniquely determined by u. Therefore, ∇u denotes one of the vector-valued functions in $\mathcal{L}^2(\Omega, Q)$, such that $\nabla u_k \to \nabla u$ in $\mathcal{L}^2(\Omega, Q)$ and u is obtained as a limit of a Cauchy sequence of Lipschitz functions $\{u_k\}$. We write $(u, \nabla u) \in \mathcal{W}_Q^{1,2}(\Omega)$ for $u \in W_Q^{1,2}(\Omega)$, so $W_Q^{1,2}(\Omega)$ is the first compact projection of $\mathcal{W}_Q^{1,2}(\Omega)$. An element $(u, \nabla u) \in \mathcal{W}_Q^{1,2}(\Omega)$ is nonnegative if $u \geq 0$.

Definition 3. *A pair* $(u, \nabla u) \in \mathcal{W}_Q^{1,2}(\Omega)$ *is a weak solution of (2.1) in* Ω *if*

$$- \int (\nabla u)^T A \nabla w = \int f w \tag{2.3}$$

for every nonngegative $w \in \left(W_Q^{1,2}\right)_0(\Omega)$. *A function* $u \in W_Q^{1,2}(\Omega)$ *is called a weak subsolution (supersolution) if the above holds with* $\geq$ $(\leq)$ *in place of equality.*

To state our main result about the regularity of weak solutions we assume the existence of a certain metric d, such that the metric balls $B(x, r) = \{y : d(x, y) < r\}$ satisfy the following two conditions

- $\forall x \in \Omega \; \exists \alpha_x : \mathbb{R}_+ \to \mathbb{R}_+$ decreasing and such that $\alpha_x(r) > 0$ for any $r > 0$ and

$$E(y, \alpha_x(R)) \subset B(y, R), \; \forall y \in \Omega, \; \forall R > 0 \tag{2.4}$$

- For any $x \in \Omega$, any $r > 0$ there exists $R > 0$ depending on r and x such that

$$B(x, r) \subset E(x, R) \tag{2.5}$$

A natural choice of such metric is the subunit metric which will be discussed in Section 5. All of the results, however, are axiomatic in the sense that if one can find *any* metric satisfying the required assumptions the results will hold true. For other choices of metric balls, e.g. flag balls and adapted noninterference balls, see [11].
Condition (2.4) essentially says that the metric balls cannot collapse to zero measure sets for non-zero radius, in particular, metric balls are open sets. Condition (2.5) assures that the metric balls do not become infinitely large, in particular, for any $x \in \Omega$ there exists $\delta > 0$ such that $B(x, r) \subset \Omega$ for any $0 < r < \delta dist(y, \partial\Omega)$.

We will require the following Sobolev inequality: there exists $\sigma > 1$ and $\delta > 0$ such that for all balls $B = B(y, r)$ with $y \in \Omega, \; 0 < r < \delta dist(y, \partial\Omega)$ there holds

$$\left\{ \frac{1}{|B|} \int\limits_B |w|^{2\sigma} \right\}^{\frac{1}{2\sigma}} \leq Cr \left\{ \frac{1}{|B|} \int\limits_B [\nabla w]_Q^2 \right\}^{\frac{1}{2}} + C \left\{ \frac{1}{|B|} \int\limits_B |w|^2 \right\}^{\frac{1}{2}} \tag{2.6}$$

for all $(w,\nabla w)\in\left(\mathcal{W}_Q^{1,2}\right)_0(B)$, the closure in $\mathcal{W}_Q^{1,2}(B)$ of $(w,\nabla w)$ where $w\in Lip_c(B)$.

We also require the following very weak Poincaré-type inequality: there exists $\lambda\geq 1$, $k\geq 0$ and $\delta>0$ such that for all balls $B=B(y,r)$ with $y\in\Omega$, $0<r<\delta dist(y,\partial\Omega)$ there holds

$$\int\limits_B |w-\langle w\rangle_B|\leq Cr\left\{\frac{|MB|^{k+1}}{|B|^k}\int\limits_{\lambda B}[\nabla w]_Q^2\right\}^{\frac{1}{2}} \tag{2.7}$$

for all $(w,\nabla w)\in\mathcal{W}_Q^{1,2}(\lambda B)$ where we denote $\langle w\rangle_B=(1/B)\int_B w$. The constant $M>1$ depends only on λ on the right hand side, and can be taken to be $M=4\lambda$. Note, that in the expression on the right we have the ratio $(|MB|/|B|)^k$ which in the case of doubling metric is equal to a dimensional constant. In the nondoubling case, however, this ratio can become unbounded as $r\to 0$. We also note, that it is a $(1,2)$ Poincaré inequality and it obviously follows from either $(1,1)$ or $(2,2)$ Poincaré inequalities.

The next assumption, which is used in implementing Moser iteration, is the existence of a certain accumulating sequence of Lipschitz cutoff functions, similar to [11], [12]. We assume there are positive constants ν, N and δ such that for each ball $B(y,r)$ with $y\in\Omega$, $0<r<\delta dist(y,\partial\Omega)$, there is a sequence of Lipschitz cutoff functions $\{\psi_j\}_{j=1}^\infty$ with the following properties:

$$\begin{cases} E_1=\mathrm{supp}\psi_1 & \subset B(y,r)\subset B(y,Mr)\\ B(y,\nu r) & \subset\{x:\psi_j(x)=1\},\ j\geq 1\\ E_{j+1}=\mathrm{supp}\psi_{j+1} & \Subset\{x:\psi_j(x)=1\},\ j\geq 1\\ \psi_j & \text{is Lipschitz},\ j\geq 1\\ ||[\nabla\psi_j]_Q||_{L^\infty(B(y,Mr))} & \leq C\frac{j^N}{r},\ j\geq 1\end{cases} \tag{2.8}$$

We can now state our main result

Theorem 1. *Suppose that $A(x,z)$ is a nonnegative semidefinite matrix in $\Omega\times\mathbb{R}$ and it satisfies (2.2). Let $d(x,y)$ be a symmetric metric in Ω, and $B(x,r)=\{y\in\Omega:d(x,y)<r\}$ with $x\in\Omega$ are the corresponding metric balls. Then every weak solution of (2.1) is continuous provided*

1. *the containment condition (2.4) holds*
2. *the boundedness condition (2.5) holds*
3. *the Sobolev and Poincaré inequalities (2.6) and (2.7) hold with the given σ*
4. *the "accumulating sequence of Lipschitz cutoff functions" condition (2.8) holds.*

More precisely, u satisfies

$$|u(x)-u(x')|\leq C\left\{\left(\frac{1}{|B(y,Mr)|}\int_{B(y,r)}u^2\right)^{\frac{1}{2}}+r^2||f||_{L^\infty}\right\}\left(\frac{d(x,x')}{r}\right)^\alpha \tag{2.9}$$

for $x,x'\in B(y,\nu_0 r)$ with some $\nu_0>0$.

Remark 1. *The integral term on the right contains a factor of $1/|B(y,Mr)|$ which in the doubling case is equivalent to $1/|B(y,r)|$ but in the nondoubling case can be* *essentially smaller. This gain, obtained from carefully performing Moser iterations, turns out to be crucial and allows us to prove continuity.*

Corollary 1. *Let all the conditions of Theorem 1 be satisfied. If, in addition, the containment condition (2.4) is satisfied with* $\alpha(r) = Cr^{1/\varepsilon}$ *then* u *is Hölder continuous, namely estimate (2.9) holds with* $d(x,x')$ *replaced by* $C|x-x'|^{\varepsilon}$.

Remark 2. *The above Corollary partially generalizes the result* [11, Theorem 8] *for the case of nondboubling metric balls. Note that containment condition (2.4) with* $\alpha(r) \sim r^{1/\varepsilon}$ *does not imply a doubling condition. We say "partially generalizes" because the operator in* [11] *includes lower order terms and more general zero-order terms.*

3. Sub and super solutions

In the process of performing Moser iterations we will need to use the equation that a nonlinear function of u satisfies. As it turns out, a nonlinear function of a weak solution to (2.1) we will be considering, satisfies a certain equation in the integral sense for a smaller class of test functions. Following [11], [12] we now introduce weaker classes of solutions to (2.1).

Definition 4. *Let* $\mathcal{W}$ *be a subset of nonnegative elements in* $\left(W_Q^{1,2}\right)_0(\Omega)$. *A function* $u \in W_Q^{1,2}(\Omega)$ *is a* $\mathcal{W}$*-weak solution (subsolution, supersolution) to (2.1) in* Ω *if the integrals in (2.3) are absolutely convergent and the indicated (in)equality holds for all* $w \in \mathcal{W}$.

We next define a class of "admissible" functions (as in [11]) that we will further compose with $u \in W_Q^{1,2}(\Omega)$.

Definition 5. *Let* $I \in \mathbb{R}$ *be an interval and* $h \in C^1(I) \cap C^2_{pw}(I)$ *be positive and monotone, where* $C^2_{pw}(I)$ *is the space of piecewise twice continuously differentiable functions on* I. *The function* h *is said to be admissible on* I *if there exists a positive constant* C *such that*

$$|h'(t)|,\ |h''(t)|,\ |th''(t)| \leq C, \quad t \in I.$$

Moreover, given $u \in W_Q^{1,2}(\Omega)$, *we say that* h *is admissible for* u *if* h *is admissible on some interval* I *containing the range of* u.

Remark 3. *If* $u \in W_Q^{1,2}(\Omega)$ *and* h *is admissible for* u, *it follows that* $\tilde{u} = h(u)$ *belongs to* $W_Q^{1,2}(\Omega)$ *provided* Ω *is bounded. For the proof see* [4, Lemma 7.5].

When composing $h' \in L^\infty$ with $u \in W_Q^{1,2}(\Omega)$ we might run into problems when u takes values in the set of discontinuities of h'. To make sense of the expressions like $(h' \circ u)\nabla u$ we will use the following proposition [12]

Proposition 1. [12, Proposition 22] *Suppose that* $(u, \nabla u) \in \mathcal{W}_Q^{1,2}(\Omega)$ *where* Ω *is bounded, and let*

$$\mathcal{R}_u = \{\alpha \in \mathbb{R} : u = \alpha \text{ on a set of positive measure}\}.$$

With $\nabla_{reg} u = \chi_{\{x\in\Omega: u(x)\notin \mathcal{R}_u\}} \nabla u$ *we have* $(u, \nabla_{reg} u) \in \mathcal{W}_Q^{1,2}(\Omega)$ *and* $(u, \nabla_{reg} u)$ *satisfies*

$$||\chi_{u=\alpha}\nabla_{reg} u|\,|_{\mathcal{L}^2(\Omega,Q)} = 0 \quad \forall \alpha \in \mathbb{R}. \tag{3.1}$$

We can now define the classes of "admissible test functions" which will be used as $\mathcal{W}$-classes in the Definition 4:

$$\mathcal{M}_Q[u,h] = \left\{ w \in \left(W_Q^{1,2}\right)_0(\Omega) : w \geq 0,\ h'(u)w \in \left(W_Q^{1,2}\right)_0(\Omega) \right\}, \tag{3.2}$$

whenever h is admissible for u,

$$\mathcal{E}[u] = \left\{ \psi^2 u : \psi \in C_0^{1,0}(\Omega) \right\}, \tag{3.3}$$

$$\mathcal{A}[u] = \left\{ \psi^2 h(u)h'(u) : \psi \in C_0^{0,1}(\Omega),\ h \text{ is admissible for } u \text{ and } h' \geq 0 \right\}. \tag{3.4}$$

In the above definitions the following convention is accepted [12]

- if $u \in W_Q^{1,2}(\Omega)$ then $h'(u)$ refers to the pair $(h'(u), h''(u)\nabla_{reg} u) \in \mathcal{W}_Q^{1,2}(\Omega)$
- if $u, v \in W_Q^{1,2}(\Omega)$ are represented by the pairs $(u, \nabla u), (v, \nabla v) \in \mathcal{W}_Q^{1,2}(\Omega)$ then the product uv is represented by the pair $(uv, u\nabla v + v\nabla u) \in \mathcal{W}_Q^{1,1}(\Omega)$

Remark 4. *Basic properties of the above classes as well as the motivation for defining them can be found in* [11, Remark 57]

Let $\tilde{u} = h(u)$ and h is admissible for u. Then $\tilde{u}$ satisfies

$$-\int (\nabla \tilde{u})^T \tilde{A} \nabla w = -\int (\nabla(wh'(u)))^T \tilde{A} \nabla u + \int w \chi_{\{x\in\Omega : u(x)\notin \mathcal{R}_u\}} h''(u) [\nabla u]_{\tilde{A}}^2$$

for all $w \in \left(W_Q^{1,2}\right)_0(\Omega)$, where $\tilde{A}(x) = A(x, u(x))$. If in addition u is a weak solution of (2.1) then $\tilde{u} = h(u)$ is a $\mathcal{M}_Q[u,h]$-weak solution of the following equation

$$\tilde{L}\tilde{u} = \nabla^T \tilde{A} \nabla \tilde{u} = h'(u) f + \chi_{\{x\in\Omega : u(x)\notin \mathcal{R}_u\}} h''(u) [\nabla u]_{\tilde{A}}^2 \tag{3.5}$$

If u is a weak subsolution (supersolution) of (2.1) and $h'(u) \geq 0 (\leq 0)$ then $\tilde{u}$ is a $\mathcal{M}_Q[u,h]$-weak subsolution of (3.5). As we will see next we only need to exploit the fact that $\tilde{u}$ is $\mathcal{E}[u]$-weak subsolution (supersolution) of (3.5). The following two lemmas (see [11]) give a relationship between different classes of weak solutions.

Lemma 1. *Let $\tilde{u} = h(u)$ and h is admissible for u. Then $\mathcal{E}[\tilde{u}] \subset \mathcal{M}_Q[u,h]$.*

In particular, if u is a weak subsolution (supersolution) of (2.1) and $h'(u) \geq 0 (\leq 0)$ then $\tilde{u}$ is a $\mathcal{E}[\tilde{u}]$-weak subsolution of (3.5).

Lemma 2. [11, Lemma 56] *Suppose u is a $\mathcal{A}[u]$-weak subsolution of (2.1) in Ω, h is admissible for u and $h'(u) \geq 0$. Then $\tilde{u} = h(u)$ is a positive $\mathcal{E}[\tilde{u}]$-weak subsolution of (3.5).*

Now, let $w = \psi^2 h(u)$ then from (3.5) we obtain

$$\int (\nabla h(u))' \tilde{A} \nabla \psi^2 h(u) + \int \chi_{\{x\in\Omega : u(x)\notin \mathcal{R}_u\}} \psi^2 h(u) h''(u) [\nabla u]_{\tilde{A}}^2 \leq -\int \psi^2 h(u) h'(u) f$$

For the left-hand side we have

$$\begin{aligned}\int (\nabla h(u))^T \tilde{A}\nabla\psi^2 h(u) + \int \chi_{\{x\in\Omega:u(x)\notin\mathcal{R}_u\}}\psi^2 h(u)h''(u)[\nabla u]^2_{\tilde{A}}\\ = \int \psi^2\Gamma(u)[\nabla u]^2_{\tilde{A}} + 2\int \langle\psi\nabla h(u), h(u)\nabla\psi\rangle\\ \geq k\int \psi^2\Gamma(u)[\nabla u]^2_Q + 2\int \langle\psi\nabla h(u), h(u)\nabla\psi\rangle\end{aligned}$$

where $\Gamma(t) = h'(t)^2 + h(t)h''(t) = \left(h(t)^2/2\right)''$. Applying Hölder inequality and then Cauchy-Schwartz inequality with some $0 < \varepsilon < 1$ to the last term and using (2.2) we obtain

$$2\left|\int \langle\psi\nabla h(u), h(u)\nabla\psi\rangle\right| \leq K\varepsilon\int \psi^2 h'(u)^2[\nabla u]^2_Q + K\varepsilon^{-1}\int h(u)^2[\nabla\psi]^2_Q$$

Now let $u \in W^{1,2}_Q(\Omega)$ and assume that $\tilde{u} = h(u)$ satisfies one of the following

- $\Gamma(u) > 0$ and $\tilde{u}$ is a $\mathcal{E}[\tilde{u}]$-weak subsolution of (3.5) in Ω,
- $\Gamma(u) < 0$ and $\tilde{u}$ is a $\mathcal{E}[\tilde{u}]$-weak supersolution of (3.5) in Ω.

Then we obtain

$$\int \psi^2(|\Gamma(u)|-\varepsilon\frac{K}{k}h'(u)^2)[\nabla u]^2_Q \leq \varepsilon^{-1}\frac{K}{k}\int h(u)^2[\nabla\psi]^2_Q + \int \psi^2 h(u)|h'(u)||f| \tag{3.6}$$

4. Proof of the main result

4.1. **Harnack inequality.** In this section we make use of the equation (3.5) for nonlinear functions of u and implement the Moser iteration to prove our main technical result, the Harnack inequality. We first prove a Caccioppoli-type inequality for the Q-gradient of the powers of u. As in [11] for $\beta > 1$ and $M > 0$ we define

$$h_{\beta,M}(t) = \begin{cases} t^\beta, & 0 < t \leq M,\\ M^\beta + \beta M^{\beta-1}(t-M), & t > M\end{cases}$$

and $h_{\beta,M}(t) = t^\beta$ for $\beta \leq 1$ and $M > 0$. One can show that $h_{\beta,M}$ is admissible for $u \in [m,\infty)$ and then $h_{\beta,M}(u) \in W^{1,2}_Q(\Omega)$ since Ω is bounded (see Remark 3). We also have

$$h'_{\beta,M}(t) = \begin{cases} \beta t^{\beta-1}, & 0 < t \leq M,\\ \beta M^{\beta-1}, & t > M\end{cases}$$

and

$$\Gamma(u) = h'_{\beta,M}(u)^2 + h_{\beta,M}(u)h''_{\beta,M}(u) = \eta_\beta(u)h'_\beta(u)^2$$

where

$$\eta_\beta(t) = \begin{cases} \frac{2\beta-1}{\beta}, & 0 \leq t \leq M \text{ or } \beta \leq 1\\ 1, & t > M \text{ and } \beta > 1\end{cases}$$

Lemma 3 (Caccioppoli inequality). *Let $u \in W^{1,2}_Q(\Omega)$ and satisfy one of the following*

(4.1a) *u is a positive weak solution of (2.1) in Ω*

(4.1b) *u is a positive $\mathcal{A}[u]$-weak subsolution of (2.1) in Ω and $\beta > \frac{1}{2}$*

and $\psi \in C_0^{0,1}(\Omega)$, $\beta \in \mathbb{R}$ *with* $\beta \neq 0, \frac{1}{2}$. *Then the following inequality holds*

$$\int \psi^2 [\nabla u^\beta]_Q^2 \leq C\mu_\beta^{-2} \int u^{2\beta} [\nabla \psi]^2 + 2\mu_\beta^{-1} |\beta| \int \psi^2 u^{2\beta - 1} |f| \tag{4.2}$$

where

$$\mu_\beta = \min \left\{ \left| \frac{2\beta - 1}{\beta} \right|, 1 \right\} \tag{4.3}$$

Proof. It follows from Lemmas 1 and 2 that $\tilde{u} = h(u) = h_{\beta,M}(u)$ is a positive $\mathcal{E}[\tilde{u}]$-weak subsolution of (3.5) when $\Gamma(u) > 0$ and a positive $\mathcal{E}[\tilde{u}]$-weak supersolution of (3.5) when $\Gamma(u) < 0$. Therefore, inequality (3.6) holds. Next, we note that $|\Gamma(u)| \geq \mu_\beta h'(u)^2$ where μ_β is defined by (4.3) and choose $\varepsilon = \frac{1}{2} \frac{k}{K} \mu_\beta$ to obtain

$$\frac{1}{2} \mu_\beta \int \psi^2 h'(u)^2 [\nabla u]_Q^2 \leq C\mu_\beta^{-1} \int \psi^2 h(u)^2 [\nabla \psi]_Q^2 + \int \psi^2 h(u) |h'(u)| |f|$$

Letting $M \to \infty$ yelds the above estimate with u^β in place of $h_{\beta,M}(u)$. Finally, using $(u^\beta)'^2 [\nabla u]_Q^2 = [\nabla u^\beta]_Q^2$ we have

$$\int \psi^2 [\nabla u^\beta]_Q^2 \leq C\mu_\beta^{-2} \int u^{2\beta} [\nabla \psi]_Q^2 + 2\mu_\beta^{-1} |\beta| \int \psi^2 u^{2\beta - 1} |f|$$

whenever $\psi \in C_0^{0,1}(\Omega)$, $\beta \in \mathbb{R}$ with $\beta \neq 0, \frac{1}{2}$ and u is bounded below by a positive constant and satisfies (4.1). □

Remark 5. *The assumption (4.1b) as well as the* $\mathcal{M}_Q[u, h]$ *class of weak solutions are only used in the proof of the local boundedness result, Proposition 3, which we do not present here and refer the reader to the proof of Proposition 59 in* [11] *instead. However, we still need to show that all the auxiliary results used in the proof hold true in our case.*

We now perform the Moser iteration on a weak solution of (2.1) using the accumulating sequence of cutoff functions (2.8) to obtain a local bound on $\sup \tilde{u}$ by an L^2-norm of $\tilde{u}$. This is a crucial step in proving Harnack inequality.

Proposition 2 (Moser iteration). *Let* $|\gamma| \leq 2$, $u \in W_Q^{1,2}(\Omega)$ *satisfy one on the following conditions*

(4.4a)
u is a nonnegative weak solution of (2.1) in Ω

(4.4b)
$u + m$ *is a positive* $\mathcal{A}[u + m]$*-weak subsolution of (2.1) in* Ω *for all* $m > 0$ *and* $\gamma > \frac{1}{2}$

Then for any metric ball $B = B(y, r)$ *with* $y \in \Omega$, $0 < r < \delta dist(y, \partial\Omega)$ *the following inequality holds*

$$\operatorname*{ess\,sup}_{x \in B(y,\nu r)} \overline{u}(x) \leq \frac{C}{(1-\nu)^\tau} \left\{ \frac{1}{|B(y, Mr)|} \int_{B(y,r)} \overline{u}^{2\gamma} \right\}^{\frac{1}{2\gamma}} \tag{4.5}$$

where $\overline{u} = u + m(r)$ *with* $m(r) = r^2 ||f||_{L^\infty}$, $\tau = \sigma/(\sigma - 1)$ *with* σ, M *and* ν *as in (2.6), (2.7) and (2.8). The constant* C *depends only on* σ *and* N *from (2.8).*

Proof. First assume that $\overline{u}$ satisfies (4.1), we will later see that for our choice of β it follows from (4.4). Therefore, we can apply (4.2) and for each $\psi \in C_0^{0,1}(\Omega)$ and $\beta \in \mathbb{R}$ with $\beta \neq 0, \frac{1}{2}$ we have

$$\begin{aligned}\int \psi^2 [\nabla \overline{u}^\beta]_Q^2 &\leq C\mu_\beta^{-2} \int \overline{u}^{2\beta} [\nabla \psi]_Q^2 + 2\mu_\beta^{-1} |\beta| \int \psi^2 \overline{u}^{2\beta-1} |f| \\ &\leq C\mu_\beta^{-2} \int \overline{u}^{2\beta} [\nabla \psi]_Q^2 + 2\mu_\beta^{-1} |\beta| \int \psi^2 \overline{u}^{2\beta} m(r)^{-1} |f|\end{aligned}$$

where the last inequality is due to the fact that $\overline{u}^{-1} \leq m(r)^{-1}$. Noting that $m(r)^{-1}|f| \leq r^{-2}$ we get

$$\int \psi^2 [\nabla \overline{u}^\beta]_Q^2 \leq C \left\{ \mu_\beta^{-2} \int \overline{u}^{2\beta} [\nabla \psi]_Q^2 + 2\mu_\beta^{-1} |\beta| r^{-2} \int \psi^2 \overline{u}^{2\beta} \right\}$$

Next, we use the above inequality with an accumulating sequence of cutoff functions ψ_j and a fixed subunit ball $B = B(y, r)$. Denote $B^* = MB = B(y, Mr)$ and note that $\mathrm{supp}\psi_j \subset B \subseteq B^*\ \forall j$. Dividing through by the measure of the ball $|B^*|$ and taking square roots we obtain

(4.6)

$$\left\{ \frac{1}{|B^*|} \int_{B^*} \psi_j^2 [\nabla \overline{u}^\beta]_Q^2 \right\}^{\frac{1}{2}} \leq C \left\{ \mu_\beta^{-2} \frac{1}{|B^*|} \int_{B^*} \overline{u}^{2\beta} [\nabla \psi_j]_Q^2 \right\}^{\frac{1}{2}} + C \left\{ \mu_\beta^{-1} |\beta| r^{-2} \frac{1}{|B^*|} \int_{B^*} \psi_j^2 \overline{u}^{2\beta} \right\}^{\frac{1}{2}}$$

We now use Sobolev inequality (2.6) for $w = \psi_j \overline{u}^\beta$. Then for some $\sigma > 1$ we obtain using (4.6)

$$\begin{aligned}\left\{ \frac{1}{|B^*|} \int_{B^*} (\psi_j \overline{u}^\beta)^{2\sigma} \right\}^{\frac{1}{2\sigma}} &\leq Cr(B^*) \left\{ \frac{1}{|B^*|} \int_{B^*} [\nabla (\psi_j \overline{u}^\beta)]_Q^2 \right\}^{\frac{1}{2}} + C \left\{ \frac{1}{|B^*|} \int_{B^*} (\psi_j \overline{u}^\beta)^2 \right\}^{\frac{1}{2}} \\ &\leq CMr \left(1 + \mu_\beta^{-1} \right) \left\{ \frac{1}{|B^*|} \int_{B^*} \overline{u}^{2\beta} [\nabla \psi_j]_Q^2 \right\}^{\frac{1}{2}} \\ &\quad + C \left(1 + \mu_\beta^{-1/2} |\beta|^{1/2} \right) \left\{ \frac{1}{|B^*|} \int_{B^*} (\overline{u}^\beta \psi_j)^2 \right\}^{\frac{1}{2}}\end{aligned}$$

Using the properties of the cutoff functions ψ_j

$$\mathrm{supp}\psi_j = E_j$$

$$[\nabla \psi_j]_Q \leq c \frac{j^N}{r(1-\nu)}$$

yields

(4.7)

$$\left\{ \frac{1}{|B^*|} \int_{E_{j+1}} (\overline{u}^\beta)^{2\sigma} \right\}^{\frac{1}{2\sigma}} \leq C \left(\frac{j^N}{(1-\nu)} + \mu_\beta^{-1/2} |\beta|^{1/2} + \frac{j^N}{(1-\nu)} \mu_\beta^{-1} \right) \left\{ \frac{1}{|B^*|} \int_{E_j} \overline{u}^{2\beta} \right\}^{\frac{1}{2}}$$

where now the constant C depends on M in (2.7). It can be shown that the constant in parenthesis is bounded by

$$\frac{j^N}{1-\nu} \cdot M(\beta) = \frac{j^N}{1-\nu} \cdot C \left(1 + |\beta|^{1/2} + \frac{|\beta|}{|2\beta - 1|^{1/2}} + \frac{|\beta|}{|2\beta - 1|} \right)$$

Next, to iterate the inequality (4.7), we fix $\gamma \neq 0$ and denote $u_j = \overline{u}^{\gamma\sigma^{j-1}}$. Take $\beta = \gamma\sigma^{j-1}$, one can see that for each $j \geq 1$ (4.1) follows from (4.4) for this choice of β. Using $u_{j+1} = u_j^\sigma$ we obtain from (4.7)

(4.8)
$$\left\{\frac{1}{|E_{j+1}|}\int_{E_{j+1}} u_{j+1}^2\right\}^{\frac{1}{2}} \leq \frac{|B^*|^{1/2}}{|E_{j+1}|^{1/2}}\frac{|E_j|^{\sigma/2}}{|B^*|^{\sigma/2}}C\frac{j^{N\sigma}}{(1-\nu)^\sigma}(M(\gamma\sigma^{j-1}))^\sigma\left\{\frac{1}{|E_j|}\int_{E_j} u_j^2\right\}^{\frac{\sigma}{2}}$$

Let

$$N_j = \left\{\frac{1}{|E_j|}\int_{E_j} u_j^2\right\}^{\frac{1}{2\sigma^{j-1}}}$$

then from (4.8)

$$N_{j+1} \leq \left[\frac{|B^*|^{(1-\sigma)/2}|E_j|^{\sigma/2}}{|E_{j+1}|^{1/2}}C\frac{j^{N\sigma}}{(1-\nu)^\sigma}(M(\gamma\sigma^{j-1}))^\sigma\right]^{\frac{1}{\sigma^j}} N_j$$

Iterating, we obtain

$$\limsup_{j\to\infty} N_j \leq C\left\{\prod_{j=1}^{\infty}\left[\frac{|B^*|^{(1-\sigma)/2}|E_j|^{\sigma/2}}{|E_{j+1}|^{1/2}}\frac{j^{N\sigma}}{(1-\nu)^\sigma}(M(\gamma\sigma^{j-1}))^\sigma\right]^{\frac{1}{\sigma^j}}\right\}N_1$$

Next, we note that the product of the measures of the sets on the right has the form

$$\prod_{j=1}^{\infty}\frac{|B^*|^{(1-\sigma)/2\sigma^j}|E_j|^{1/2\sigma^{j-1}}}{|E_{j+1}|^{1/2\sigma^j}}$$

Further,

$$\begin{aligned}\prod_{j=1}^{m}|B^*|^{(1-\sigma)/2\sigma^j} &= \exp\left(\log(|B^*|)\sum_{j=1}^{m}\left(\frac{1}{2\sigma^j}-\frac{1}{2\sigma^{j-1}}\right)\right)\\ &= |B^*|^{\frac{1}{2\sigma^m}-\frac{1}{2}} \longrightarrow |B^*|^{-\frac{1}{2}} \text{ as } m\to\infty\end{aligned}$$

and

$$\begin{aligned}\prod_{j=1}^{m}\frac{|E_j|^{1/2\sigma^{j-1}}}{|E_{j+1}|^{1/2\sigma^j}} &= \prod_{j=1}^{m}|E_j|^{1/2\sigma^{j-1}}\Big/\prod_{j=2}^{m+1}|E_j|^{1/2\sigma^{j-1}}\\ &= \frac{|E_1|^{\frac{1}{2}}}{|E_{m+1}|^{\frac{1}{2\sigma^m}}} \leq \frac{|E_1|^{\frac{1}{2}}}{|B(y,\nu r)|^{\frac{1}{2\sigma^m}}} \longrightarrow |E_1|^{\frac{1}{2}} \text{ as } m\to\infty\end{aligned}$$

Therefore

$$\limsup_{j\to\infty} N_j \leq C\frac{1}{(1-\nu)^{\frac{\sigma}{\sigma-1}}}\prod_{j=1}^{\infty}\left(j^{N/\sigma^{j-1}}(M(\gamma\sigma^{j-1}))^{\frac{1}{\sigma^{j-1}}}\right)\left\{\frac{1}{|B^*|}\int_{E_1}\overline{u}^{2\gamma}\right\}^{\frac{1}{2}} \tag{4.9}$$

Also

$$\operatorname*{ess\,sup}_{x\in B(y,\nu r)} \overline{u}^{\gamma} \leq \limsup_{j\to\infty} N_j \tag{4.10}$$

and therefore we only need to estimate the constant

$$C_\gamma = \prod_{j=1}^{\infty}\left(j^{N/\sigma^{j-1}}(M(\gamma\sigma^{j-1}))^{\frac{1}{\sigma^{j-1}}}\right)$$

where

$$M(\gamma\sigma^{j-1}) = C\left(1+|\gamma\sigma^{j-1}|^{1/2}+\frac{|\gamma\sigma^{j-1}|}{|2\gamma\sigma^{j-1}-1|^{1/2}}+\frac{|\gamma\sigma^{j-1}|}{|2\gamma\sigma^{j-1}-1|}\right)$$

We can write

$$C_\gamma = C\exp\sum_{j=1}^{\infty}\frac{\ln\left[j^N\left(1+|\gamma\sigma^{j-1}|^{1/2}+\frac{|\gamma\sigma^{j-1}|}{|2\gamma\sigma^{j-1}-1|^{1/2}}+\frac{|\gamma\sigma^{j-1}|}{|2\gamma\sigma^{j-1}-1|}\right)\right]}{\sigma^{j-1}}$$

In order to have $C_\gamma \leq \infty$ we must ensure that $\gamma\sigma^{j-1}$ does not approach $1/2$ for any values of j. We first note that for any γ we have $\tilde{\gamma} \leq \gamma < \tilde{\gamma}\sigma$ with $\tilde{\gamma}$ of the form $\tilde{\gamma} = \frac{1}{4}\sigma^{\eta}(\sigma+1)$ for some integer η. Therefore due to the monotonicity of

$$\left(\frac{1}{|B^*|}\int_B \overline{u}^{\gamma}\right)^{1/\gamma}$$

it is enough to prove (4.5) for $\gamma = \tilde{\gamma}$. For this choice of γ we have

$$|2\tilde{\gamma}\sigma^{j-1}-1| \geq \frac{1}{2}(1-\sigma^{-1})$$

Indeed, if $\eta \geq -j+1$ then

$$2\tilde{\gamma}\sigma^{j-1}-1 = \frac{1}{2}\sigma^{\eta+j-1}(\sigma+1)-1 \geq \frac{\sigma}{2}-\frac{1}{2} \geq \frac{1}{2}(1-\sigma^{-1})$$

while for $\eta \leq -j$ we similarly check that

$$2\tilde{\gamma}\sigma^{j-1}-1 \leq -\frac{1}{2}(1-\sigma^{-1})$$

We therefore obtain

$$C_\gamma \leq C_\sigma(1+|\gamma|^{p(\sigma)})$$

and for $|\gamma| \leq 2$ from (4.9), (4.10)

$$\operatorname*{ess\,sup}_{x\in B(y,\nu r)} \overline{u}^{\gamma} \leq \frac{C_\sigma}{(1-\nu)^{\frac{\sigma}{\sigma-1}}}\left\{\frac{1}{|B^*|}\int_{E_1}\overline{u}^{2\gamma}\right\}^{\frac{1}{2}} \leq \frac{C_\sigma}{(1-\nu)^{\frac{\sigma}{\sigma-1}}}\left\{\frac{1}{|MB|}\int_{B}\overline{u}^{2\gamma}\right\}^{\frac{1}{2}} \tag{4.11}$$

Finally, we note that if $\gamma > 1/2$ then for $\beta = \gamma\sigma^{j-1}$ we also have $\beta > 1/2$. Therefore, $\overline{u} = u + m$ satisfies (4.1) as long as u satisfies (4.4). □

Harnack inequality will follow from the following auxiliary lemma (see Lemma [7]).

Lemma 4. *Let $w > 0$ be a measurable function defined in a neighborhood of $B_0 = B(x, r_0)$. Suppose there exist positive constants τ, c_1, c_2, $M \geq 1$ such that the following two conditions hold*

(1)

$$\sup_{x\in\nu B} w^{\gamma} \leq \frac{c_1}{(1-\nu)^{\tau}} \left\{ \frac{1}{|MB|} \int\limits_B w^{2\gamma} \right\}^{1/2} \tag{4.12}$$

for every $0 < \nu_0 < \nu < 1$, $0 < \gamma \leq 2$, where $B = B(x,r) \subset B_0$, $MB = B(x, Mr) \subset B_0$.

(2)

$$s^{\frac{1}{p}} |\{x \in B : \log w > s + a\}| < c_2|MB| \tag{4.13}$$

for every $s > 0$, some $a \geq 0$, and some $p \geq 1$.

Then there exist a constant $b = b(\nu_0, c_1, c_2, \tau, p)$ such that

$$\operatorname*{ess\,sup}_{\nu_0 B} w < be^a \tag{4.14}$$

Proof. Define

$$\varphi(\rho) = \operatorname*{ess\,sup}_{y\in B(x,\rho)} (\log w(y) - a) \quad \text{for} \quad \nu_0 r \leq \rho < r$$

We can decompose the ball B in the following way

$$B = \{y \in B : \log w(y) - a > \frac{1}{2}\varphi(r)\} \cup \{y \in B : \log w(y) - a \leq \frac{1}{2}\varphi(r)\}$$

Then, using condition (4.13) we have

$$e^{-2\gamma a} \int\limits_B w^{2\gamma} \leq \frac{2^{1/p} c_2}{\varphi^{1/p}} e^{2\gamma\varphi} |MB| + e^{\gamma\varphi}|B|$$

or

$$\frac{e^{-2\gamma a}}{|MB|} \int\limits_B w^{2\gamma} \leq \frac{2^{1/p} c_2}{\varphi^{1/p}} e^{2\gamma\varphi} + e^{\gamma\varphi} \frac{|B|}{|MB|} \leq \frac{2^{1/p} c_2}{\varphi^{1/p}} e^{2\gamma\varphi} + e^{\gamma\varphi}$$

where $\varphi = \varphi(r)$.

Next, choose γ so that the two terms on the right-hand side are equal $(2^{1/p}c_2/\varphi^{1/p})e^{2\gamma\varphi} = e^{\gamma\varphi}$, i.e.

$$\gamma = \frac{1}{\varphi} \log\left(\frac{\varphi^{1/p}}{2^{1/p} c_2}\right) = \frac{1}{p\varphi} \log\left(\frac{\varphi}{2c_2^p}\right) \tag{4.15}$$

To satisfy the condition $0 < \gamma \leq 2$ we must then require

$$\varphi = \varphi(r) > c_3 \tag{4.16}$$

with c_3 depending only on c_2. In that case we have

$$\left(\frac{e^{-2\gamma a}}{|MB|} \int\limits_B w^{2\gamma}\right)^{\frac{1}{2}} \leq \sqrt{2} e^{\gamma\varphi/2}$$

and using (4.12)

$$\varphi(\nu r) \leq \frac{1}{\gamma}\log\left(\frac{c_1\sqrt{2}}{(1-\nu)^{\tau}}e^{\gamma\varphi/2}\right) = \frac{1}{\gamma}\log\left(\frac{c_1\sqrt{2}}{(1-\nu)^{\tau}}\right) + \frac{1}{2}\varphi(r)$$

hence, by (4.15),

$$\varphi(\nu r) < \varphi(r)\left(\frac{p\log(c_1\sqrt{2}/(1-\nu)^{\tau})}{\log(\varphi/2c_2^p)} + \frac{1}{2}\right).$$

If

$$\varphi(r) > \frac{2^{1+2p}c_1^{4p}c_2^p}{(1-\nu)^{4p\tau}} \tag{4.17}$$

then the first term in parenthesis is less than 1/4 and therefore we have $\varphi(\nu r) < 3/4\varphi(r)$. On the other hand, if (4.16) or (4.17) do not hold, then there exists c_4 such that

$$\varphi(r) \leq \frac{c_4}{(1-\nu)^{4p\tau}}$$

Therefore, in any case we have

$$\varphi(\nu r) < \frac{3}{4}\varphi(r) + \frac{c_4}{(1-\nu)^{4p\tau}} \tag{4.18}$$

since $\varphi(\nu r) \leq \varphi(r)$. We now iterate this inequality with

$$0 < \nu_0 < \nu_1 < \nu_2 < \ldots < \nu_k \leq 1$$

to obtain

$$\varphi(\nu_0 r) < \left(\frac{3}{4}\right)^k \varphi(r) + c_4\sum_{j=0}^{k-1}\left(\frac{3}{4}\right)^j (\nu_{j+1}-\nu_j)^{-4p\tau}$$

Choosing

$$\nu_j = 1 - \frac{1-\nu_0}{1+j}$$

and letting $k \to \infty$ we obtain

$$\varphi(\nu_0 r) \leq C(\nu_0, c_1, c_2, \tau, p)$$

which concludes the proof. □

We now want to apply Lemma 4 to a positive subsolution $\overline{u}$. It has been shown in Proposition 2 that condition (4.12) is satisfied. The following lemma shows that (4.13) holds for $\overline{u}$ and $1/\overline{u}$.

Lemma 5. *Let $\overline{u} = u + m(r)$, $m(r) = r^2\|f\|_{L^\infty}$ with $r \leq r_0$. Then for any $p \geq k/2+1$ where k is as in (2.7) there holds*

$$s^{1/p}|\{x \in B : \log\overline{u} > s + \langle\log\overline{u}\rangle_B\}| < C|MB| \tag{4.19}$$

$$s^{1/p}|\{x \in B : \log(1/\overline{u}) > s - \langle\log\overline{u}\rangle_B\}| < C|MB| \tag{4.20}$$

with $M \geq 4\lambda$ as in (2.7).

Proof. Denote $v = \ln \overline{u}$, for any $s > 0$, $v \in L^1(B)$ nonnegative we have

$$s^{1/p}|\{x \in B : v - \langle v\rangle_B > s\}| \leq \int_B |v - \langle v\rangle_B|^{1/p}$$

Using Hölder inequality we have

$$\int_B |v - \langle v\rangle_B|^{1/p} \leq \left(\int_B |v - \langle v\rangle_B|\right)^{\frac{1}{p}} |B|^{\frac{p-1}{p}}$$

and applying Poincaré inequality (2.7)

$$s^{1/p}|\{x \in B : v - \langle v\rangle_B > s\}| \leq (Cr)^{1/p} \frac{|MB|^{\frac{k+1}{2p}}}{|B|\frac{k}{2p}} |B|^{\frac{p-1}{p}} \left\{\int\limits_{\lambda B} [\nabla v]_Q^2\right\}^{\frac{1}{2p}}$$

Therefore, in order to prove (4.19) we need to show

$$\int\limits_{\lambda B} [\nabla v]_Q^2 \leq \frac{c}{r^2}|MB|. \tag{4.21}$$

Consider equation (2.3) and substitute $w = \frac{\varphi^2}{\overline{u}}$ with $\varphi \in W_0^{1,2}(2\lambda B)$, $\varphi = 1$ in λB and $||[\nabla\varphi]_Q||_{L^\infty} \leq C/r$. Here r is the radius of the ball $B = B(x, r)$. We obtain

$$\int_{2\lambda B} \varphi^2(\nabla \ln(\overline{u}))^T \tilde{A} \nabla \ln(\overline{u}) = 2\int_{2\lambda B} \varphi\nabla\varphi\tilde{A}\nabla\ln(\overline{u}) + \int_{2\lambda B} \frac{f\varphi^2}{\overline{u}}$$

by using Hölder inequality on the first term on the right, we obtain

$$\begin{aligned}\int_{2\lambda B} \varphi^2(\nabla \ln(\overline{u}))^T \tilde{A} \nabla \ln(\overline{u}) &\leq C\left(\int_{2\lambda B} [\nabla\varphi]_{\tilde{A}}^2\right)^{1/2}\left(\int_{2\lambda B} \varphi^2[\nabla\ln(\overline{u})]_{\tilde{A}}^2\right)^{1/2} + \int_{2\lambda B}\frac{\varphi^2}{r^2}\\ &\leq \varepsilon\int_{2\lambda B} \varphi^2[\nabla\ln(\overline{u})]_{\tilde{A}}^2 + \frac{C}{\varepsilon}\int_{2\lambda B}[\nabla\varphi]_{\tilde{A}}^2 + \frac{|2\lambda B|}{r^2}\end{aligned}$$

where in the last inequality we used Cauchy-Schwartz inequality. From (2.2) and the estimate $||[\nabla\varphi]_Q||_{L^\infty} \leq C/r$ it follows

$$\int_{2\lambda B} \varphi^2[\nabla\ln(\overline{u})]_Q^2 \leq \varepsilon\frac{k}{K}\int_{2\lambda B}\varphi^2[\nabla\ln(\overline{u})]_Q^2 + \frac{C|2\lambda B|}{\varepsilon r^2} + \frac{|2\lambda B|}{r^2}$$

Now, we choose ε small enough that the first term on the right can be absorbed into the left-hand side to obtain

$$\int_{2\lambda B}\varphi^2[\nabla\ln(\overline{u})]_Q^2 \leq C\frac{|2\lambda B|}{r^2}$$

which implies (4.21).
The proof of (4.20) follows in a similar way. □

The following Harnack inequality is now almost immediate

Theorem 2. *Let u be a nonnegative weak solution of (2.1) in $B(y, r)$. Then u satisfies the strong Harnack inequality*

$$\operatorname*{ess\,sup}_{x\in B(y,\nu_0 r)} (u(x) + m(r)) \leq C_{Har} \operatorname*{ess\,inf}_{x\in B(y,\nu_0 r)} (u(x) + m(r)) \tag{4.22}$$

Proof. It follows from (4.5), (4.19), and (4.20) that the functions $w_1 = \overline{u}$ and $w_2 = 1/\overline{u}$ satisfy the conditions of Lemma 4 with $a_1 = \langle \ln \overline{u} \rangle_B$ and $a_2 = -\langle \ln \overline{u} \langle_B$ respectively. Therefore we have

$$\operatorname*{ess\,sup}_{\nu_0 B} \overline{u} < be^{\langle \ln \overline{u} \rangle_B}$$

$$\operatorname*{ess\,sup}_{\nu_0 B} \frac{1}{\overline{u}} < be^{-\langle \ln \overline{u} \rangle_B}$$

Multiplying the above inequalities we obtain (4.22). □

4.2. **"Metric" Hölder continuity.** We will now establish Hölder continuity of solutions with respect to the subunit metric. This and the containment condition (2.4) then imply uniform continuity with a local estimate of the modulus of continuity. We first formulate the result on the local boundedness of weak solutions.

Proposition 3 (Local boundedness)**.** *Suppose that u is a weak solution to (2.1) in Ω and that (2.6), (2.7), and (2.8) hold. Then u is locally bounded in Ω. More precisely, if $y \in \Omega$, $0 < r < \delta dist(y, \partial\Omega)$ for sufficiently small δ, then*

$$||u||_{L^\infty(B(y,\nu r))} \leq C \left\{ \left(\frac{1}{|B(y, Mr)|} \int_{B(y,r)} u^2 \right)^{\frac{1}{2}} + r^2 ||f||_{L^\infty} \right\} \tag{4.23}$$

where $\nu > 0$ is as in (2.8) and $M > 1$ as in (2.7).

The proof is similar to the proof of Proposition 59 in [11] and follows from (4.5). Similar result has been recently established in a more general setting [13]. Note, however that we have a factor of $1/|MB|$ on the right hand side, and in the case of nondoubling metric it can be much smaller than the factor $1/|B|$ obtained in [13]. The next proposition follows from Harnack inequality and the proof can be found, for example, in [4, Theorem 8.22], [11].

Proposition 4. *Let u be a nonnegative weak solution of (2.1) in $B(y, R)$. For $0 < r < R$ denote*

$$\omega(r) = \operatorname*{ess\,sup}_{x \in B(y,r)} u(x) - \operatorname*{ess\,inf}_{x \in B(y,r)} u(x)$$

Then we have

$$\omega(r) \leq C(\omega(R) + R^2 ||f||_{L^\infty}) \left(\frac{r}{R} \right)^\alpha, \quad 0 < r \leq R$$

Hölder continuity of solutions with respect to metric d follows in a usual way from Harnack inequality using Lemma 8.23 of [4], which concludes the proof of Theorem 1.

5. Subunit metric

5.1. **Definitions.** Let $Q(x, \xi)$ be nonnegative semidefinite continuous quadratic form.

Definition 6. *A Lipschitz curve $\gamma : [0, r] \to \Omega$ is subunit with respect to $Q(x, \xi)$ if*

$$(\gamma'(t) \cdot \xi)^2 \leq Q(\gamma(t), \xi) \text{ a.e. } t \in [0, r],\ \xi \in \mathbb{R}^n.$$

Definition 7. *We define*

$$d(x, y) = \inf\{r > 0 \ :\ \gamma(0) = x,\ \gamma(r) = y,\ \gamma \text{ is subunit in } \Omega\}.$$

Definition 8. *If d is finite on $\Omega \times \Omega$ we define subunit balls $B(x,r)$ by*

$$B(x,r) = \{y \in \Omega \ : \ d(x,y) < r\}, \quad x \in \Omega,\ 0 < r < \infty.$$

We will denote $E(x,r)$ the usual Euclidean ball

$$E(x,r) = \{y \in \Omega \ : \ |x-y| < r\}, \quad x \in \Omega,\ 0 < r < \infty.$$

We will assume that subunit balls satisfy the following conditions.

(1) $\forall x \in \Omega\ \exists \alpha_x : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\alpha_x(r) > 0$ for any $r > 0$ and

$$E(y, \alpha_x(R)) \subset B(x,R),\ \forall y \in \Omega,\ \forall R > 0 \tag{5.1}$$

(2) For any $x \in \Omega$, any $r > 0$

$$|B(x,r)| \le C_n r^n \tag{5.2}$$

5.2. **Accumulating sequence of cutoff functions.** The following lemma is a variant of Proposition 68 in [11].

Lemma 6. *Let $Q(x,\xi) = \xi' Q(x)\xi$ be continuous nonnegative semidefinite quadratic form. Suppose that subunit metrics associated to $Q(x)$ satisfies (2.4), (2.5). Then there exist an accumulating sequence of cutoff functions satisfying (2.8).*

Proof. For any $\varepsilon \ge 0$ let $Q^\varepsilon(x,\xi) = Q(x,\xi) + \varepsilon^2|\xi|^2$. Then as it has been shown in [11, Lemma 66] the subunit metric $d^\varepsilon(x,y)$ associated to Q^ε satisfies

$$[\nabla_x d^\varepsilon(x,y)]_Q \le \sqrt{n}, \quad x,y \in \Omega$$

uniformly in $\varepsilon > 0$. Define $\phi_j(t)$, $j \ge 1$, to vanish for $t \ge r_j$, to equal 1 for $t \le r_{j+1}$ and to be linear on the interval $[r_{j+1}, r_j]$, where $r_j - r_{j+1} = r/(kj^2)$, $r_1 = r$. Here the coefficient k is to be chosen later depending on the size of the "inner" ball νB. Let $\psi_j(x) = \phi_{3j}(d^{\varepsilon_{3j}})$, we have

$$[\nabla \psi_j]_Q \le ||\nabla \phi_j||_\infty [\nabla d^{\varepsilon_{3j}}]_Q \le \frac{kj^2}{r}\sqrt{n}.$$

Let us denote $r_\infty = \lim_{j\to\infty} r_j$, we want to have $r_\infty > \nu r$. Writing

$$r - r_\infty = \sum_{j=1}^{\infty}(r_j - r_{j+1}) = \sum_{j=1}^{\infty} \frac{r}{kj^2} = \frac{r}{k}\frac{\pi^2}{6}$$

one can see that with $k = 2/(1-\nu)$ this condition is satisfied. Therefore we have

$$[\nabla \psi_j]_Q \le c_n \frac{j^2}{r(1-\nu)} \tag{5.3}$$

Therefore, for ε small enough the conditions in (2.8) are satisfied. □

Remark 6. *Note that the condition that $Q(x)$ is continuous cannot be easily omitted. In* [14] *the author constructs an example of a discontinuous solution to a degenerate linear elliptic equation (see Theorem 1.3 and Conjecture 6). However, the matrix Q in that case is discontinuous and this requirement seems to be essential for the construction.*

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