

# SUMS OF SQUARES I: SCALAR FUNCTIONS

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ABSTRACT. This is the first in a series of three papers dealing with sums of squares and hypoellipticity in the infinitely degenerate regime. A result of C. Fefferman and D. H. Phong shows that every  $C^{3,1}$  nonnegative function on  $\mathbb{R}^n$  can be written as a finite sum of squares of  $C^{1,1}$  functions, and was used by them to improve Gårding's inequality, and subsequently by P. Guan to prove regularity for certain degenerate operators.

In this paper we investigate sharp criteria sufficient for writing a smooth nonnegative function  $f$  on  $\mathbb{R}^n$  as a finite sum of squares of  $C^{2,\delta}$  functions for some  $\delta > 0$ , and we denote this property by saying  $f$  is  $SOS_{\text{regular}}$ . The emphasis on  $C^{2,\delta}$ , as opposed to  $C^{1,1}$ , arises because of applications to hypoellipticity for smooth infinitely degenerate operators in the spirit of M. Christ, which are pursued in the third paper of this series.

Thus we consider the case where  $f$  is smooth and flat at the origin, and positive away from the origin. Our sufficient condition for such an  $f$  to be  $SOS_{\text{regular}}$  is that  $f$  is  $\omega$ -monotone for some modulus of continuity  $\omega_s(t) = t^s$ ,  $0 < s \leq 1$ , where  $\omega$ -monotone means

$$f(y) \leq C\omega(f(x)), \quad y \in B_x,$$

and where  $B_x = B\left(\frac{x}{2}, \frac{|x|}{2}\right)$  is the ball having a diameter with endpoints 0 and  $x$  (this is the interval  $(0, x)$  in dimension  $n = 1$ ). On the other hand, we show that if  $\omega$  is any modulus of continuity with  $\lim_{t \rightarrow 0} \frac{\omega(t)}{\omega_s(t)} = \infty$  for all  $s > 0$ , then there exists a smooth nonnegative function  $f$  that is flat at the origin, and positive away from the origin, that is *not*  $SOS_{\text{regular}}$ , answering in particular a question left open by Bony.

Refinements of this result are given for  $f \in C^{4,2\delta}$ , and the related problem of extracting smooth positive roots from such smooth functions is also considered.

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## 1. INTRODUCTION

It is an open problem whether or not there are smooth nonnegative functions  $\lambda$  on the real line (even vanishing only at the origin, and to infinite order there), such that they **cannot** be written as a finite sum  $\lambda = \sum_{n=1}^N f_n^2$  of squares of smooth functions  $f_n$ . Examples of such functions are attributed to Paul Cohen in both [Bru] and [BoCoRo], but apparently no example has ever appeared in the literature, and the existence

of such an example is an open problem, see [Pie, Remark 5.1]<sup>1</sup>. Such sum of squares decompositions are relevant to hypoellipticity questions in partial differential equations, see e.g. Hörmander [Ho], and especially in the infinitely degenerate regime, see e.g. Christ [Chr] and references given there. In particular we point to the theorem of Christ there that asserts hypoellipticity for a second order differential operator  $L$  if it is a finite sum  $\sum X_k^{\text{tr}} X_k$  of squares of smooth vector fields  $X_k$  satisfying certain conditions relevant to hypoellipticity in the infinitely degenerate regime. In the third paper [KoSa3] of this series, the authors have extended this theorem to  $C^{2,\delta}$  vector fields, which is essentially optimal for second order operators. Thus for partial differential equations, the crucial sum of squares question is this.

**Problem 1.** *When can a nonnegative scalar or matrix function  $f(x)$  on  $\mathbb{R}^n$  be written as a sum of squares of  $C^{2,\delta}$  scalar or vector functions for some  $\delta > 0$ ?*<sup>2</sup>

A well known and important construction of Fefferman and Phong in 1978, with only a bare sketch of a proof given in [FePh], was used by Guan [Gua, see the end of the paper] in the mid 1990's to prove the following result that Guan attributed to Fefferman: every smooth (even  $C^{3,1}$ ) nonnegative function  $f$  on  $\mathbb{R}^n$  can be written as a sum of squares of  $C^{1,1}$  functions. However, while this decomposition was a perfect fit for the  $C^2$  *a priori* estimates proved for the Monge-Ampere equation by Guan, this decomposition falls short for applications which require  $C^{2,\delta}$  coefficients or vector fields for some  $\delta > 0$ . The classical such application is Schauder theory, where  $C^{2,\delta}$  coefficients play a pivotal role, and more importantly for us is a generalization of a sum of squares theorem of Christ that we prove in [KoSa3] using  $C^{2,\delta}$  vector fields. As a consequence, we will refer to a function  $g$  in  $\bigcup_{\delta>0} C^{2,\delta}$  as a *regular* function, so that in the context of partial differential

equations in the infinitely degenerate regime at the origin, where some of the coefficients are flat (i.e. vanish to an infinite order) at the origin, the scalar question becomes this. We say that a scalar or matrix function is *elliptical* if it is positive definite away from the origin.

**Problem 2.** *When can an elliptical flat smooth scalar function  $f$  on  $\mathbb{R}^n$  be written as a sum of squares of regular scalar functions?*

The corresponding question for elliptical *finite type* smooth scalar functions has been well studied in the wake of Hilbert's 17<sup>th</sup> problem, and there are algebraic obstructions to writing a smooth function as a sum of squares of smooth functions. For example, the homogeneous Motzkin polynomial  $M$  in  $n = 3$  dimensions, and a generalization  $L$  to dimension  $n = 4$ ,

$$\begin{aligned} M(x, y, z) &= z^6 + x^2 y^2 (x^2 + y^2 - 3\lambda z^2), & (x, y, z) \in \mathbb{R}^3, \\ L(x, y, z, w) &= w^4 + x^2 y^2 + y^2 z^2 + z^2 x^2 - 4\lambda x y z w, & (x, y, z, w) \in \mathbb{R}^4, \end{aligned}$$

are nonnegative for  $0 \leq \lambda \leq 1$ , vanish only at the origin for  $0 < \lambda < 1$ , and are *not* finite sums of squares of polynomials for  $0 < \lambda \leq 1$ , see [BoBrCoPe]. As pointed out by Bony [Bon], Taylor expansions can then be used to show that  $M$  cannot be written as a finite sum of squares of  $C^3$  functions, and that  $L$  cannot be written as a finite sum of squares of  $C^2$  functions. This latter observation, along with  $L(x, y, z, w)$  itself, will play a critical role in establishing sharpness for finite sums of squares of regular functions.

Our main sum of squares theorem for scalar nonnegative functions gives a sharp answer to this question in terms of an  $\omega$ -*monotone property*, defined below for any modulus of continuity  $\omega$ , namely that the answer to Problem 2 is affirmative if  $f$  is *Hölder monotone*. On the other hand, part (2) of Theorem 2.5 below, shows that the sum of squares decomposition can fail for any  $\omega$ -monotone property weaker than Hölder monotone. In particular, this settles a question left open in [BoBrCoPe, Remark 1.4 on page 141], that asked if there exists an *elliptical* flat smooth function that is not a sum of squares of  $C^{2,\omega}$  functions.

There are several notions of monotonicity for nonnegative functions of several variables used in this paper, and we illustrate them here by giving these definitions for functions  $f(x)$  defined on the unit interval  $[0, 1]$ , with higher dimensional definitions given later. Let  $\omega(t)$  be a modulus of continuity defined on  $[0, 1]$ , i.e.  $\omega$  is continuous, nondecreasing and strictly concave, and satisfies  $\omega(0) = 0$  and  $\omega(1) = 1$ . Then we define varying degrees of monotonicity that are weaker than traditional monotonicity as follows.

**Definition 1.1.** *Suppose  $f : [0, 1] \rightarrow [0, \infty)$  and that  $\omega(t)$  is a modulus of continuity on  $[0, 1]$ .*

<sup>1</sup>See also <https://mathoverflow.net/a/106072>

<sup>2</sup>In this paper we consider the scalar problem. The equally relevant problem of writing a nonnegative matrix function as a sum of squares is treated in [KoSa2].

- (1)  $f$  is  $\omega$ -monotone if  $0 \leq f(y) \leq C\omega(f(x))$  for  $0 \leq y \leq x \leq 1$  and some positive constant  $C$ ,
- (2)  $\omega_s(t) = \begin{cases} t(1 + \ln \frac{1}{t}) & \text{if } s = 1 \\ t^s & \text{if } 0 < s < 1 \\ \frac{1}{1 + \ln \frac{1}{t}} & \text{if } s = 0 \end{cases}$ , for  $0 \leq s, t \leq 1$ ,
- (3)  $f$  is nearly monotone if  $f$  is  $\omega_s$ -monotone for every  $0 \leq s < 1$ ,
- (4)  $f$  is Hölder monotone if  $f$  is  $\omega_s$ -monotone for some  $0 < s \leq 1$ .

Then  $\omega_s$  is a modulus of continuity for all  $0 \leq s \leq 1$ , and

$$t \ll \omega_1(t) \ll \omega_{s'}(t) \ll \omega_s(t) \ll \omega_0(t) \ll 1, \quad 0 < s < s' \leq 1,$$

where for positive functions on  $(0, 1)$ ,  $a(t) \ll b(t)$  means  $\lim_{t \searrow 0} \frac{a(t)}{b(t)} = 0$ .

We now extend the definition of  $\omega$ -monotone to several variables.

**Definition 1.2.** Given a modulus of continuity  $\omega$ , we say that a function  $f : \mathbb{R}^n \rightarrow [0, \infty)$  is  $\omega$ -monotone if  $0 \leq f(y) \leq C\omega(f(x))$  for  $y \in B\left(\frac{x}{2}, \frac{|x|}{2}\right)$  and some positive constant  $C$ , and we set

$$(1.1) \quad \|f\|_{\omega\text{-mon}} \equiv \sup_{x \in B(0,1), y \in B\left(\frac{x}{2}, \frac{|x|}{2}\right)} \frac{f(y)}{\omega(f(x))}.$$

We say that  $f$  is Hölder monotone if  $f$  is  $\omega_s$ -monotone for some  $0 < s \leq 1$ .

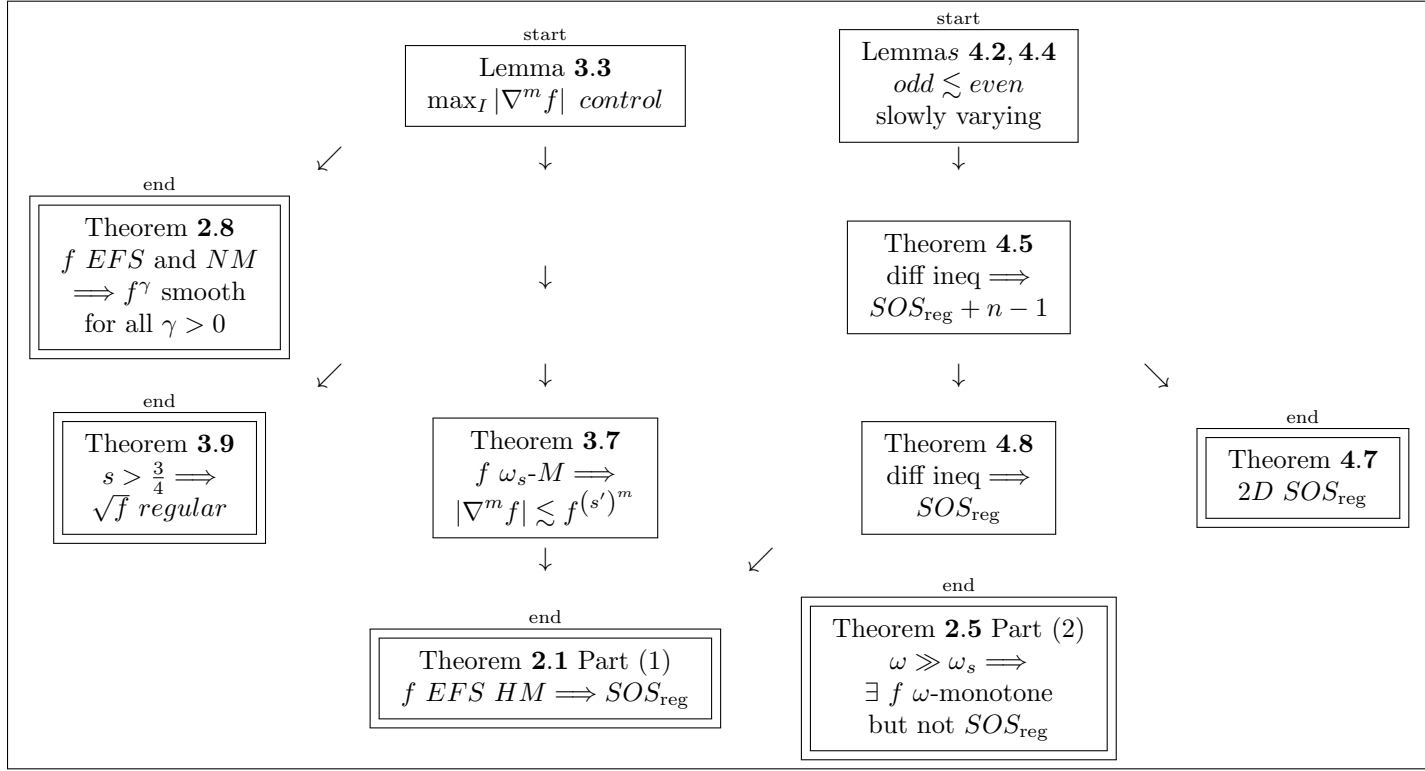
See Remark 3.6 below for a discussion of the condition  $y \in B\left(\frac{x}{2}, \frac{|x|}{2}\right)$  in (1.1).

We begin by stating our main results on sums of squares in Section 2. Section 3 is devoted to connections between vanishing to infinite order, derivative estimates, the notion of  $\omega$ -monotone for a modulus of continuity  $\omega$ , and smoothness of positive roots. Section 4 then uses ideas of Fefferman-Phong [FePh], Tataru [Tat] and Bony [Bon] to establish conditions under which smooth nonnegative functions can be written as a sum of squares of regular functions. In the final Section 5 of the paper, we construct examples that demonstrate sharpness of our sums of regular squares results.

For the reader's convenience we include a schematic diagram of connections between some of the lemmas and theorems in this paper. Results in a double box are logical ends. Theorem 2.1 is the main result that will be used in the subsequent papers [KoSa2] and [KoSa3].

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<sup>3</sup>We define  $\omega_1(t)$  to be dual to  $\omega_0(t)$  in the sense that  $\omega_1(t)\omega_0(t) = t$  (note that  $\omega_{\frac{1}{p}}(t)\omega_{\frac{1}{p'}}(t) = 1$ ), and  $\omega_1$  plays a role in Theorem 2.8 below.



## 2. STATEMENTS OF MAIN THEOREMS ON SUMS OF SQUARES AND EXTRACTING ROOTS

Here is our adaptation of the Fefferman-Phong algorithm, following Tataru [Tat] and Bony [Bon], to sums of squares of regular functions.

**Theorem 2.1.** *Suppose  $0 < \delta < 1$  and that  $f$  is a nonnegative  $C^{4,2\delta}$  function on  $\mathbb{R}^n$ . If*

$$(2.1) \quad |\nabla^4 f(x)| \leq C f(x)^{\frac{\delta}{2+\delta}} \quad \text{and} \quad |\nabla^2 f(x)| \leq C f(x)^{\frac{2\delta(1+\delta)}{2+\delta}},$$

*then  $f$  can be decomposed as a finite sum of squares of functions  $g_\ell \in C^{2,\delta}(\mathbb{R}^n)$ ,*

$$f(x) = \sum_{\ell=1}^N g_\ell(x)^2, \quad x \in \mathbb{R}^n.$$

*In the case that  $f$  doesn't vanish, except possibly at the origin, both of the above differential inequalities in (2.1) hold provided one of the following three conditions hold:*

- (1)  *$f$  is flat, smooth and  $\omega_s$ -monotone for some  $s < 1$  satisfying*

$$s > \sqrt[4]{\delta} \sqrt[4]{\frac{2}{4+2\delta}}, \quad \text{and} \quad s > \sqrt{\delta} \sqrt{\frac{2+2\delta}{2+\delta}}.$$

- (2)  *$f$  is strongly finite type, i.e.  $f(x) \geq |x|^N$  for some  $N \in \mathbb{N}$ , and vanishes to order at least four.*

- (3)  *$f$  is bounded below by a positive constant.*

More detailed information on the size and smoothness of the functions  $g_\ell(x)$  is given in Theorem 4.8 below.

**Definition 2.2.** *A nonnegative function  $f : B(0, a) \rightarrow \mathbb{R}$  is flat, or vanishes to infinite order at the origin in  $\mathbb{R}^n$ , if*

$$\lim_{x \rightarrow 0} |x|^{-N} f(x) = 0, \quad \text{for all } N \in \mathbb{N}.$$

**Definition 2.3.** *A function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  is regular if  $g \in \bigcup_{\delta > 0} C^{2,\delta}(\mathbb{R}^n)$ , i.e.  $g$  is  $C^{2,\delta}$  for some  $0 < \delta < 1$ .*

**Definition 2.4.** A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is elliptical if  $f(x) > 0$  for  $x \neq 0$ , and more generally, an  $N \times N$  matrix-valued function  $F : \mathbb{R}^n \rightarrow \mathbb{R}^{N \times N}$  is elliptical if  $F(x)$  is positive definite for  $x \neq 0$ .

Here is a strengthening of the counterexample in [BoBrCoPe, Theorem 1.2 (d)].

**Theorem 2.5.** Let  $n \geq 5$ .

- (1) If  $\beta > s > 0$ , there is an elliptical, flat, smooth  $\omega_s$ -monotone function  $f$  that cannot be written as a finite sum of squares of  $C^{2,\beta}(\mathbb{R}^n)$ .
- (2) Suppose  $\omega$  is a modulus of continuity such that  $\omega_s \ll \omega$  for all  $0 < s < 1$ . Then there is an elliptical, flat, smooth  $\omega$ -monotone function  $f$  that cannot be written as a finite sum of squares of regular functions. In particular we can take  $\omega = \omega_0$ .

The following corollary highlights the sharpness of the above results within the scale of  $\omega$ -monotone conditions.

**Corollary 2.6.** Suppose that  $f : \mathbb{R}^n \rightarrow [0, \infty)$  is elliptical, flat and smooth.

- (1) Then  $f$  can be written as a finite sum of squares of regular functions if  $f$  is Hölder monotone.
- (2) Conversely, for any modulus of continuity  $\omega$  satisfying  $\omega \gg \omega_s$  for all  $0 < s < 1$ , there is an  $\omega$ -monotone function  $f$  that cannot be written as a finite sum of squares of regular functions.

**Remark 2.7.** The answer to Problem 2 is affirmative in all dimensions  $n \geq 1$  if  $f$  is Hölder monotone. However, in dimension  $n = 1$ , the answer is affirmative without any additional  $\omega$ -monotone assumptions at all [Bon], while in dimension  $n \geq 5$ , the assumption of Hölder monotone is essentially sharp. We do not know if significantly weaker monotonicity assumptions will imply an affirmative answer to Problem 2 in the remaining dimensions  $n = 2, 3, 4$ .

The following theorem on extracting smooth roots motivates our definition of nearly monotone, and is what initially led us to consider  $\omega_s$ -monotone functions, and which then ultimately played a key role in the above decompositions into sums of regular functions.

**Theorem 2.8.** Let  $n \geq 1$ . Suppose that  $f : B(0, a) \rightarrow [0, \infty)$  is an elliptical flat smooth function on  $B(0, a) \subset \mathbb{R}^n$ . Then the first three of the following four conditions are equivalent. Moreover, the fourth condition, which holds in particular if  $f$  is  $\omega_1$  monotone, implies the first three conditions, but not conversely. Finally, for any  $0 < s < 1$ , there is an  $\omega_s$ -monotone function  $f$  such that  $f^{\frac{1}{s}-1}$  is not smooth.

- (1) There is  $\delta > 0$  such that  $f(x)^\gamma$  is smooth on  $B(0, a)$  for all  $0 < \gamma < \delta$ .
- (2) For every  $m \geq 1$  and  $0 < s < 1$ , there is a positive constant  $\Gamma_{n,m,s}$  such that

$$|\nabla^m f(x)| \leq \Gamma_{n,m,s} f(x)^s, \quad \text{for } x \in B(0, a).$$

- (3) The functions  $f(x)^\gamma$  are flat smooth functions on  $B(0, a)$  for all  $\gamma > 0$ .
- (4) The function  $f$  is nearly monotone.

**Terminology:** If for  $0 < s < 1$ , we set  $\mathcal{M}_s$  to be the collection of all  $\omega_s$ -monotone functions, then the **intersection** of all of these collections

$$\mathcal{NM} \equiv \bigcap_{0 < s < 1} \mathcal{M}_s,$$

which is the set of *nearly monotone* functions, is closely related to the set of *elliptical flat smooth* functions all of whose positive powers are smooth. On the other hand, the **union** of all of these collections

$$\mathcal{HM} \equiv \bigcup_{0 < s < 1} \mathcal{M}_s,$$

which is the set of *Hölder monotone* functions, is closely related to the set of *elliptical flat smooth* functions that can be written as a finite sum of squares of regular functions.

## 3. SMOOTH NONNEGATIVE FUNCTIONS

Here we discuss some connections between vanishing to infinite order, derivative estimates, and the notion of  $\omega$ -monotone for a modulus of continuity  $\omega$ , and finish with the proof of Theorem 2.8 on smoothness of positive roots. We use the following notation,

$$\begin{aligned}
 (3.1) \quad \nabla^m &= \nabla \otimes \nabla \otimes \dots \otimes \nabla = [\partial_{i_1} \partial_{i_2} \dots \partial_{i_m}]_{1 \leq i_1, i_2, \dots, i_m \leq n}, \\
 (\nabla h)^m &= \nabla h \otimes \nabla h \otimes \dots \otimes \nabla h = [(\partial_{i_1} h) (\partial_{i_2} h) \dots (\partial_{i_m} h)]_{1 \leq i_1, i_2, \dots, i_m \leq n}, \\
 (x \cdot \nabla)^m &= (x_1 \partial_1 + \dots + x_n \partial_n)^m = \sum_{i=(i_1, i_2, \dots, i_m) \in \{1, 2, \dots, n\}^m \equiv \Gamma^m} x_{i_1} x_{i_2} \dots x_{i_m} \partial_{i_1} \partial_{i_2} \dots \partial_{i_m} = \sum_{i \in \Gamma^m} x^i \partial_i.
 \end{aligned}$$

## 3.1. Infinite order vanishing.

**Lemma 3.1.** *Suppose that  $f : B(0, a) \rightarrow (-\infty, \infty)$  is a flat smooth function. Then  $D^\mu f$  is a flat smooth function for all multiindices  $\mu \in \mathbb{Z}_+^n$ .*

*Proof.* Clearly  $D^\mu f$  is smooth for all multiindices  $\mu \in \mathbb{Z}_+^n$ . We first use induction on  $m$  to establish that

$$(3.2) \quad D^\mu f(0) = 0 \quad \text{for all } |\mu| \leq m,$$

holds for all  $m \in \mathbb{N}$ . The case  $m = 0$  is a tautology, so suppose  $M \in \mathbb{N}$  and that (3.2) holds for all  $m < M$ . Then Taylor's formula of order  $M$  yields

$$f(x) = \sum_{|\mu|=M} \binom{|\mu|}{\mu} D^\mu f(0) x^\mu + O(|x|^{M+1}),$$

and thus

$$\left| \sum_{|\mu|=M} \binom{|\mu|}{\mu} D^\mu f(0) x^\mu \right| \leq C |x|^{M+1},$$

since  $f$  vanishes to infinite order. It follows that the polynomial  $\sum_{|\mu|=M} \binom{|\mu|}{\mu} D^\mu f(0) x^\mu$  vanishes identically, and so  $D^\mu f(0) = 0$  for all  $|\mu| = M$ , which gives (3.2) for  $m = M$ , and completes the inductive proof.

Now apply Taylor's formula of order  $k-1$  to the smooth function  $t \rightarrow D^\mu f(tx)$  to obtain

$$D^\mu f(x) = \sum_{\ell=0}^{k-1} \frac{1}{\ell!} (x \cdot \nabla)^\ell D^\mu f(0) + \frac{(x \cdot \nabla)^k}{k!} D^\mu f(\theta_x x) = \frac{(x \cdot \nabla)^k}{k!} D^\mu f(\theta_x x),$$

where  $0 < \theta_x < 1$ , and thus the smoothness of  $f$  implies

$$|D^\mu f(x)| \leq C_{k,\mu} |x|^k,$$

which shows that  $D^\mu f$  is flat for all multiindices  $\mu \in \mathbb{Z}_+^n$ . ■

**Remark 3.2.** *If  $f$  is a flat function that is smooth only on  $(-a, a) \setminus \{0\}$ , then its derivative need not be bounded in any open interval  $(0, \varepsilon)$  for  $\varepsilon > 0$ , e.g.*

$$\begin{aligned}
 f(x) &= e^{-\frac{1}{x}} \left( \sin^2 e^{\frac{1}{x}} + 1 \right); \\
 f'(x) &= \frac{e^{-\frac{1}{x}}}{x^2} \left( \sin^2 e^{\frac{1}{x}} + 1 \right) - e^{-\frac{1}{x}} \left( 2 \sin e^{\frac{1}{x}} \cos e^{\frac{1}{x}} \right) \frac{e^{\frac{1}{x}}}{x^2} \\
 &= \frac{e^{-\frac{1}{x}}}{x^2} \left( \sin^2 e^{\frac{1}{x}} + 1 \right) - \frac{1}{x^2} \left( \sin 2e^{\frac{1}{x}} \right),
 \end{aligned}$$

where  $\lim_{n \rightarrow \infty} f'(x_n) = -\infty$  if  $x_n \searrow 0$  and  $\sin 2e^{\frac{1}{x_n}} = 1$  for all  $n \geq 1$ .

**3.2. Derivative estimates.** The proof of our first main theorem will use a generalization of Lemma 5.13 from [GuSa], which is the case  $n = m = 1$  of the following lemma. We denote the diameter of a ball  $B$  by  $\ell(B)$ .

**Lemma 3.3.** *For each triple of integers  $k, m, n \in \mathbb{N}$  with  $k \geq m \geq 1$  and  $n \geq 1$ , there is a constant  $C_{k,m,n} > 0$  such that for any ball  $B$  in  $\mathbb{R}^n$  and  $f \in C^{k-1,1}(B)$ , we have*

$$\begin{aligned} \max_{z \in B} |\nabla^m f(z)| &\leq C_{k,m,n} \frac{1}{\ell(B)^m} \max_{t_1, t_2 \in B} \left| f(t_1) - f(t_2) - (t_1 - t_2) \cdot \nabla f(t_2) - \dots - \frac{[(t_1 - t_2) \cdot \nabla]^{m-1}}{(m-1)!} f(t_2) \right| \\ &+ C_{k,m,n} \left( \max_{t_1, t_2 \in B} \left| f(t_1) - f(t_2) - (t_1 - t_2) \cdot \nabla f(t_2) - \dots - \frac{[(t_1 - t_2) \cdot \nabla]^{m-1}}{(m-1)!} f(t_2) \right| \right)^{1 - \frac{m}{k}} \\ &\quad \times \left( \max_{t \in B} |\nabla^k f(t)| \right)^{\frac{m}{k}}. \end{aligned}$$

*Proof.* Let  $k \geq m \geq 1$  and  $n \geq 1$ , and fix  $z \in B$ . Taylor's formula gives

$$f(t) = f(z) + (t - z) \cdot \nabla f(z) + \dots + \frac{[(t - z) \cdot \nabla]^m}{m!} f(z) + \dots + \frac{[(t - z) \cdot \nabla]^{k-1}}{(k-1)!} f(z) + r(t).$$

Define

$$\begin{aligned} P(t) &= \frac{[(t - z) \cdot \nabla]^m}{m!} f(z) + \dots + \frac{[(t - z) \cdot \nabla]^{k-1}}{(k-1)!} f(z) \\ &= f(t) - f(z) - (t - z) \cdot \nabla f(z) - \dots - \frac{[(t - z) \cdot \nabla]^{m-1}}{(m-1)!} f(z) - r(t). \end{aligned}$$

Let  $D$  be a ball such that  $z \in D \subset B$  and  $\ell(D) = \min\{\ell(B), \delta\}$  where  $\ell(D)$  denotes the radius of the ball  $D$  and where

$$\delta = \left( \frac{\max_{t_1, t_2 \in B} \left| f(t_1) - f(t_2) - (t_1 - t_2) \cdot \nabla f(t_2) - \dots - \frac{[(t_1 - t_2) \cdot \nabla]^{m-1}}{(m-1)!} f(t_2) \right|}{\max_{t \in B} |\nabla^k f(t)|} \right)^{\frac{1}{k}}.$$

Since  $P$  is a polynomial of degree  $k-1$  there is a constant  $C_{k,m} > 0$ , independent of  $D$  and  $B$ , such that

$$\max_{t \in D} |\nabla^m P(t)| \leq C_{k,m} \frac{1}{\ell(D)^m} \max_{t \in D} |P(t)|.$$

Then using a standard estimate for the remainder in Taylor's formula we get

$$\begin{aligned} |\nabla^m f(z)| &= |\nabla^m P(z)| \leq \max_{t \in D} |\nabla^m P(t)| \leq C_{k,m} \frac{1}{\ell(D)^m} \max_{t \in D} |P(t)| \\ (3.3) \quad &\leq \frac{1}{\ell(D)^m} \max_{t_1, t_2 \in D} \left| f(t_1) - f(t_2) - (t_1 - t_2) \cdot \nabla f(t_2) - \dots - \frac{[(t_1 - t_2) \cdot \nabla]^{m-1}}{(m-1)!} f(t_2) \right| \\ &\quad + \frac{1}{\ell(D)^m} \max_{t \in D} |r(t)| \\ &\leq \frac{1}{\ell(D)^m} \max_{t_1, t_2 \in D} \left| f(t_1) - f(t_2) - \dots - \frac{[(t_1 - t_2) \cdot \nabla]^{m-1}}{(m-1)!} f(t_2) \right| + \frac{1}{\ell(D)^m} \max_{t \in D} |\nabla^k f(t)| \ell(D)^k. \end{aligned}$$

Moreover,  $\ell(D)^{k-m} \leq \delta^{k-m}$  gives

$$\begin{aligned} \frac{1}{\ell(D)^m} \max_{t \in D} |\nabla^k f(t)| \ell(D)^k &= \max_{t \in D} |\nabla^k f(t)| \ell(D)^{k-m} \leq \max_{t \in D} |\nabla^k f(t)| \delta^{k-m} \\ &= \max_{t \in D} |\nabla^k f(t)| \left( \frac{\max_{t_1, t_2 \in B} \left| f(t_1) - f(t_2) - (t_1 - t_2) \cdot \nabla f(t_2) - \dots - \frac{[(t_1 - t_2) \cdot \nabla]^{m-1}}{(m-1)!} f(t_2) \right|}{\max_{t \in B} |\nabla^k f(t)|} \right)^{\frac{k-m}{k}} \\ &\leq \left( \max_{t \in B} |\nabla^k f(t)| \right)^{\frac{m}{k}} \left( \max_{t_1, t_2 \in B} \left| f(t_1) - f(t_2) - (t_1 - t_2) \cdot \nabla f(t_2) - \dots - \frac{[(t_1 - t_2) \cdot \nabla]^{m-1}}{(m-1)!} f(t_2) \right| \right)^{1 - \frac{m}{k}}. \end{aligned}$$

Now in the case that  $\delta \leq \ell(B)$ , we are done since then  $\ell(D) = \delta$  and

$$\begin{aligned} \frac{1}{\ell(D)^m} \max_{t_1, t_2 \in D} \left| f(t_1) - f(t_2) - (t_1 - t_2) \cdot \nabla f(t_2) - \dots - \frac{[(t_1 - t_2) \cdot \nabla]^{m-1}}{(m-1)!} f(t_2) \right| \\ = \left( \frac{\max_{t_1, t_2 \in B} \left| f(t_1) - f(t_2) - (t_1 - t_2) \cdot \nabla f(t_2) - \dots - \frac{[(t_1 - t_2) \cdot \nabla]^{m-1}}{(m-1)!} f(t_2) \right|}{\max_{t \in B} |\nabla^k f(t)|} \right)^{-\frac{m}{k}} \\ \times \max_{t_1, t_2 \in D} \left| f(t_1) - f(t_2) - (t_1 - t_2) \cdot \nabla f(t_2) - \dots - \frac{[(t_1 - t_2) \cdot \nabla]^{m-1}}{(m-1)!} f(t_2) \right| \\ \leq \left( \max_{t_1, t_2 \in B} \left| f(t_1) - f(t_2) - (t_1 - t_2) \cdot \nabla f(t_2) - \dots - \frac{[(t_1 - t_2) \cdot \nabla]^{m-1}}{(m-1)!} f(t_2) \right| \right)^{1 - \frac{m}{k}} \left( \max_{t \in B} |\nabla^k f(t)| \right)^{\frac{m}{k}}. \end{aligned}$$

On the other hand, in the case  $\delta > \ell(B)$ , we have  $D = B$  and the inequality

$$\max_{t \in B} |\nabla^k f(t)| \ell(B)^k < \max_{t_1, t_2 \in B} \left| f(t_1) - f(t_2) - (t_1 - t_2) \cdot \nabla f(t_2) - \dots - \frac{[(t_1 - t_2) \cdot \nabla]^{m-1}}{(m-1)!} f(t_2) \right|.$$

Thus from (3.3) with  $D = B$  we conclude that

$$\begin{aligned} |\nabla^m f(z)| &\leq \frac{1}{\ell(B)^m} \max_{t_1, t_2 \in B} \left| f(t_1) - f(t_2) - (t_1 - t_2) \cdot \nabla f(t_2) - \dots - \frac{[(t_1 - t_2) \cdot \nabla]^{m-1}}{(m-1)!} f(t_2) \right| \\ &\quad + \frac{1}{\ell(B)^m} \max_{t \in B} |\nabla^k f(t)| \ell(B)^k \\ &\leq 2 \frac{1}{\ell(B)^m} \max_{t_1, t_2 \in B} \left| f(t_1) - f(t_2) - (t_1 - t_2) \cdot \nabla f(t_2) - \dots - \frac{[(t_1 - t_2) \cdot \nabla]^{m-1}}{(m-1)!} f(t_2) \right|, \end{aligned}$$

which completes the proof.  $\blacksquare$

**3.3. Nearly monotone functions.** In the introduction, we have defined the notion of an  $\omega_s$ -monotone function on intervals of the real line. The extension of this definition to higher dimensions is for the most part straightforward, with the only wrinkle being the region over which the supremum is to be taken. Here is the generalization of  $\omega$ -monotone to higher dimensions.

**Definition 3.4.** Let  $B(0, a)$  be the ball of radius  $a$  centered at the origin  $\mathbb{R}^n$ . Define a nonnegative function  $f : B(0, a) \rightarrow [0, \infty)$  to be  $\omega$ -monotone in  $B(0, a)$  if there is a positive  $C$  such that

$$f(t) \leq C\omega(f(x)), \quad \text{for all } t \in B\left(\frac{x}{2}, \frac{|x|}{2}\right), \text{ and } x \in B(0, a).$$

**Remark 3.5.** If  $f(x) = g(|x|)$  is a radial function, then  $f$  is  $\omega$ -monotone in  $B(0, a)$  if and only if  $g$  is  $\omega$ -monotone in  $(0, a)$ .

**Remark 3.6.** We make some comments on the role played by the ball  $B_x = B\left(\frac{x}{2}, \frac{|x|}{2}\right)$  in the higher dimensional definition of  $\omega$ -monotone. First we point out that the family of balls  $\{B_x\}_{x \in \mathbb{R}^n}$  is dilation and

rotation invariant in the sense that  $B_{\delta\Theta x} = \delta\Theta B_x$  for all rotations  $\Theta$  and dilations  $\delta$ . We now claim that for the purposes of this paper, the family of balls can be replaced by any dilation and rotation invariant family of open convex sets  $\{E_x\}_{x \in \mathbb{R}^n}$  satisfying (i)  $0, x \in \partial E_x$ , (ii) the eccentricity of  $E_x$  is uniformly controlled in  $x$ , and (iii) the set  $E_x$  is starlike with respect to each of its boundary points. Indeed, such sets  $E_x$  are not zero sets for polynomials, and so the rescaling argument used in Lemma 3.3 remains in force. The starlike property is used in the proof of Lemma 3.3 to show that given any point  $z \in E_x$  and any number  $0 < \varepsilon \leq 1$ , there is a set  $D$  that is a translate, dilate and rotation of  $E_x$ , and that satisfies  $z \in D \subset E_x$  and  $\text{diam } D = \varepsilon \text{diam } E_x$ . Finally, the important property in Theorem 4.8 below that (4.14) implies (4.5) also remains in force. However, it appears that the definitions of  $\omega$ -monotonicity using these more general families of convex sets are essentially equivalent when restricted to elliptical flat smooth functions, and so nothing significant appears to be gained by their use.

As an example of such a family in the plane, we mention the case when  $E_x$  is the tilted square having opposite corners at 0 and  $x$ .

A nearly monotone function on the line is quite close to being monotone, while a Hölder monotone function can be far removed from being monotone, but not as far removed from monotone as is an  $\omega_0$ -monotone function with logarithmic modulus of continuity  $\omega_0(t) = \frac{1}{1+\ln \frac{1}{t}}$ . Each of these notions in higher dimensions will play a role in this paper.

We need two more results in preparation for the proof of our near characterization of elliptical flat smooth functions having smooth positive powers.

First, we recall a more general version of an elementary composition formula from [MaSaUrVu]. Let  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  and  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  be two smooth functions. With  $\psi^{(k)}$  understood to be  $\psi^{(k)} \circ h$  on the right hand side, we have

$$\begin{aligned} [\partial_i (\psi \circ h)]_{1 \leq i \leq n} &= \psi' [\partial_i h]_{1 \leq i \leq n}, \\ [\partial_j \partial_i (\psi \circ h)]_{1 \leq i, j \leq n} &= \psi'' [(\partial_j h) (\partial_i h)]_{1 \leq i, j \leq n} + \psi' [\partial_j \partial_i h]_{1 \leq i, j \leq n}, \\ [\partial_k \partial_j \partial_i (\psi \circ h)]_{1 \leq i, j, k \leq n} &= \psi''' [(\partial_k h) (\partial_j h) (\partial_i h)]_{1 \leq i, j, k \leq n} + \psi'' [(\partial_k \partial_j h) (\partial_i h) + (\partial_j h) (\partial_k \partial_i h)]_{1 \leq i, j \leq n} \\ &\quad + \psi'' [(\partial_k h) (\partial_j \partial_i h)]_{1 \leq i, j, k \leq n} + \psi' [\partial_k \partial_j \partial_i h]_{1 \leq i, j, k \leq n}. \end{aligned}$$

We can write this more compactly using the notation of (3.1) and symmetrizing products, to obtain

$$\begin{aligned} \nabla (\psi \circ h) &= \psi' \nabla h, \\ \nabla^2 (\psi \circ h) &= \psi'' (\nabla h)^2 + \psi' \nabla^2 h, \\ \nabla^3 (\psi \circ h) &= \psi''' (\nabla h)^3 + 3\psi'' (\nabla^2 h) \otimes (\nabla h) + \psi' (\nabla^3 h). \end{aligned}$$

In general we have the formula

$$(3.4) \quad \nabla^M (\psi \circ h) = \sum_{m=1}^M \left( \psi^{(m)} \circ h \right) \left( \sum_{\substack{\alpha = (\alpha_1, \dots, \alpha_M) \in \mathbb{Z}_+^M \\ \alpha_1 + \alpha_2 + \dots + \alpha_M = m \\ \alpha_1 + 2\alpha_2 + \dots + M\alpha_M = M}} \left[ \begin{matrix} M \\ \alpha \end{matrix} \right] (\nabla h)^{\alpha_1} \otimes (\nabla^2 h)^{\alpha_2} \otimes \dots \otimes (\nabla^M h)^{\alpha_M} \right),$$

where  $\left[ \begin{matrix} M \\ \alpha \end{matrix} \right]$  is defined for  $\alpha = (\alpha_1, \dots, \alpha_M) \in \mathbb{Z}_+^M$  satisfying  $\alpha_1 + 2\alpha_2 + \dots + M\alpha_M = M$ . We will not need to evaluate or even estimate  $\left[ \begin{matrix} M \\ \alpha \end{matrix} \right]$  for our purposes in this paper, but a recursion formula for these coefficients in a special case can be found in [MaSaUrVu].

Second, we give the explicit dependence on bounding an  $m^{\text{th}}$  derivative of an  $\omega_s$ -monotone function, a fact which will be used later in our sum of squares theorem.

**Theorem 3.7.** *Let  $m, n \in \mathbb{N}$  and  $0 < s' < s < 1$  be given. Fix a ball  $B(0, a) \subset \mathbb{R}^n$  with radius  $a > 0$ . Then there are positive constants  $\Gamma_{m, n, s, s', a}$  such that*

$$(3.5) \quad |\nabla^m f(x)| \leq \Gamma_{m, n, s, s', a} f(x)^{(s')^m}, \quad \text{for all } x \in B(0, a),$$

and for all  $f$  that are elliptical, flat, smooth,  $\omega_s$ -monotone, and satisfy  $|f(x)| \leq 1$  on  $B(0, a)$ .

*Proof.* First using Lemma 3.3 and the fact that  $f$  is  $\omega_s$ -monotone, we have

$$\begin{aligned}
\max_{t \in B(0, |x|)} |\nabla f(t)| &\leq C_{k,1,n} \left\{ \frac{1}{|x|} \max_{t_1, t_2 \in B(\frac{x}{2}, \frac{|x|}{2})} |f(t_1) - f(t_2)| \right. \\
&\quad \left. + \left( \max_{t_1, t_2 \in B(\frac{x}{2}, \frac{|x|}{2})} |f(t_1) - f(t_2)| \right)^{1-\frac{1}{k}} \left( \max_{t \in B(\frac{x}{2}, \frac{|x|}{2})} |\nabla^k f(t)| \right)^{\frac{1}{k}} \right\} \\
&\leq C_{k,1,n} \left\{ \frac{1}{|x|} \Gamma_s f(x)^s + (\Gamma_s f(x)^s)^{1-\frac{1}{k}} \left( \max_{t \in B(\frac{x}{2}, \frac{|x|}{2})} |\nabla^k f(t)| \right)^{\frac{1}{k}} \right\} \\
&\leq C_{k,1,n} \left\{ \frac{1}{|x|} \Gamma_s f(x)^s + M_k^{\frac{1}{k}} (\Gamma_s f(x)^s)^{1-\frac{1}{k}} \right\},
\end{aligned}$$

where  $\max_{t \in B(0, |x|)} |\nabla^k f(t)| \leq M_k$  for  $x \in \overline{B}(0, a)$  follows since  $f$  is smooth. Moreover, since  $f$  is flat, we have  $f(x) \leq A_k |x|^k$ , and thus  $f(x)^{\frac{1}{k}} \leq A_k^{\frac{1}{k}} |x|$ , and so for  $x \in B(0, a)$ ,  $x \neq 0$ , and  $0 < f(x) \leq 1$ , we have

$$\begin{aligned}
|\nabla f(x)| &\leq C_{k,1,n} \left\{ \Gamma_\varepsilon f(x)^{s-\frac{1}{k}} \frac{f(x)^{\frac{1}{k}}}{|x|} + M_k^{\frac{1}{k}} \Gamma_s^{1-\frac{1}{k}} f(x)^{(s)(1-\frac{1}{k})} \right\} \\
&\leq B_{k,n,s} f(x)^{s-\frac{1}{k}} + D_{k,n,\varepsilon} f(x)^{s(1-\frac{1}{k})},
\end{aligned}$$

where  $B_{k,n,s} = C_{k,1,n} \Gamma_\varepsilon A_k^{\frac{1}{k}}$  and  $D_{k,n,\varepsilon} = C_{k,1,n} \Gamma_s^{1-\frac{1}{k}} M_k^{\frac{1}{k}}$ . Thus by choosing  $k$  sufficiently large, we see that for every  $0 < s' < s < 1$ , there is a positive constant  $\Gamma_{n,s,s',a}$  such that

$$(3.6) \quad |\nabla f(x)| \leq \Gamma_{n,s,s',a} f(x)^{s'}, \quad x \neq 0, \quad 0 < s' < s < 1,$$

which proves the case  $m = 1$  of (3.5). We now prove the general case by induction on  $m$ . Fix  $m \geq 1$  and suppose that for all  $1 \leq \ell \leq m$  and all  $0 < s' < s < 1$  there holds

$$(3.7) \quad |\nabla^\ell f(x)| \leq \Gamma_{\ell,n,s,s'} f(x)^{(s')^\ell}, \quad \text{for } x \in B(0, a).$$

Since  $f$  is a flat smooth function we have from Lemma 3.3 that

$$\begin{aligned}
\max_{t \in B} |\nabla^{m+1} f(t)| &\leq C_{k,m,n} \frac{1}{|x|^{m+1}} \max_{t_1, t_2 \in B} \left| f(t_1) - f(t_2) - (t_1 - t_2) \cdot \nabla f(t_2) - \dots - \frac{[(t_1 - t_2) \cdot \nabla]^m}{m!} f(t_2) \right| \\
&\quad + C_{k,m,n} \left( \max_{t_1, t_2 \in B} \left| f(t_1) - f(t_2) - (t_1 - t_2) \cdot \nabla f(t_2) - \dots - \frac{[(t_1 - t_2) \cdot \nabla]^m}{m!} f(t_2) \right| \right)^{1-\frac{m+1}{k}} \\
&\quad \times \left( \max_{t \in B} |\nabla^k f(t)| \right)^{\frac{m+1}{k}}.
\end{aligned}$$

Now fix  $s' < s$  and let  $\varepsilon = (s - s')/2$ , so that  $s' + \varepsilon < s$ . Then using (3.7) with  $s' + \varepsilon$  in place of  $s'$  and the fact that  $f$  is  $\omega_s$ -monotone, we conclude that for  $B = B(0, |x|)$  we have

$$\begin{aligned}
&\left| f(t_1) - f(t_2) - (t_1 - t_2) \cdot \nabla f(t_2) - \dots - \frac{[(t_1 - t_2) \cdot \nabla]^m}{m!} f(t_2) \right| \\
&\leq |f(t_1) - f(t_2)| + |\nabla f(t_2) \cdot (t_1 - t_2)| + \dots + \left| \frac{[(t_1 - t_2) \cdot \nabla]^m}{m!} f(t_2) \right| \\
&\leq C_{n,s} f(x)^s + \sum_{\ell=1}^m \Gamma_{\ell,n,s,s'} f(t_2)^{(s'+\varepsilon)^\ell} |x|^\ell \\
&\leq C_{n,s} f(x)^s + \sum_{\ell=1}^m \Gamma_{\ell,n,s,s'} (f(x)^s)^{(s'+\varepsilon)^\ell} |x|^\ell \leq \sum_{\ell=0}^m \Gamma_{\ell,n,s,s'} f(x)^{s(s'+\varepsilon)^\ell} |x|^\ell.
\end{aligned}$$

Thus we have

$$\begin{aligned} & \frac{1}{|x|^{m+1}} \max_{t_1, t_2 \in B} \left| f(t_1) - f(t_2) - (t_1 - t_2) \cdot \nabla f(t_2) - \dots - \frac{[(t_1 - t_2) \cdot \nabla]^m}{m!} f(t_2) \right| \\ & \leq \sum_{\ell=0}^m \frac{\Gamma_{\ell, n, s, s'} f(x)^{s(s'+\varepsilon)^\ell}}{|x|^{m+1-\ell}} = \sum_{\ell=1}^{m+1} \Gamma_{\ell, n, s, s'} f(x)^{(s'-\varepsilon)^\ell} \frac{f(x)^{s-(s'+\varepsilon)}}{|x|^{m+1-\ell}} \leq \sum_{\ell=1}^{m+1} \Gamma_{\ell, n, s, s', a} f(x)^{(s'+\varepsilon)^\ell}, \end{aligned}$$

where in the last inequality we used the fact that  $f$  is flat, and thus  $f(x) \leq C_M |x|^M$  for all  $M > 0$ . Therefore we obtain

$$\begin{aligned} |\nabla^{m+1} f(x)| & \leq \sum_{\ell=1}^{m+1} \Gamma_{\ell, n, s, s', a} f(x)^{(s'+\varepsilon)^\ell} + \left( \sum_{\ell=1}^{m+1} \Gamma_{\ell, n, s, s', a} f(x)^{(s'+\varepsilon)^\ell} \right)^{1 - \frac{m+1}{k}} M_k^{\frac{m+1}{k}} \\ & \leq \Gamma_{m, n, s, s', a} f(x)^{(s')^{m+1}}, \end{aligned}$$

upon taking  $k$  sufficiently large so that  $(s' + \varepsilon)^{m+1} (1 - \frac{m+1}{k}) \geq (s')^{m+1}$ . ■

We are now ready to proceed with the proof of Theorem 2.8.

*Proof of Theorem 2.8.* First note that **(3)**  $\implies$  **(1)** is trivial and **(4)**  $\implies$  **(2)** follows from Theorem 3.7.

**(1)**  $\implies$  **(2)**: Since  $f(x)^\beta$  is smooth and nonnegative for  $0 < \beta < \delta$ , we have the classical inequality of Malgrange, see e.g. [Gla, Lemme I],

$$|\nabla [f(x)^\beta]| \leq C \sqrt{f(x)^\beta},$$

which implies that

$$\begin{aligned} \beta f(x)^{\beta-1} |\nabla f(x)| & \leq C f(x)^{\frac{\beta}{2}}, \\ \text{hence } |\nabla f(x)|^2 & \leq C f(x)^{2-\beta}. \end{aligned}$$

Next we compute

$$\nabla^2 [f(x)^\beta] = \nabla (\beta f^{\beta-1} \nabla f(x)) = \beta(\beta-1) f(x)^{\beta-2} (\nabla f(x))^2 + \beta f(x)^{\beta-1} \nabla^2 f(x),$$

which implies that

$$\begin{aligned} \beta f(x)^{\beta-1} |\nabla^2 f(x)| & \leq C f(x)^{\beta-2} |\nabla f(x)|^2 + C |\nabla^2 [f(x)^\beta]| \\ \text{hence } |\nabla^2 f(x)| & \leq f(x)^{1-\beta} \left\{ C f(x)^{\beta-2} f(x)^{2-\beta} + C_\beta \right\} \\ & \leq C f(x)^{1-\beta} + C_\beta f(x)^{1-\beta} = C_\beta f(x)^{1-\beta}, \end{aligned}$$

where we have used the fact that  $f(x)^\beta$  is smooth, hence  $|\nabla^2 [f(x)^\beta]|$  is bounded on compact subsets of  $B(0, a)$ . We now prove by induction that

$$|\nabla^M f(x)| \leq \Gamma_{M, \gamma} f(x)^{1-M\gamma}, \quad \text{where } \gamma = \frac{\beta}{2}.$$

Define the nonnegative power functions  $s_\gamma : [0, \infty) \rightarrow [0, \infty)$  by  $s_\gamma(t) = t^\gamma$  for  $t \in [0, \infty)$ , and note that

$$(3.8) \quad s_\gamma^{(k)}(t) = \binom{\gamma}{k} t^{\gamma-k}, \quad \text{for } k \geq 0 \text{ and } t \in (0, \infty).$$

Indeed, with  $g(x) = f(x)^\gamma = s_\gamma \circ f(x)$  we have using the composition formula (3.4)

$$\begin{aligned} \nabla^M g(x) & = \nabla^M (s_\gamma \circ f)(x) \\ & = \sum_{m=1}^M (s_\gamma^{(m)} \circ f)(x) \left( \sum_{\substack{\alpha=(\alpha_1, \dots, \alpha_M) \in \mathbb{Z}_+^M \\ \alpha_1 + \alpha_2 + \dots + \alpha_M = m \\ \alpha_1 + 2\alpha_2 + \dots + M\alpha_M = M}} \begin{bmatrix} M \\ \alpha \end{bmatrix} (\nabla f(x))^{\alpha_1} \dots (\nabla^M f(x))^{\alpha_M} \right), \end{aligned}$$

and since  $\alpha_M > 0$  implies  $\alpha_M = m = 1$ , we obtain that

$$\begin{aligned} & \left[ \begin{matrix} M \\ \alpha \end{matrix} \right] (s_\gamma^{(1)} \circ f)(x) \nabla^M f(x) \\ &= \nabla^M g(x) - \sum_{m=2}^M (s_\gamma^{(m)} \circ f)(x) \left( \sum_{\substack{\alpha=(\alpha_1, \dots, \alpha_M) \in \mathbb{Z}_+^M \\ \alpha_1 + \alpha_2 + \dots + \alpha_M = m \\ \alpha_1 + 2\alpha_2 + \dots + M\alpha_M = M}} \left[ \begin{matrix} M \\ \alpha \end{matrix} \right] (\nabla f(x))^{\alpha_1} \dots (\nabla^{M-1} f(x))^{\alpha_{M-1}} \right), \end{aligned}$$

hence using the inductive assumption and the fact that  $g = f^\gamma$  is smooth,

$$\begin{aligned} & \left| (s_\gamma^{(1)} \circ f)(x) \nabla^M f(x) \right| \\ & \leq C + C_{\gamma, M} \sum_{m=2}^M \binom{\gamma}{m} f(x)^{\gamma-m} \left( \sum_{\substack{\alpha=(\alpha_1, \dots, \alpha_{M-1}) \in \mathbb{Z}_+^{M-1} \\ \alpha_1 + \alpha_2 + \dots + \alpha_{M-1} = m \\ \alpha_1 + 2\alpha_2 + \dots + (M-1)\alpha_{M-1} = M}} \left[ \begin{matrix} M \\ \alpha \end{matrix} \right] f(x)^{(1-\gamma)\alpha_1} \dots f(x)^{(1-(M-1)\gamma)\alpha_{M-1}} \right), \end{aligned}$$

where  $C$  above is a bound for  $|\nabla^M g(x)|$ , and finally that

$$|\nabla^M f(x)| \leq C f(x)^{1-\gamma} \left\{ C + C_{\gamma, M} \sum_{m=2}^M f(x)^{\gamma-m} f(x)^{m-M\gamma} \right\} \leq C'_{\gamma, M} f(x)^{1-\gamma M}.$$

**(2)  $\implies$  (3):** Again set  $g(x) = f(x)^\gamma = (s_\gamma \circ f)(x)$ , and as before we have

$$\begin{aligned} \nabla^M g(x) &= \nabla^M (s_\gamma \circ f)(x) \\ &= \sum_{m=1}^M (s_\gamma^{(m)} \circ f)(x) \left( \sum_{\substack{\alpha=(\alpha_1, \dots, \alpha_M) \in \mathbb{Z}_+^M \\ \alpha_1 + \alpha_2 + \dots + \alpha_M = m \\ \alpha_1 + 2\alpha_2 + \dots + M\alpha_M = M}} \left[ \begin{matrix} M \\ \alpha \end{matrix} \right] (\nabla f(x))^{\alpha_1} \dots (\nabla^M f(x))^{\alpha_M} \right), \end{aligned}$$

Now we use (3.8), i.e.

$$\left| (s_\gamma^{(m)} \circ f)(x) \right| = \left| s_\gamma^{(m)}(f(x)) \right| = \left| \binom{\gamma}{m} f(x)^{\gamma-m} \right|,$$

and condition **(2)** with  $s = 1 - \varepsilon$ , i.e.  $|\nabla^k f(x)| \leq \Gamma_{k, \varepsilon} f(x)^{1-\varepsilon}$ , to obtain

$$\begin{aligned} |\nabla^M g(x)| &\leq C \sum_{m=1}^M f(x)^{\gamma-m} \left( \sum_{\substack{\alpha=(\alpha_1, \dots, \alpha_M) \in \mathbb{Z}_+^M \\ \alpha_1 + \alpha_2 + \dots + \alpha_M = m \\ \alpha_1 + 2\alpha_2 + \dots + M\alpha_M = M}} \left[ \begin{matrix} M \\ \alpha \end{matrix} \right] (f(x)^{1-\varepsilon})^{\alpha_1} \dots (f(x)^{1-\varepsilon})^{\alpha_M} \right) \\ &\leq C \sum_{m=1}^M f(x)^{\gamma-m} (f(x)^{m(1-\varepsilon)}) = C \sum_{m=1}^M f(x)^{\gamma-m\varepsilon} \leq CM f(x)^{\gamma-M\varepsilon}. \end{aligned}$$

If we choose  $\varepsilon < \frac{\gamma}{M}$ , then we see that  $\nabla^M g(x)$  is a flat function for each  $M \geq 0$ , and it follows that  $g$  is a flat smooth function.

**(3)  $\nRightarrow$  (4):** Let  $g$  be any elliptical flat smooth function that fails to be nearly monotone, or even just fails the inequality  $g(t) \leq 4g(x)$  for some  $0 \leq t < x < a$ . Then if  $f(x) = e^{-\frac{1}{g(x)}}$ , the functions  $f(x)^\alpha$  are smooth for all  $\alpha > 0$ , but  $f$  is clearly not nearly monotone since in particular,  $f$  fails the inequality

$f(t) \leq C_s f(x)^s$ ,  $0 \leq t < x < a$  for every  $\frac{1}{2} \leq s < 1$ . Indeed, if this inequality holds for some  $s \geq \frac{1}{2}$ , then

$$\begin{aligned} f(t) &\leq C_s f(x)^s, \quad 0 \leq t < x < a, \\ \implies \ln \frac{1}{f(t)} &\geq \ln \frac{1}{C_s} + s \ln \frac{1}{f(x)}, \quad 0 \leq t < x < a, \\ \implies \frac{1}{g(t)} &\geq -\ln C_s + \frac{s}{g(x)}, \quad 0 \leq t < x < a, \\ \implies g(t) &\leq \frac{1}{\frac{s}{g(x)} - \ln C_s} = \frac{g(x)}{s - (\ln C_s)g(x)}, \quad 0 \leq t < x < a, \end{aligned}$$

which shows that for  $x$  small enough, namely  $g(x) < \frac{s}{2 \ln C_s}$ , we have  $g(t) \leq \frac{2}{s} g(x) \leq 4g(x)$ , contradicting our assumption on  $g$ .

Finally, we give a modification of Glaseser's example in [Gla] that shows that for any  $0 \leq s < 1$ , there is an  $\omega_s$ -monotone function  $f$  such that  $f^\alpha$  is not smooth if  $0 < \alpha \leq \frac{1}{s} - 1$ . Suppose  $\varphi$  is an elliptical flat smooth function on  $(-1, 1)$  that is decreasing on  $(-1, 0]$  and increasing on  $[0, 1)$ . Suppose further that  $\varphi$  is constant in a neighbourhood of  $\frac{1}{n}$  for each  $n \in \mathbb{N}$ , say in  $(\frac{1}{n} - \varepsilon_n, \frac{1}{n} + \varepsilon_n)$ . See [Gla, page 206] for a construction of such a function. Let  $0 < \gamma < 1$  and define

$$f_\gamma(x) \equiv \varphi(x)^{\frac{1}{\gamma}-1} \left( \sin^2 \frac{\pi}{x} + \varphi(x) \right), \quad \text{for } -1 < x < 1.$$

Then  $f_\gamma$  is a flat smooth function vanishing only at 0. Indeed, Theorem 2.8 shows in particular that  $\varphi(x)^{\frac{1}{\gamma}-1}$  is a smooth flat function for  $0 < \gamma < 1$ , and then the smoothness of  $f_\gamma$  at the origin follows easily from the inequalities

$$\left| \frac{d^n}{dx^n} \sin^2 \frac{\pi}{x} \right| \leq C_n |x|^{-2n}.$$

The assumption that  $\varphi$  is positive away from the origin shows that  $f_\gamma$  is as well. Following Glaeser's argument, we now show that  $g_\gamma(x) = (f_\gamma(x))^\gamma$  doesn't have a bounded second derivative in any neighbourhood of the origin. Indeed, if  $x = \frac{1}{n} + y$  where  $y \in (-\varepsilon_n, \varepsilon_n)$ , then

$$\begin{aligned} \sin^2 \frac{\pi}{x} &= \sin^2 \frac{\pi}{\frac{1}{n} + y} = \sin^2 \left( n\pi - n^2\pi \frac{y}{1 + ny} \right) \\ &= \sin^2 \left( n^2\pi \frac{y}{1 + ny} \right) = (n^2\pi y)^2 + o(y^2), \end{aligned}$$

and so

$$\begin{aligned} f_\gamma \left( \frac{1}{n} + y \right) &= \varphi \left( \frac{1}{n} + y \right)^{\frac{1}{\gamma}-1} \left( \sin^2 \frac{\pi}{\frac{1}{n} + y} + \varphi \left( \frac{1}{n} + y \right) \right) \\ &= \varphi \left( \frac{1}{n} \right)^{\frac{1}{\gamma}-1} \left( n^4\pi^2 y^2 + o(y^2) + \varphi \left( \frac{1}{n} \right) \right) \\ &= \varphi \left( \frac{1}{n} \right)^{\frac{1}{\gamma}} \left( \frac{n^4\pi^2 y^2}{\varphi \left( \frac{1}{n} \right)} + 1 + \frac{o(y^2)}{\varphi \left( \frac{1}{n} \right)} \right), \end{aligned}$$

implies that for  $y$  sufficiently small depending on  $n$ , we have

$$\begin{aligned} g_\gamma \left( \frac{1}{n} + y \right) &= \left( f_\gamma \left( \frac{1}{n} + y \right) \right)^\gamma = \varphi \left( \frac{1}{n} \right)^\gamma \left( 1 + \frac{n^4\pi^2 y^2}{\varphi \left( \frac{1}{n} \right)} + \frac{o(y^2)}{\varphi \left( \frac{1}{n} \right)} \right)^\gamma \\ &= \varphi \left( \frac{1}{n} \right)^\gamma \left( 1 + \gamma \frac{n^4\pi^2 y^2}{\varphi \left( \frac{1}{n} \right)} + \frac{o(y^2)}{\varphi \left( \frac{1}{n} \right)} \right) = \varphi \left( \frac{1}{n} \right)^\gamma + \gamma n^4\pi^2 y^2 + o(y^2), \end{aligned}$$

which in turn shows that  $g_\gamma'' \left( \frac{1}{n} \right) = 2\gamma n^4\pi^2$ .

On the other hand, if  $\frac{\pi}{t} = n\pi + \frac{\pi}{2}$  and  $\frac{\pi}{x} = n\pi$ , then  $t < x$ ,  $\sin^2 \frac{\pi}{t} = 1$ ,  $\sin^2 \frac{\pi}{x} = 0$  and

$$\begin{aligned} \frac{f_\gamma(t)}{f_\gamma(x)^{1-\varepsilon}} &= \frac{\varphi(t)^{\frac{1}{\gamma}} (\sin^2 \frac{\pi}{t} + \varphi(t))}{\varphi(x)^{\frac{1}{\gamma}(1-\varepsilon)} (\sin^2 \frac{\pi}{x} + \varphi(x))^{1-\varepsilon}} \\ &= \frac{\varphi(t)^{\frac{1}{\gamma}} (1 + \varphi(t))}{\varphi(x)^{\frac{1}{\gamma}(1-\varepsilon)+1-\varepsilon}} > \frac{\varphi(t)^{\frac{1}{\gamma}}}{\varphi(x)^{(\frac{1}{\gamma}+1)(1-\varepsilon)}} = \varphi(t)^{\varepsilon(\frac{1}{\gamma}+1)-1} \end{aligned}$$

is bounded as  $n \rightarrow \infty$  if  $\varepsilon \left( \frac{1}{\gamma} + 1 \right) \geq 1$ , i.e.  $\varepsilon \geq \frac{\gamma}{1+\gamma}$ . Since these pairs  $(t, x)$  are the worst choices, it follows easily that

$$\sup_{0 < t < x < 1} \frac{f_\gamma(t)}{f_\gamma(x)^s} < \infty \iff s \leq \frac{1}{1+\gamma}.$$

Thus for  $0 < s = \frac{1}{1+\gamma} < 1$ , this gives an example of an elliptical flat smooth function  $f_\gamma$  that satisfies  $\omega_s$ -monotonicity, but the power function  $(f_\gamma(x))^\gamma$  is not smooth.

This completes the proof of Theorem 2.8. ■

**Remark 3.8.** Let  $M \geq 0$  and  $0 < s < 1$ . If  $|\nabla^k f(x)| \leq \Gamma_{k,s} f(x)^s$  holds for  $0 \leq k \leq M$ , then  $f^\gamma \in C^{M-1,1}$  for all  $\gamma \geq M(1-s)$ . For this, see the end of the proof of (2)  $\implies$  (3) above. In particular  $\sqrt{f} \in C^{1,1}$  if  $s \geq \frac{3}{4}$  and  $\sqrt{f} \in C^{2,1}$  if  $s \geq \frac{5}{6}$ . When  $s > \frac{3}{4}$ , we show in Theorem 3.9 just below that  $\sqrt{f} \in C^{2,\delta}$  for some  $\delta > 0$ . Finally, we see that if we assume  $f$  is  $\omega_s$ -monotone for some  $s > 1 - \frac{\gamma}{M}$ , then we conclude that  $f^\gamma \in C^{M-1,1}$ .

But we can do better than the previous remark indicates, as the next and last theorem in this section shows.

**Theorem 3.9.** Let  $M \geq 2$ . Suppose that  $f$  is elliptical, flat, smooth and  $\omega_s$ -monotone on  $\mathbb{R}^d$  for some  $1 - \frac{1}{2M} < s \leq 1$ . Then there is  $\delta > 0$  and  $g \in C^{M,\delta}(\mathbb{R}^d)$  such that  $f = g^2$ .

For the proof, we follow Bony [Bon, Subsection 5.1], and define for a multiindex  $\alpha$  and  $0 < \delta < 1$ ,

$$(3.9) \quad [h]_{\alpha,\delta}(x) \equiv \limsup_{y,z \rightarrow x} \frac{|D^\alpha h(y) - D^\alpha h(z)|}{|y-z|^\delta}.$$

There is a subproduct rule,

$$(3.10) \quad [fg]_{\alpha,\delta}(x) \lesssim \sum_{\beta \leq \alpha} [f]_{\alpha-\beta,\delta}(x) |D^\beta g(x)| + \sum_{\beta \leq \alpha} |D^{\alpha-\beta} f(x)| [g]_{\beta,\delta}(x),$$

which follows using the product rule  $D^\alpha(fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (D^{\alpha-\beta} f)(D^\beta g)$  and the decomposition

$$\begin{aligned} & (D^{\alpha-\beta} f)(y) (D^\beta g)(y) - (D^{\alpha-\beta} f)(z) (D^\beta g)(z) \\ &= [(D^{\alpha-\beta} f)(y) - (D^{\alpha-\beta} f)(z)] (D^\beta g)(y) + (D^{\alpha-\beta} f)(z) [(D^\beta g)(y) - (D^\beta g)(z)], \end{aligned}$$

after then dividing by  $|y-z|^\delta$  and taking  $\limsup_{y,z \rightarrow x}$  inside the sum.

To derive a subchain rule we start by considering the case of  $\alpha = \mathbf{e}_1 = (1, 0, \dots, 0)$  and  $\alpha = \mathbf{e}_1 + \mathbf{e}_2 = (1, 1, 0, \dots, 0)$ . We have  $\partial_1(\psi(h(x))) = \psi'(h(x)) \partial_1 h(x)$ , and therefore

$$\begin{aligned} [\psi \circ h]_{\mathbf{e}_1,\delta}(x) &\equiv \limsup_{y,z \rightarrow x} \frac{|\partial_1 \psi \circ h(y) - \partial_1 \psi \circ h(z)|}{|y-z|^\delta} \\ &= \limsup_{y,z \rightarrow x} \frac{|\psi'(h(y)) \partial_1 h(y) - \psi'(h(z)) \partial_1 h(z)|}{|y-z|^\delta} \\ &\leq \limsup_{y,z \rightarrow x} \frac{|\psi'(h(y)) [\partial_1 h(y) - \partial_1 h(z)]|}{|y-z|^\delta} + \limsup_{y,z \rightarrow x} \frac{|[\psi'(h(y)) - \psi'(h(z))] \partial_1 h(z)|}{|y-z|^\delta} \\ &= |\psi' \circ h(x)| [h]_{\mathbf{e}_1,\delta}(x) + [\psi' \circ h]_{\mathbf{0},\delta}(x) |\partial_1 h(x)|. \end{aligned}$$

Next we have

$$\partial_1 \partial_2(\psi \circ h) = \psi''(\partial_1 h)(\partial_2 h) + \psi' \partial_2 \partial_1 h$$

and

$$\begin{aligned}
[\psi \circ h]_{\mathbf{e}_1 + \mathbf{e}_2, \delta}(x) &\equiv \limsup_{y, z \rightarrow x} \frac{|\partial_1 \partial_2 (\psi \circ h)(y) - \partial_1 \partial_2 (\psi \circ h)(z)|}{|y - z|^\delta} \\
&\leq \limsup_{y, z \rightarrow x} \frac{|\{(\psi'' \circ h)(\partial_1 h)(\partial_2 h)\}(y) - \{(\psi'' \circ h)(\partial_1 h)(\partial_2 h)\}(z)|}{|y - z|^\delta} \\
&\quad + \limsup_{y, z \rightarrow x} \frac{|\{(\psi' \circ h) \partial_2 \partial_1 h\}(y) - \{(\psi' \circ h) \partial_2 \partial_1 h\}(z)|}{|y - z|^\delta} \\
&\leq [\psi'' \circ h]_{\mathbf{0}, \delta}(x) |\partial_1 h(x) \partial_2 h(x)| + |\psi'' \circ h(x)| \left( [h]_{\mathbf{e}_1, \delta}(x) \partial_2 h(x) + [h]_{\mathbf{e}_2, \delta}(x) \partial_1 h(x) \right) \\
&\quad + [\psi' \circ h]_{\mathbf{0}, \delta}(x) |\partial_1 h(x) \partial_2 h(x)| + |\psi' \circ h(x)| [h]_{\mathbf{e}_1 + \mathbf{e}_2, \delta}(x).
\end{aligned}$$

Generalizing to  $[\psi \circ h]_{\alpha, \delta}$  with  $|\alpha| = M$  one obtains

$$\begin{aligned}
[\psi \circ h]_{\alpha, \delta}(x) &\lesssim \sum_{m=1}^M \left( [\psi^{(m)} \circ h]_{\mathbf{0}, \delta}(x) \sum_{\substack{0 < \beta_i \leq \alpha \\ |\beta_1| + \dots + |\beta_m| = M}} D^{\beta_1} h \dots D^{\beta_m} h \right) \\
&\quad + \sum_{m=1}^M \left( |\psi^{(m)} \circ h(x)| \sum_{\substack{0 < \beta_i \leq \alpha \\ |\beta_1| + \dots + |\beta_m| = M}} \sum_{j=1}^m [h]_{\beta_j, \delta} D^{\beta_1} h \dots D^{\beta_{j-1}} h D^{\beta_{j+1}} h \dots D^{\beta_m} h \right).
\end{aligned}$$

Indeed,  $m$  indicates how many factors we will have in the product of derivatives of  $h$ ; each  $\beta_i$  is a multiindex, which is nonzero and does not exceed  $\alpha$ ; the total number of derivatives we take is  $|\beta_1| + \dots + |\beta_m| = M = |\alpha|$ .

In the first line above we will replace  $[\psi^{(m)} \circ h]_{\mathbf{0}, \delta}(x)$  with

$$\begin{aligned}
[\psi^{(m)} \circ h]_{\mathbf{0}, \delta}(x) &= \limsup_{y, z \rightarrow x} \frac{|\psi^{(m)} \circ h(y) - \psi^{(m)} \circ h(z)|}{|y - z|^\delta} \\
&= \limsup_{y, z \rightarrow x} \frac{|\psi^{(m)}(h(y)) - \psi^{(m)}(h(z))|}{|h(y) - h(z)|} \frac{|h(y) - h(z)|}{|y - z|^\delta} \\
&= \psi^{(m+1)}(h(x)) [h]_{\mathbf{0}, \delta}(x),
\end{aligned}$$

to obtain

$$\begin{aligned}
[\psi \circ h]_{\alpha, \delta}(x) &\lesssim \sum_{m=1}^M \left( \psi^{(m+1)}(h(x)) [h]_{\mathbf{0}, \delta}(x) \sum_{\substack{0 < \beta_i \leq \alpha \\ |\beta_1| + \dots + |\beta_m| = M}} D^{\beta_1} h \dots D^{\beta_m} h \right) \\
&\quad + \sum_{m=1}^M \left( |\psi^{(m)} \circ h(x)| \sum_{\substack{0 < \beta_i \leq \alpha \\ |\beta_1| + \dots + |\beta_m| = M}} \sum_{j=1}^m [h]_{\beta_j, \delta} D^{\beta_1} h \dots D^{\beta_{j-1}} h D^{\beta_{j+1}} h \dots D^{\beta_m} h \right).
\end{aligned}$$

*Proof of Theorem 3.9.* In the special case  $\psi(t) = s_{1/2}(t) = t^{1/2}$  and  $h = f$  we have

$$|D^k \psi(t)| \leq C_k t^{1/2-k} \text{ and } [\psi]_{k, \delta}(t) \leq C_k t^{1/2-k-\delta},$$

and therefore

$$\begin{aligned} \left[ \sqrt{f} \right]_{\alpha, \delta} (x) &\lesssim \sum_{m=1}^M \left( f(x)^{1/2-m-1} f(x)^{(s')^\delta} \sum_{\substack{0 < \beta_i \leq \alpha \\ |\beta_1| + \dots + |\beta_m| = M}} f(x)^{(s')^{|\beta_1|}} \dots f(x)^{(s')^{|\beta_m|}} \right) \\ &+ \sum_{m=1}^M \left( f(x)^{1/2-m} \sum_{\substack{0 < \beta_i \leq \alpha \\ |\beta_1| + \dots + |\beta_m| = M}} \sum_{j=1}^m f(x)^{(s')^{|\beta_1|}} \dots f(x)^{(s')^{|\beta_j|+\delta}} \dots f(x)^{(s')^{|\beta_m|}} \right). \end{aligned}$$

We now combine this inequality with the inequalities from (3.5) and their analogues for  $|D|$ , namely

$$|D^\ell f(x)| \lesssim f(x)^{(s')^\ell} \text{ and } [f]_{\ell, \delta}(x) \lesssim f(x)^{(s')^{\ell+\delta}},$$

to see that  $[\sqrt{f}]_{\alpha, \delta}(x) \lesssim 1$  for a sufficiently small  $\delta > 0$  when  $s > s' > 1 - \frac{1}{2M}$ . Let  $\varepsilon' = 1 - s'$  so  $\varepsilon' = 1 - s' \in (0, \frac{1}{2M})$ . We use the estimate

$$(s')^k = (1 - \varepsilon')^k \geq 1 - k\varepsilon'$$

to obtain

$$(s')^{|\beta_1|} + \dots + (s')^{|\beta_m|} \geq m - M\varepsilon', \quad (s')^{|\beta_1|} + \dots + (s')^{|\beta_j|+\delta} + \dots + (s')^{|\beta_m|} \geq m - M\varepsilon' - \delta\varepsilon'$$

since  $|\beta_1| + \dots + |\beta_m| = M$ . This gives

$$\begin{aligned} \left[ \sqrt{f} \right]_{\alpha, \delta} (x) &\lesssim \sum_{m=1}^M \left( f(x)^{1/2-m-1+1-\delta\varepsilon'} f(x)^{m-M\varepsilon'} \right) + \sum_{m=1}^M \left( f(x)^{1/2-m} f(x)^{m-M\varepsilon'-\delta\varepsilon'} \right) \\ &\lesssim f(x)^{1/2-M\varepsilon'-\delta\varepsilon'}, \end{aligned}$$

which is bounded if  $\delta > 0$  is chosen sufficiently small since  $\varepsilon' < \frac{1}{2M}$ . This completes the proof that  $\sqrt{f} \in C^{M, \delta}$ . ■

#### 4. SUM OF SQUARES VIA BONY'S HÖLDER ADAPTATION OF FEFFERMAN-PHONG

Here we will follow Tataru's adaptation of the Fefferman-Phong argument, incorporating Bony's Hölder modification, that uses the implicit function theorem and Lemma 4.2 below on controlling odd derivatives by even derivatives, plus a bit more. But we begin here by stating and proving the implicit function theorem in the form we will use it, and then giving the control of odd derivatives by even derivatives for nonnegative functions.

**Theorem 4.1.** *Let  $H : \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}$  be  $C^1$  and let  $y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$  satisfy*

$$H(y', y_n) = 0 \text{ and } \frac{\partial H}{\partial x_n}(y', y_n) \neq 0.$$

(1) *Then there is a ball  $U = B(y', r) \subset \mathbb{R}^{n-1}$  and an interval  $V = (y_n - r, y_n + r)$  such that there is a unique function  $h : U \rightarrow V$  so that  $z = h(x')$  satisfies*

$$H(x', h(x')) = 0, \quad x' \in U.$$

(2) *Moreover  $h$  is continuously differentiable and*

$$\begin{aligned} Dh(x') &= -\frac{1}{\frac{\partial H}{\partial x_n}(x', h(x'))} (D_{x'} H)(x', h(x')), \\ \text{i.e. } \frac{\partial h}{\partial x_i} &= -\frac{\frac{\partial H}{\partial x_i}(x', h(x'))}{\frac{\partial H}{\partial x_n}(x', h(x'))}, \quad \text{for } 1 \leq i \leq n-1. \end{aligned}$$

(3) If in addition  $H$  is  $C^2$ , then  $h$  is also  $C^2$  and

$$\frac{\partial^2 h}{\partial x_i \partial x_j}(x') = -\frac{\frac{\partial^2 H}{\partial x_i \partial x_j}}{\frac{\partial H}{\partial x_n}} + \frac{\frac{\partial H}{\partial x_j} \frac{\partial^2 H}{\partial x_i \partial x_n} + \frac{\partial H}{\partial x_i} \frac{\partial^2 H}{\partial x_j \partial x_n}}{\left(\frac{\partial H}{\partial x_n}\right)^2} - \frac{\frac{\partial H}{\partial x_i} \frac{\partial H}{\partial x_j} \frac{\partial^2 H}{\partial x_n^2}}{\left(\frac{\partial H}{\partial x_n}\right)^3},$$

where  $H$  and its partial derivatives are evaluated at  $(x', h(x'))$  for  $x' \in U$ .

(4) If in addition  $H$  is  $C^m$  for some  $m \in \mathbb{N}$ , then  $h$  is also  $C^m$  and there is a formula for the  $m^{\text{th}}$  order partial derivatives of  $h$  having the following form for  $\alpha \in \mathbb{Z}_+^{n-1}$  with  $|\alpha| = m$ ,

$$\frac{\partial^m h}{\partial x^\alpha}(x') = \sum_{\ell=0}^m \frac{(-1)^{\ell+1} \ell!}{\left(\frac{\partial H}{\partial x_n}\right)^{\ell+1}} \sum_{\beta \in \mathbb{Z}_+^{n-1}: |\beta|=\ell} \frac{\partial^m H}{\partial^{\alpha-\beta} x \partial x_n^\ell} \prod_{\substack{\gamma_j \in \mathbb{Z}_+^{n-1} \\ \gamma_1 + \dots + \gamma_k = \beta}} \frac{\partial^{|\gamma_j|} H}{\partial x^{\gamma_j}}.$$

*Proof.* Parts (1) and (2) are the classical implicit function theorem. For part (3), if  $H$  is  $C^2$  we have,

$$\begin{aligned} \frac{\partial^2 h}{\partial x_i \partial x_j} &= -\frac{\partial}{\partial x_j} \frac{\frac{\partial H}{\partial x_i}(x', h(x'))}{\frac{\partial H}{\partial x_n}(x', h(x'))} \\ &= -\frac{\frac{\partial H}{\partial x_n}(x', h(x')) \frac{\partial}{\partial x_j} \left\{ \frac{\partial H}{\partial x_i}(x', h(x')) \right\} - \frac{\partial H}{\partial x_i}(x', h(x')) \frac{\partial}{\partial x_j} \left\{ \frac{\partial H}{\partial x_n}(x', h(x')) \right\}}{\left(\frac{\partial H}{\partial x_n}(x', h(x'))\right)^2} \\ &= -\frac{\frac{\partial H}{\partial x_n}(x', h(x')) \left\{ \frac{\partial^2 H}{\partial x_i \partial x_j}(x', h(x')) + \frac{\partial^2 H}{\partial x_i \partial x_n}(x', h(x')) \left( -\frac{\frac{\partial H}{\partial x_j}(x', h(x'))}{\frac{\partial H}{\partial x_n}(x', h(x'))} \right) \right\}}{\left(\frac{\partial H}{\partial x_n}(x', h(x'))\right)^2} \\ &\quad + \frac{\frac{\partial H}{\partial x_i}(x', h(x')) \left\{ \frac{\partial^2 H}{\partial x_j \partial x_n}(x', h(x')) + \frac{\partial^2 H}{\partial x_n^2}(x', h(x')) \left( -\frac{\frac{\partial H}{\partial x_j}(x', h(x'))}{\frac{\partial H}{\partial x_n}(x', h(x'))} \right) \right\}}{\left(\frac{\partial H}{\partial x_n}(x', h(x'))\right)^2}, \end{aligned}$$

which gives

$$\begin{aligned} \frac{\partial^2 h}{\partial x_i \partial x_j} &= -\frac{\frac{\partial H}{\partial x_n} \frac{\partial^2 H}{\partial x_i \partial x_j} - \frac{\partial H}{\partial x_n} \frac{\partial^2 H}{\partial x_i \partial x_n} \left( \frac{\frac{\partial H}{\partial x_j}}{\frac{\partial H}{\partial x_n}} \right)}{\left(\frac{\partial H}{\partial x_n}\right)^2} + \frac{\frac{\partial H}{\partial x_i} \frac{\partial^2 H}{\partial x_j \partial x_n} - \frac{\partial H}{\partial x_i} \frac{\partial^2 H}{\partial x_n^2} \left( \frac{\frac{\partial H}{\partial x_j}}{\frac{\partial H}{\partial x_n}} \right)}{\left(\frac{\partial H}{\partial x_n}\right)^2} \\ &= -\frac{\frac{\partial^2 H}{\partial x_i \partial x_j}}{\frac{\partial H}{\partial x_n}} + \frac{\frac{\partial H}{\partial x_j} \frac{\partial^2 H}{\partial x_i \partial x_n}}{\left(\frac{\partial H}{\partial x_n}\right)^2} + \frac{\frac{\partial H}{\partial x_i} \frac{\partial^2 H}{\partial x_j \partial x_n}}{\left(\frac{\partial H}{\partial x_n}\right)^2} - \frac{\frac{\partial H}{\partial x_i} \frac{\partial H}{\partial x_j} \frac{\partial^2 H}{\partial x_n^2}}{\left(\frac{\partial H}{\partial x_n}\right)^3}. \end{aligned}$$

Part (4) is established in a similar fashion. ■

Now we recall from Fefferman-Phong [FePh] and Tataru [Tat, Lemma 5.1], the control of odd derivatives in terms of even derivatives for a *nonnegative*  $C^{3,1}$  function  $f$ . For the convenience of the reader, we repeat the argument of Tataru [Tat, Lemma 5.1] in slightly greater detail here.

**Lemma 4.2.** *Suppose  $f(x) \geq 0$  and  $|f'''(x)| \leq 1$  for  $x \in \mathbb{R}$ . Then*

$$\begin{aligned} (4.1) \quad |f'(x)| &\leq \frac{8}{3} f(x)^{\frac{3}{4}} + \frac{8}{3} f(x)^{\frac{1}{2}} |f''(x)|^{\frac{1}{2}}, \\ |f'''(x)| &\leq 8 f(x)^{\frac{1}{4}} + 8 |f''(x)|^{\frac{1}{2}}, \\ -f''(x) &\leq \frac{5}{3} f(x)^{\frac{1}{2}}, \end{aligned}$$

Due to the control of the negative part of  $f''$  in the third line of (4.1), we can rewrite the first two lines in terms of the positive part of  $f''$ .

**Corollary 4.3.** *If  $f \geq 0$  and  $|f'''(x)| \leq 1$ , then for  $f(x) \leq 1$ ,*

$$\begin{aligned} |f'(x)| &\leq \max \left\{ 8f(x)^{\frac{3}{4}}, \frac{8}{3}f(x)^{\frac{3}{4}} + \frac{8}{3}f(x)^{\frac{1}{2}}f''(x)^{\frac{1}{2}}_+ \right\} \\ &\leq 8f(x)^{\frac{3}{4}} + \frac{8}{3}f(x)^{\frac{1}{2}}f''(x)^{\frac{1}{2}}_+, \\ |f'''(x)| &\leq \max \left\{ 24f(x)^{\frac{1}{4}}, 8f(x)^{\frac{1}{4}} + 8f''(x)^{\frac{1}{2}}_+ \right\} \\ &\leq 24f(x)^{\frac{1}{4}} + 8f''(x)^{\frac{1}{2}}_+. \end{aligned}$$

Finally we note that these inequalities extend to  $x \in \mathbb{R}^n$  in the form

$$\begin{aligned} (4.2) \quad |\nabla f(x)| &\lesssim f(x)^{\frac{3}{4}} + f(x)^{\frac{1}{2}} |\nabla^2 f(x)|^{\frac{1}{2}}, \\ |\nabla^3 f(x)| &\lesssim f(x)^{\frac{1}{4}} + |\nabla^2 f(x)|^{\frac{1}{2}}, \\ |\nabla^2 f(x)| &\lesssim \sup_{\Theta \in \mathbb{S}^{n-1}} [\partial_{\Theta}^2 f(x)]_+ + f(x)^{\frac{1}{2}}, \end{aligned}$$

provided  $|\nabla^4 f(x)| \leq 1$  on  $\mathbb{R}^n$ , upon using the equivalence of norms,

$$\sup_{\Theta \in \mathbb{S}^{n-1}} |\partial_{\Theta}^k f(x)| \approx |\nabla^k f(x)|, \quad x \in \mathbb{R}^n, 1 \leq k \leq 4,$$

on the finite dimensional vector space of homogeneous polynomials on  $\mathbb{R}^n$  of degree  $k$ . Here  $\partial_{\Theta}$  denotes the directional derivative in the direction of the unit vector  $\Theta$  in the sphere  $\mathbb{S}^{n-1}$ . For example, when  $n = 2$ , we can identify  $\Theta = \theta$  with  $(\cos \theta, \sin \theta)$  and we have

$$\partial_{\theta} f = (\cos \theta, \sin \theta) \cdot \nabla f = \cos \theta \frac{\partial f}{\partial x_1} + \sin \theta \frac{\partial f}{\partial x_2}.$$

*Proof.* To see the inequalities in (4.1) we may suppose that  $x = 0$ . Since the inequalities are invariant under the rescalings  $f(x) \rightarrow \lambda^{-4} f(\lambda x)$  for  $\lambda > 0$ , we may also assume  $f(0) \leq 1$ . We write

$$0 \leq f(y) \leq f(0) + f'(0)y + \frac{1}{2}f''(0)y^2 + \frac{1}{6}f'''(0)y^3 + \frac{1}{24}y^4,$$

to obtain

$$\left| f'(0)y + \frac{1}{6}f'''(0)y^3 \right| \leq f(0) + \frac{1}{2}f''(0)y^2 + \frac{1}{24}y^4.$$

The same bound for  $2y$  is

$$\left| f'(0)2y + \frac{4}{3}f'''(0)y^3 \right| \leq f(0) + 2f''(0)y^2 + \frac{2}{3}y^4,$$

and we claim that combining the bounds yields

$$\begin{aligned} (4.3) \quad |f'(0)y| &\leq \frac{3}{2}f(0) + f''(0)y^2 + \frac{1}{6}y^4, \\ |f'''(0)y^3| &\leq 3f(0) + 3f''(0)y^2 + y^4. \end{aligned}$$

Indeed, we have both

$$\begin{aligned} -f'(0)y - \frac{1}{6}f'''(0)y^3 &\leq f(0) + \frac{1}{2}f''(0)y^2 + \frac{1}{24}y^4, \\ f'(0)2y + \frac{4}{3}f'''(0)y^3 &\leq f(0) + 2f''(0)y^2 + \frac{2}{3}y^4, \end{aligned}$$

and adding 8 times the first inequality to the second gives

$$\begin{aligned} -6f'(0)y &\leq 9f(0) + 6f''(0)y^2 + y^4; \\ -f'(0)y &\leq \frac{3}{2}f(0) + f''(0)y^2 + \frac{1}{6}y^4. \end{aligned}$$

On the other hand, we also have both

$$\begin{aligned} f'(0)y + \frac{1}{6}f'''(0)y^3 &\leq f(0) + \frac{1}{2}f''(0)y^2 + \frac{1}{24}y^4, \\ -f'(0)2y - \frac{4}{3}f'''(0)y^3 &\leq f(0) + 2f''(0)y^2 + \frac{2}{3}y^4, \end{aligned}$$

and adding 8 times the first inequality to the second gives

$$\begin{aligned} 6f'(0)y &\leq 9f(0) + 6f''(0)y^2 + y^4; \\ f'(0)y &\leq \frac{3}{2}f(0) + f''(0)y^2 + \frac{1}{6}y^4. \end{aligned}$$

Altogether this gives the first inequality in (4.3), and the second inequality is proved similarly.

Now set  $y = \frac{f(0)^{\frac{1}{2}}}{f(0)^{\frac{1}{4}} + |f''(0)|^{\frac{1}{2}}}$  in the first inequality in (4.3) to obtain

$$\begin{aligned} |f'(0)| \frac{f(0)^{\frac{1}{2}}}{f(0)^{\frac{1}{4}} + |f''(0)|^{\frac{1}{2}}} &\leq \frac{3}{2}f(0) \frac{f(0)^{\frac{1}{2}}}{f(0)^{\frac{1}{4}}} + |f''(0)| \left( \frac{f(0)^{\frac{1}{2}}}{|f''(0)|^{\frac{1}{2}}} \right)^2 + \frac{1}{6} \left( \frac{f(0)^{\frac{1}{2}}}{f(0)^{\frac{1}{4}}} \right)^4 \\ &\leq \frac{3}{2}f(0)^{\frac{5}{4}} + f(0) + \frac{1}{6}f(0), \end{aligned}$$

which gives

$$\begin{aligned} |f'(0)| &\leq \left( f(0)^{\frac{1}{4}} + |f''(0)|^{\frac{1}{2}} \right) \left( \frac{3}{2}f(0)^{\frac{3}{4}} + \frac{7}{6}f(0)^{\frac{1}{2}} \right) \\ &= \left( f(0)^{\frac{3}{4}} + f(0)^{\frac{1}{2}} |f''(0)|^{\frac{1}{2}} \right) \left( \frac{3}{2}f(0)^{\frac{1}{4}} + \frac{7}{6} \right). \end{aligned}$$

Using  $f(0) \leq 1$  we thus obtain

$$|f'(0)| \leq \frac{8}{3} \left( f(0)^{\frac{3}{4}} + f(0)^{\frac{1}{2}} |f''(0)|^{\frac{1}{2}} \right),$$

which is the first line in (4.1).

The second line in (4.1) is proved by setting

$$y = \max \left\{ f(0)^{\frac{1}{4}}, |f''(0)|^{\frac{1}{2}} \right\},$$

and is left for the reader.

Finally, the third line in (4.1) is obtained by setting  $y = f(0)^{\frac{1}{4}}$  in the first line of (4.3), which gives

$$0 \leq \frac{3}{2}f(0) + f''(0)f(0)^{\frac{1}{2}} + \frac{1}{6}f(0).$$

■

For  $\delta > 0$  define

$$r_\delta(x) \equiv \max \left\{ f(x)^{\frac{1}{4+2\delta}}, \left( \sup_{\Theta \in \mathbb{S}^{n-1}} [\partial_\Theta^2 f(x)]_+ \right)^{\frac{1}{2+2\delta}} \right\}, \quad x \in \mathbb{R}^n.$$

Following Tataru [Tat] we now show that  $r_\delta$  is *slowly varying*, i.e. there are  $0 < c, \gamma < 1$  such that

$$|r_\delta(x) - r_\delta(y)| \leq \gamma r_\delta(x), \quad \text{for } |x - y| \leq cr(x).$$

We prove this only in  $\mathbb{R}$ , and leave the straightforward extension to higher dimensions for the reader.

**Lemma 4.4.** *Let  $\delta > 0$ . If  $f(x) \geq 0$  and  $|\nabla^4 f(x)| \leq 1$  for  $x \in \mathbb{R}$ , then*

$$|r_\delta(x) - r_\delta(y)| \leq \left( \frac{1}{2} \right)^{\frac{1}{4+2\delta}} r_\delta(x), \quad \text{for } |x - y| \leq \frac{1}{200} r(x).$$

*Proof.* By translation and rescaling we can assume that  $x = 0$  and  $r_\delta(0) = 1$ . Then  $f(0)^{\frac{1}{4+2\delta}}, f''(0)^{\frac{1}{2+2\delta}} \leq 1$  and by Corollary 4.3 we have

$$|f(0)| \leq 1, \quad |f'(0)| \leq 11, \quad |f''(0)| \leq \frac{5}{3}, \quad |f'''(0)| \leq 32,$$

and so with  $|y| = |y - x| \leq \frac{1}{200}$ , Taylor's formula shows that both  $f$  and  $f''_+$  are slowly varying, i.e.

$$|f(y) - f(0)| \leq \frac{11}{200} + \frac{\frac{5}{3}}{2(200)^2} + \frac{32}{6(200)^3} + \frac{1}{24(200)^4} < \frac{1}{2},$$

and

$$|f''(y)_+ - f''(0)_+| \leq \frac{32}{200} + \frac{1}{2(200)^2} < \frac{1}{2},$$

which yields

$$\begin{aligned} |r_\delta(x) - r_\delta(y)| &= \left| \max \left\{ f(0)^{\frac{1}{4+2\delta}}, |f''(0)|^{\frac{1}{2+2\delta}} \right\} - \max \left\{ f(y)^{\frac{1}{4+2\delta}}, |f''(y)|^{\frac{1}{2+2\delta}} \right\} \right| \\ &\leq \max \left\{ \left| f(0)^{\frac{1}{4+2\delta}} - f(y)^{\frac{1}{4+2\delta}} \right|, \left| |f''(0)|^{\frac{1}{2+2\delta}} - |f''(y)|^{\frac{1}{2+2\delta}} \right| \right\} \leq \gamma < 1. \end{aligned}$$

Indeed, if  $|f''(0)| = 0$  and  $|f''(y)| = \frac{1}{2}$ , then  $\left| |f''(0)|^{\frac{1}{2+2\delta}} - |f''(y)|^{\frac{1}{2+2\delta}} \right| = \left( \frac{1}{2} \right)^{\frac{1}{2+2\delta}}$ , while if  $f(0) = 0$  and  $f(y) = \frac{1}{2}$ , then  $\left| f(0)^{\frac{1}{4+2\delta}} - f(y)^{\frac{1}{4+2\delta}} \right| = \left( \frac{1}{2} \right)^{\frac{1}{4+2\delta}}$ , and since these cases are optimal, we have the above inequality with  $\gamma = \left( \frac{1}{2} \right)^{\frac{1}{4+2\delta}} < 1$ . ■

**4.1. A provisional SOS theorem.** Here we begin with the following provisional sum of squares theorem, an analogue of Lemmas 1 and 2 in [FePh], which will be used to prove our main Theorems 4.7 and 4.8 below. For any  $0 \leq \beta \leq 1$ , and any continuous function  $h$  defined on a ball  $B$  in  $\mathbb{R}^n$ , we define

$$\|h\|_{\text{Lip}_\beta(B)} \equiv \sup_{x,y \in B} \frac{|h(x) - h(y)|}{|x - y|^\beta},$$

and for  $k \in \mathbb{Z}_+$  we denote by  $C^{k,\beta}(B)$  the space of functions  $f$  on  $B$  normed by

$$\|f\|_{C^{k,\beta}(B)} \equiv \sum_{\ell=0}^k \left\| \nabla^\ell f \right\|_{L^\infty(B)} + \left\| \nabla^k f \right\|_{\text{Lip}_\beta(B)}.$$

We will use the following ‘distance function’ related to derivatives of  $f$  that was used in Tataru [Tat] and Bony [Bon]:

$$(4.4) \quad \rho(x) = \rho_f(x) \equiv \max \left\{ f(x)^{\frac{1}{4+2\delta}}, \left( \sup_{\Theta \in \mathbb{S}^{n-1}} [\partial_\Theta^2 f(x)]_+ \right)^{\frac{1}{2+2\delta}}, |\nabla^4 f(x)|^{\frac{1}{2\delta}} \right\}, \quad x \in \mathbb{R}^n.$$

**Theorem 4.5.** *Suppose  $0 < \delta < 1$  and  $n \geq 1$ . Then there exists a constant  $N = N(\delta, n)$  depending on  $\delta$  and  $n$  with the following property. For every nonnegative  $f \in C^{4,2\delta}(\mathbb{R}^n)$  satisfying*

$$(4.5) \quad |\nabla^4 f(x)| \leq C f(x)^{\frac{\delta}{2+\delta}} \quad \text{and} \quad |\nabla^2 f(x)| \leq C f(x)^{\frac{2\delta(1+\delta)}{2+\delta}},$$

*there are functions  $g_\ell \in C^{2,\delta}(\mathbb{R}^n)$  satisfying*

$$(4.6) \quad \begin{aligned} |D^\alpha g_\ell(x)| &\leq C \rho(x)^{2+\delta-|\alpha|}, & 0 \leq |\alpha| \leq 2, \\ [g_\ell]_{\alpha,\delta}(x) &\leq C, & |\alpha| = 2, \end{aligned}$$

*and*

$$(4.7) \quad \begin{aligned} |D^\alpha g_\ell^2(x)| &\leq C \rho(x)^{4+2\delta-|\alpha|}, & 0 \leq |\alpha| \leq 2, \\ [g_\ell^2]_{\alpha,2\delta}(x) &\leq C, & |\alpha| = 2, \end{aligned}$$

*and nonnegative functions  $h_\ell \in C^{4,2\delta}(\mathbb{R}^n)$ , for  $1 \leq \ell \leq N$ , such that*

$$f(x) = \sum_{\ell=1}^N g_\ell(x)^2 + \sum_{\ell=1}^N h_\ell(x), \quad x \in \mathbb{R}^n,$$

and where each function  $h_\ell$  can be further decomposed into a countable sum of functions from a bounded set in  $C^{4,2\delta}(\mathbb{R}^n)$  with pairwise disjoint supports, i.e.

$$(4.8) \quad \begin{aligned} h_\ell &= \sum_{m=1}^{\infty} k_{\ell,m}, \\ \text{Supp } k_{\ell,m} \cap \text{Supp } k_{\ell,m'} &= \emptyset \text{ for } m \neq m', \\ \{k_{\ell,m}\}_{\ell,m} &\text{ is a bounded set of functions in } C^{4,2\delta}(\mathbb{R}^n). \end{aligned}$$

Moreover, each  $k_{\ell,m}$  has the two critical properties that

(i)  $k_{\ell,m}$  can be factored as

$$k_{\ell,m}(x) = \Phi_{\ell,m}(x)^2 \kappa_{\ell,m}(x), \quad \Phi_{\ell,m} \in C^{2,\delta}, \kappa_{\ell,m} \in C^{4,2\delta}$$

where  $\kappa_{\ell,m}(x)$  is a function of just  $n-1$  variables, i.e. there is a rotation  $R = R_{\ell,m}$  depending on  $\ell, m$  such that in the rotated variables  $y = Rx$ , the function  $\kappa_{\ell,m}$  is independent of  $y_n$ , and

(ii) The function  $\kappa_{\ell,m}$  of  $n-1$  variables satisfies the differential inequalities (4.5).

Here the families  $\{\Phi_{\ell,m}\}_{\ell,m}$  and  $\{\kappa_{\ell,m}\}_{\ell,m}$  lie in bounded sets in  $C^{2,\delta}$  and  $C^{4,2\delta}$  respectively.

**Remark 4.6.** The purpose of the first differential inequality in (4.5) is to limit our analysis to only the cases when  $|\nabla^4 f|^{\frac{1}{2\delta}}$  is not the dominant term in the definition of  $\rho_f$  (the implicit function theorem is not decisive if  $|\nabla^4 f|^{\frac{1}{2\delta}}$  dominates), while the purpose of the second differential inequality is to show that the first inequality is inherited by the functions  $\kappa_{\ell,m}$  that arise in the induction step in applications of Theorem 4.5. In dimension  $n = 1$  both of these differential inequalities can be dropped by results of Bony in [Bon], in dimension  $n = 2$  the second inequality can be dropped by Theorem 4.7 below, while in dimension  $n \geq 5$ , we will see in Remark 4.10 below that both inequalities are in general required.

*Proof.* We begin the proof of Theorem 4.5 by further adapting the version of the Fefferman-Phong argument due to Bony in [Bon, Théorème 2] using the ‘distance function’  $\rho(x)$ . Define  $\Gamma \equiv \{x \in \mathbb{R}^n : f(x) = 0\}$  and  $d(x) \equiv \text{dist}(x, \Gamma)$ . Recall the sublinear operators  $[\cdot]_{\alpha,\delta}$  defined in (3.9) above,

$$[\varphi]_{\alpha,\delta}(x) \equiv \limsup_{y,z \rightarrow x} \frac{|D^\alpha \varphi(y) - D^\alpha \varphi(z)|}{|y - z|^\delta}.$$

One now writes  $U \equiv \Gamma^c$  as a countable union of cubes  $Q_\nu$  with center  $x^\nu$  and diameter comparable to  $\rho(x^\nu)$ . Now for all  $s > 0$  sufficiently small, there is a collection of balls  $\{B_\nu\}_{\nu=1}^\infty$  covering  $U$  with centers  $x^\nu$  and radii

$$r_\nu \equiv s\rho(x^\nu),$$

having bounded overlap  $\sum_{\nu=1}^\infty \mathbf{1}_{B_\nu} \leq C\mathbf{1}_U$ , as well as a partition of unity  $\{\Phi_\nu\}_{\nu=1}^\infty$  subordinate to this collection satisfying

$$\begin{aligned} \sum_{\nu=1}^\infty \Phi_\nu(x)^2 &= 1, \quad \text{Supp } \Phi_\nu \subset B_\nu, \\ \sup_{x \in U} |D^\alpha \Phi_\nu(x)| &\leq C_{\alpha,s} \frac{1}{r_\nu^{|\alpha|}}, \quad \alpha \in \mathbb{Z}_+^n. \end{aligned}$$

We now wish to show that each function  $\Phi_\nu^2 f$  can be decomposed as a sum of a square with control, and a nonnegative function  $h_\nu \in C^{4,2\delta}$  with the special decomposition property as in (4.8). For this we will use the following inequalities for  $x \in B_\nu$ ,

$$(4.9) \quad \begin{aligned} 0 &< \rho(x) \leq Cd(x), \\ |D^\alpha f(x)| &\leq C\rho(x)^{4+2\delta-|\alpha|}, \quad 0 \leq |\alpha| \leq 4, \\ [f]_{\alpha,2\delta}(x) &\leq C, \quad |\alpha| = 4, \end{aligned}$$

We now prove (4.9). From (4.5) we see that  $|\nabla^2 f(x)| = |\nabla^4 f(x)| = 0$  for all  $x \in \Gamma$ , and then the first two lines in (4.2), together with  $f \in C^{4,2\delta}(\mathbb{R}^n)$ , show that  $f(x) \leq C|x - x_0|^{4+2\delta}$  and  $|\nabla^2 f(x)| \leq C|x - x_0|^{2+2\delta}$

for all  $x_0 \in \Gamma$ . Thus we have  $f(x)^{\frac{1}{4+2\delta}} \leq C \inf_{x_0 \in \Gamma} |x - x_0| = Cd(x)$  and  $\left( \sup_{\Theta \in \mathbb{S}^{n-1}} [\partial_{\Theta}^2 f(x)]_+ \right)^{\frac{1}{2+2\delta}} \leq C \inf_{x_0 \in \Gamma} |x - x_0| = Cd(x)$ , and so

$$\begin{aligned} f(x)^{\frac{1}{4+2\delta}} &\leq Cd(x) \text{ and } \left( \sup_{\Theta \in \mathbb{S}^{n-1}} [\partial_{\Theta}^2 f(x)]_+ \right)^{\frac{1}{2+2\delta}} \leq Cd(x), \\ \text{and } \rho(x) &= \max \left\{ f(x)^{\frac{1}{4+2\delta}}, \left( \sup_{\Theta \in \mathbb{S}^{n-1}} [\partial_{\Theta}^2 f(x)]_+ \right)^{\frac{1}{2+2\delta}} \right\} \leq Cd(x), \end{aligned}$$

which is the first line in (4.9).

On the other hand,

$$\begin{aligned} |D^0 f(x)| &= f(x) \leq \rho(x)^{4+2\delta}, \\ \text{and } |D^2 f(x)| &\leq C \sup_{\Theta \in \mathbb{S}^{n-1}} [\partial_{\Theta}^2 f(x)]_+ + Cf(x)^{\frac{1}{2}} \rho(x)^{\delta} \leq C\rho(x)^{2+2\delta}, \end{aligned}$$

where the first inequality in the second line above follows using the third line in (4.2) applied to  $\eta(x) \equiv \rho(x^{\nu})^{-2\delta} f(x)$ ,

$$\left| \rho(x^{\nu})^{-2\delta} \nabla^2 f(x) \right| \lesssim \rho(x^{\nu})^{-2\delta} \sup_{\Theta \in \mathbb{S}^{n-1}} [\partial_{\Theta}^2 f(x)]_+ + \left| \rho(x^{\nu})^{-2\delta} f(x) \right|^{\frac{1}{2}} \lesssim \rho(x^{\nu})^{-2\delta},$$

since  $\rho = r_{\delta}$  is slowly varying. From the control of odd order derivatives by those of even order in the first two lines of Lemma 4.2 applied to  $\eta(x) \equiv \rho(x^{\nu})^{-2\delta} f(x)$ , we then obtain

$$\begin{aligned} \left| \rho(x^{\nu})^{-2\delta} \nabla f(x) \right| &\lesssim \left( \rho(x^{\nu})^{-2\delta} f(x) \right)^{\frac{3}{4}} + \left( \rho(x^{\nu})^{-2\delta} f(x) \right)^{\frac{1}{2}} \left( \rho(x^{\nu})^{-2\delta} \nabla^2 f(x) \right)^{\frac{1}{2}} \\ &\lesssim \left( \rho(x^{\nu})^{-2\delta} \rho(x)^{4+2\delta} \right)^{\frac{3}{4}} + \left( \rho(x^{\nu})^{-2\delta} \rho(x)^{4+2\delta} \right)^{\frac{1}{2}} \left( \rho(x^{\nu})^{-2\delta} \rho(x)^{2+2\delta} \right)^{\frac{1}{2}} \\ &\lesssim \rho(x^{\nu})^3, \quad x \in B_{\nu}, \end{aligned}$$

since  $\rho$  is slowly varying on  $B_{\nu}$ , and similarly

$$\begin{aligned} \left| \rho(x^{\nu})^{-2\delta} \nabla^3 f(x) \right| &\lesssim \left( \rho(x^{\nu})^{-2\delta} f(x) \right)^{\frac{1}{4}} + \left( \rho(x^{\nu})^{-2\delta} f(x) \right)^{\frac{1}{2}} \left| \rho(x^{\nu})^{-2\delta} \nabla^2 f(x) \right|^{\frac{1}{2}} \\ &\lesssim \left( \rho(x^{\nu})^{-2\delta} \rho(x)^{4+2\delta} \right)^{\frac{1}{4}} + \left( \rho(x^{\nu})^{-2\delta} \rho(x)^{4+2\delta} \right)^{\frac{1}{2}} \left| \rho(x^{\nu})^{-2\delta} \rho(x)^{2+2\delta} \right|^{\frac{1}{2}} \\ &\lesssim \rho(x) + \rho(x)^3 \lesssim \rho(x), \quad x \in B_{\nu}. \end{aligned}$$

Combined with  $|\nabla^4 f(x)| \leq C\rho(x)^{2\delta}$ , this gives the second line in (4.9), and the subproduct rule (3.10) yields the third line. This completes the proof of (4.9).

We claim

$$\begin{aligned} \Phi_{\nu} f &= g_{\nu}^2 + h_{\nu}, \\ \|D^{\alpha} g_{\nu}\|_{\text{Lip}_{\delta}(B_{\nu})} &\leq C_{\alpha,s} r_{\nu}^{2-|\alpha|}, \quad 0 \leq |\alpha| \leq 2, \\ \left\| [g_{\nu}]_{\alpha,\delta} \right\|_{\infty} &\leq C_{\alpha,s}, \quad |\alpha| = 2, \\ \|D^{\alpha} h_{\nu}\|_{\text{Lip}_{2\delta}(B_{\nu})} &\leq C_{\alpha,s} r_{\nu}^{4-|\alpha|}, \quad 0 \leq |\alpha| \leq 4, \\ \left\| [h_{\nu}]_{\alpha,2\delta} \right\|_{\infty} &\leq C_{\alpha,s}, \quad |\alpha| = 4, \end{aligned}$$

where the constant  $C_{\alpha,s}$  is independent of  $\nu$ . Moreover, we also have analogous inequalities for  $g_{\nu}^2$  that mirror those of  $f$ :

$$\begin{aligned} \|D^{\alpha} g_{\nu}^2\|_{\text{Lip}_{2\delta}(B_{\nu})} &\leq C_{\alpha,s} r_{\nu}^{4-|\alpha|}, \quad 0 \leq |\alpha| \leq 4, \\ \left\| [g_{\nu}^2]_{\alpha,2\delta} \right\|_{\infty} &\leq C_{\alpha,s}, \quad |\alpha| = 4. \end{aligned}$$

**Case I:**  $f(x^{\nu}) \geq c\rho(x^{\nu})^{4+2\delta}$ .

In this case we define

$$f_1(x) = \Phi_\nu(x) \sqrt{f(x)},$$

and use the inequalities in (4.9). A first order partial derivative  $D^\mu f_1$  of  $f_1$  is

$$D^\mu f_1(x) = D^\mu \Phi_\nu(x) \sqrt{f(x)} + \Phi_\nu(x) \frac{D^\mu f(x)}{2\sqrt{f(x)}},$$

and its modulus is bounded by

$$\left| D^\mu \Phi_\nu(x) \sqrt{f(x)} \right| + \left| \Phi_\nu(x) \frac{D^\mu f(x)}{2\sqrt{f(x)}} \right| \lesssim \frac{1}{r(x^\nu)} \rho(x^\nu)^{2+\delta} + \frac{\rho(x)^{3+2\delta}}{2\rho(x)^{2+\delta}} \lesssim \rho(x)^{1+\delta},$$

by the assumption of Case I, together with the slowly varying property of  $\rho \approx r$ . A second order partial derivative  $D^\mu f_1$  is

$$D^\mu f_1(x) = D^\mu \Phi_\nu(x) \sqrt{f(x)} + D^\alpha \Phi_\nu(x) \frac{D^\beta f(x)}{\sqrt{f(x)}} + \Phi_\nu(x) \frac{D^\mu f(x)}{2\sqrt{f(x)}} - \Phi_\nu(x) \frac{D^\alpha f(x) D^\beta f(x)}{4f(x)^{\frac{3}{2}}},$$

where  $|\alpha| = |\beta| = 1$ . Now for  $x \in B_\nu$  we have

$$\begin{aligned} \left| D^\mu \Phi_\nu(x) \sqrt{f(x)} \right| &\lesssim \frac{1}{r(x^\nu)^2} \rho(x)^{2+\delta} \approx \rho(x)^\delta, \\ \left| D^\alpha \Phi_\nu(x) \frac{D^\beta f(x)}{\sqrt{f(x)}} \right| &\lesssim \frac{1}{r(x^\nu)} \frac{\rho(x)^{3+2\delta}}{\rho(x)^{2+\delta}} \lesssim \rho(x)^\delta, \\ \Phi_\nu(x) \frac{D^\mu f(x)}{2\sqrt{f(x)}} &\lesssim \frac{\rho(x)^{2+2\delta}}{\rho(x)^{2+\delta}} = \rho(x)^\delta, \\ \left| \Phi_\nu(x) \frac{D^\alpha f(x) D^\beta f(x)}{4f(x)^{\frac{3}{2}}} \right| &\lesssim \frac{(\rho(x)^{3+2\delta})^2}{(\rho(x)^{4+2\delta})^{\frac{3}{2}}} = \rho(x)^\delta, \end{aligned}$$

and we conclude that  $|\nabla^2 f_1(x)|$  is bounded by a multiple of  $\rho(x)^\delta$  for  $x \in B_\nu$ . Finally, using the subproduct rule (3.10) for  $[\cdot]_{\alpha,\delta}$ , we obtain that  $f_1 \in C^{2,\delta}$  uniformly in  $\nu$ .

Similarly, we have for a first order partial derivative  $D^\mu (f_1(x)^2)$  and  $x \in B_\nu$ ,

$$\begin{aligned} \left| D^\mu (f_1(x)^2) \right| &= \left| D^\mu (\Phi_\nu(x)^2) f(x) + \Phi_\nu(x)^2 D^\mu f(x) \right| \\ &\lesssim r(x^\nu)^{-1} \rho(x)^{4+2\delta} + \rho(x)^{3+2\delta} \lesssim \rho(x)^{3+2\delta}, \end{aligned}$$

which is the case  $|\mu| = 1$  of

$$\left| D^\mu (f_1(x)^2) \right| \lesssim r(x^\nu)^{4-|\mu|+2\delta}, \quad 0 \leq |\mu| \leq 4,$$

and the remaining cases  $|\mu| = 2, 3, 4$  are proved in the same way. Finally, using the subproduct rule (3.10) for  $[\cdot]_{\alpha,2\delta}$ , together with the third line in (4.9), we obtain that  $f_1^2 \in C^{4,2\delta}$  uniformly in  $\nu$ .

**Case II:**  $f(x^\nu) < c\rho(x^\nu)^{4+2\delta}$ .

In this case we have without loss of generality that  $\partial_{x_n}^2 f(x^\nu) = \rho(x^\nu)^{2+2\delta}$ , and hence that  $\partial_{x_n}^2 f(x) \geq \frac{1}{2}\rho(x^\nu)^{2+2\delta}$  for  $x \in B_\nu$ , provided  $c$  is chosen sufficiently small independent of  $\nu$ . Let us write  $x = (\xi, x_n)$  and  $x^\nu = (\xi^\nu, x_n^\nu)$ . Then for  $|\xi - \xi^\nu| < \frac{1}{\sqrt{2}}r_\nu = \frac{s}{\sqrt{2}}\rho(x^\nu)$ , the function  $x_n \rightarrow f(\xi, x_n)$  has its second derivative bounded below by  $\frac{1}{2}\rho(x^\nu)^{2+2\delta}$  on the closed interval  $[a_n^\nu, b_n^\nu] \equiv \left[ x_n^\nu - \frac{1}{\sqrt{2}}r_\nu, x_n^\nu + \frac{1}{\sqrt{2}}r_\nu \right]$ , and hence has a unique minimum point in  $[a_n^\nu, b_n^\nu]$ , say at  $x_n = X(\xi)$ . If moreover,  $c$  is chosen to be at most  $\frac{s^2}{8}$ , then the minimum is actually attained at  $x_n = X(\xi)$  in the *open* interval  $(a_n^\nu, b_n^\nu)$ . Indeed, if not, say  $f(\xi, a_n^\nu)$  is the

minimum of  $f$  on the closed interval  $[a_n^\nu, b_n^\nu]$ , then  $\partial_{x_n} f(\xi, a_n^\nu) \geq 0$  as well as  $f(\xi, a_n^\nu) \geq 0$ , and so Taylor's formula gives for an intermediate point  $c_n^\nu$  between  $a_n^\nu$  and  $x_n^\nu$ ,

$$\begin{aligned} f(\xi, x_n^\nu) &= f(\xi, a_n^\nu) + \partial_{x_n} f(\xi, a_n^\nu)(x_n^\nu - a_n^\nu) + \partial_{x_n}^2 f(\xi, c_n^\nu) \frac{(x_n^\nu - a_n^\nu)^2}{2} \\ &\geq \frac{1}{2} \rho(x^\nu)^{2+2\delta} \frac{\left(\frac{1}{\sqrt{2}} r_\nu\right)^2}{2} = \frac{s^2}{8} \rho(x^\nu)^{4+2\delta} \geq c \rho(x^\nu)^{4+2\delta}, \end{aligned}$$

contradicting the Case II assumption.

Set  $F(\xi) \equiv f(\xi, X(\xi))$ . Then

$$\begin{aligned} f(\xi, x_n) &= f(\xi, X(\xi)) + \partial_{x_n} f(\xi, X(\xi)) [x_n - X(\xi)] \\ &\quad + \int_0^1 (1-t) \partial_{x_n}^2 f(\xi, (1-t)X(\xi) + tx_n) dt \frac{(x_n - X(\xi))^2}{2} \\ &= F(\xi) + H(\xi, x_n) (x_n - X(\xi))^2; \\ \text{where } F(\xi) &= f(\xi, X(\xi)), \\ \text{and } H(\xi, x_n) &\equiv \frac{1}{2} \int_0^1 (1-t) \partial_{x_n}^2 f(\xi, (1-t)X(\xi) + tx_n) dt, \end{aligned}$$

where  $H(\xi, x_n)$  satisfies

$$\begin{aligned} H(\xi, x_n) &\geq \frac{1}{2} \int_0^1 (1-t) \frac{1}{2} \rho(x^\nu)^{2+2\delta} dt = \frac{1}{8} \rho(x^\nu)^{2+2\delta}. \\ |\partial_{x_n}^2 H(\xi, x_n)| &\leq \frac{1}{2} \int_0^1 (1-t) |\partial_{x_n}^4 f(\xi, (1-t)X(\xi) + tx_n)| t^2 dt \\ &\lesssim \int_0^1 (1-t) t^2 \rho(x^\nu)^{2\delta} dt \lesssim \rho(x^\nu)^{2\delta}, \end{aligned}$$

Now we wish to bound  $|\nabla^2 H(\xi, x_n)|$  by  $C \rho(x^\nu)^{2\delta}$ . For this, we first note that by parts (2) and (3) of Theorem 4.1 applied to the function  $\partial_{x_n} f$ , we have

$$\begin{aligned} |\partial_{\xi_i} X(\xi)| &= \left| \frac{\partial_{\xi_i} \partial_{x_n} f}{\partial_{x_n}^2 f} \right| \lesssim \frac{\rho(x^\nu)^{2+2\delta}}{\rho(x^\nu)^{2+2\delta}} = 1, \\ |\partial_{\xi_i} \partial_{\xi_j} X(\xi)| &\leq \left| \frac{\partial_{\xi_i} \partial_{\xi_j} \partial_{x_n} f}{\partial_{x_n}^2 f} \right| + \left| \frac{(\partial_{\xi_j} \partial_{x_n} f)(\partial_{\xi_i} \partial_{x_n}^2 f) + (\partial_{\xi_i} \partial_{x_n} f)(\partial_{\xi_j} \partial_{x_n}^2 f)}{(\partial_{x_n}^2 f)^2} \right| + \left| \frac{(\partial_{\xi_j} \partial_{x_n} f)(\partial_{\xi_j} \partial_{x_n} f)(\partial_{x_n}^3 f)}{(\partial_{x_n}^2 f)^3} \right| \\ &\lesssim \frac{\rho(x^\nu)^{1+2\delta}}{\rho(x^\nu)^{2+2\delta}} + \frac{\rho(x^\nu)^{2+2\delta} \rho(x^\nu)^{1+2\delta}}{(\rho(x^\nu)^{2+2\delta})^2} + \frac{(\rho(x^\nu)^{2+2\delta})^2 \rho(x^\nu)^{1+2\delta}}{(\rho(x^\nu)^{2+2\delta})^3} \\ &\lesssim \rho(x^\nu)^{-1}. \end{aligned}$$

Then we compute

$$\begin{aligned} \partial_{x_n} [\partial_{x_n}^2 f(\xi, (1-t)X(\xi) + tx_n)] &= \partial_{x_n}^3 f(\xi, (1-t)X(\xi) + tx_n) t, \\ \nabla_\xi [\partial_{x_n}^2 f(\xi, (1-t)X(\xi) + tx_n)] &= \nabla_\xi \partial_{x_n}^2 f(\xi, (1-t)X(\xi) + tx_n) \\ &\quad + \partial_{x_n}^3 f(\xi, (1-t)X(\xi) + tx_n) (1-t) \nabla X(\xi), \end{aligned}$$

and obtain that their moduli are bounded by

$$\begin{aligned} |\partial_{x_n} [\partial_{x_n}^2 f(\xi, (1-t)X(\xi) + tx_n)]| &\lesssim \rho(x^\nu)^{1+2\delta}, \\ |\nabla_\xi [\partial_{x_n}^2 f(\xi, (1-t)X(\xi) + tx_n)]| &\lesssim \rho(x^\nu)^{1+2\delta}. \end{aligned}$$

Similarly,

$$\begin{aligned} |\nabla_\xi^2 [\partial_{x_n}^2 f(\xi, (1-t)X(\xi) + tx_n)]| &\lesssim \rho(x^\nu)^{2\delta}, \\ |\partial_{x_n} \nabla_\xi [\partial_{x_n}^2 f(\xi, (1-t)X(\xi) + tx_n)]| &\lesssim \rho(x^\nu)^{2\delta}, \\ |\partial_{x_n}^2 [\partial_{x_n}^2 f(\xi, (1-t)X(\xi) + tx_n)]| &\lesssim \rho(x^\nu)^{2\delta}. \end{aligned}$$

Thus for  $1 \leq |\mu| \leq 2$ , we have that

$$D^\mu H(\xi, x_n) = \frac{1}{2} \int_0^1 (1-t) D^\mu [\partial_{x_n}^2 f(\xi, (1-t)X(\xi) + tx_n)] dt$$

satisfies

$$|D^\mu H(\xi, x_n)| \lesssim \rho(x^\nu)^{2-|\mu|+2\delta}.$$

Thus  $K(\xi, x_n) \equiv H(\xi, x_n)(x_n - X(\xi))^2$  has a  $C^{2,\delta}$  square root  $G(\xi, x_n)(x_n - X(\xi))$  where  $G(\xi, x_n) \equiv \sqrt{H(\xi, x_n)}$ . Indeed,

$$\begin{aligned} \nabla [G(\xi, x_n)(x_n - X(\xi))] &= \nabla G(\xi, x_n)(x_n - X(\xi)) + G(\xi, x_n) \nabla (x_n - X(\xi)) \\ &= \frac{1}{2} \frac{\nabla H(\xi, x_n)}{H(\xi, x_n)^{\frac{1}{2}}} (x_n - X(\xi)) + H(\xi, x_n)^{\frac{1}{2}} \nabla (x_n - X(\xi)) \end{aligned}$$

satisfies

$$|\nabla [G(\xi, x_n)(x_n - X(\xi))]| \lesssim \frac{\rho(x^\nu)^{1+2\delta}}{\rho(x^\nu)^{1+\delta}} \rho(x^\nu) + \rho(x^\nu)^{1+\delta} \approx \rho(x^\nu)^{1+\delta},$$

and for  $\mu = \alpha + \beta$  with  $|\alpha| = |\beta| = 1$ ,

$$\begin{aligned} D^\mu [G(\xi, x_n)(x_n - X(\xi))] &= D^\mu G(\xi, x_n)(x_n - X(\xi)) + 2D^\alpha G(\xi, x_n) D^\beta [(x_n - X(\xi))] \\ &\quad + G(\xi, x_n) D^\mu [(x_n - X(\xi))] \\ &= \frac{1}{2} \left( \frac{D^\mu H(\xi, x_n)}{H(\xi, x_n)^{\frac{1}{2}}} - \frac{|D^\alpha H(\xi, x_n)| |D^\beta H(\xi, x_n)|}{H(\xi, x_n)^{\frac{3}{2}}} \right) (x_n - X(\xi)) \\ &\quad + \frac{D^\alpha H(\xi, x_n)}{H(\xi, x_n)^{\frac{1}{2}}} D^\beta (x_n - X(\xi)) \\ &\quad + H(\xi, x_n)^{\frac{1}{2}} D^\mu (x_n - X(\xi)), \end{aligned}$$

and so

$$\begin{aligned} |D^\mu [G(\xi, x_n)(x_n - X(\xi))]| &\lesssim \left( \frac{\rho(x^\nu)^{2\delta}}{\rho(x^\nu)^{1+\delta}} + \frac{(\rho(x^\nu)^{1+2\delta})^2}{\rho(x^\nu)^{3+3\delta}} \right) \rho(x^\nu) \\ &\quad + \frac{\rho(x^\nu)^{1+2\delta}}{\rho(x^\nu)^{1+\delta}} + \rho(x^\nu)^\delta \\ &\lesssim \rho(x^\nu)^\delta, \end{aligned}$$

and finally also

$$[G(\xi, x_n)(x_n - X(\xi))^2]_{2,\delta} \leq C.$$

It thus follows that

$$\Phi_\nu(\xi, x_n)^2 [f(\xi, x_n) - F(\xi)] = \{\Phi_\nu(\xi, x_n)(x_n - X(\xi)) G(\xi, x_n)\}^2$$

has a  $C^{2,\delta}$  square root  $f_2(x) \equiv \Phi_\nu(\xi, x_n)(x_n - X(\xi)) G(\xi, x_n)$ .

Similarly, the function

$$f_2(x)^2 = \Phi_\nu(x)^2 \{f(\xi, x_n) - F(\xi)\}$$

satisfies the estimates

$$|D^\mu (f_2(x)^2)| \lesssim r(x^\nu)^{4-|\mu|+2\delta}, \quad 0 \leq |\mu| \leq 4,$$

upon using the estimates obtained below for  $D^\mu F(\xi)$ ,  $0 \leq m \leq 4$ , and this then leads to the conclusion that  $f_2^2 \in C^{4,2\delta}$  uniformly in  $\nu$ .

Now we note that all cases have been exhausted by the second assumption in (4.5).

**Note:** *This is a key juncture in the proof since we have thus eliminated consideration of the difficult case in which  $|\nabla^4 f(x)|^{\frac{1}{2\delta}}$  is the dominant term in the definition of  $\rho(x)$ , and where the implicit function is no longer decisive. However, see Bony [Bon] for how to proceed when  $x \in \mathbb{R}$  is one-dimensional.*

Thus altogether we have shown so far that

$$f(x) = \sum_{\nu=1}^{\infty} \Phi_{\nu}(x)^2 f(x),$$

where for each  $\nu$ , after a rotation of coordinates depending on  $\nu$ , either

$$\begin{aligned} \Phi_{\nu}(x)^2 f(x) &= g_{\nu}(x)^2, \\ \text{where } g_{\nu} &\in C^{2+\delta}(B_{\nu}), \end{aligned}$$

or

$$\begin{aligned} \Phi_{\nu}(x)^2 f(x) &= \Phi_{\nu}(x)^2 F(\xi) + \Phi_{\nu}(x)^2 H(\xi, x_n) (x_n - X(\xi))^2 \\ &= \Phi_{\nu}(x)^2 \kappa_{\nu}(x) + g_{\nu}(x)^2, \\ \text{where } \kappa_{\nu} &\in C^{4+2\delta}(B_{\nu}) \text{ and } \Phi_{\nu}, g_{\nu} \in C^{2+\delta}(B_{\nu}). \end{aligned}$$

Finally we use the bounded overlap of the balls  $B_{\nu}$  to write  $\mathbb{N} = \bigcup_{\ell=1}^N A_{\ell}$  as a finite pairwise disjoint union of index sets  $A_{\ell}$  such that for each  $\ell$  the balls  $\{B_{\nu}\}_{\nu \in A_{\ell}}$  have pairwise disjoint triples. Then we group the sum of all the functions into finitely many functions  $h_{\ell}(x) = \sum_{\nu \in A_{\ell}} \Phi_{\nu}(x)^2 \kappa_{\nu}(x)$  and  $g_{\ell}(x) = \sum_{\nu \in B_{\ell}} g_{\nu}(x)^2$  that satisfy the conclusions of the theorem, save for the assertion that  $\kappa_{\nu}$  satisfies (4.5), to which we now turn.

In order to prove assertion (ii) of Theorem 4.5, we suppose for the moment, and only for the sake of simplicity of calculation, that the dimension is  $n = 2$  and the variable is  $(x, y) \in \mathbb{R}^2$ . Then for a function arising from Case II, which is the only case that is nontrivial, we have that

$$(f_{22}(x_{\nu}))^{\frac{1}{2+2\delta}} \approx \rho_f = \max \left\{ f^{\frac{1}{4+2\delta}}, \left( \sup_{\theta \in \mathbb{S}^{n-1}} [\partial_{\theta}^2 f]_+ \right)^{\frac{1}{2+2\delta}}, |\nabla^4 f|^{\frac{1}{2\delta}} \right\}$$

and with  $X = (x, h(x))$  and  $F(x) \equiv f(X)$ ,

$$\begin{aligned} f_2(X) &= 0, \\ h'(x) &= -\frac{f_{12}(X)}{f_{22}(X)}, \\ F'(x) &= f_1(X) + f_2(X) h'(x) = f_1(X) - \frac{f_2(X) f_{12}(X)}{f_{22}(X)}, \\ F''(x) &= f_{11}(X) + f_{12}(X) h'(x) - \partial_1 \left( \frac{f_2(X) f_{12}(X)}{f_{22}(X)} \right) - \partial_2 \left( \frac{f_2(X) f_{12}(X)}{f_{22}(X)} \right) h'(x), \end{aligned}$$

which gives

$$\begin{aligned} F''(x) &= f_{11}(X) - \frac{f_{12}(X) f_{12}(X)}{f_{22}(X)} \\ &\quad - \left( \frac{f_{12}(X) f_{12}(X)}{f_{22}(X)} + \frac{f_2(X) f_{112}(X)}{f_{22}(X)} - \frac{f_2(X) f_{12}(X) f_{122}(X)}{f_{22}(X)^2} \right) \\ &\quad + \left( \frac{f_{22}(X) f_{12}(X)}{f_{22}(X)} + \frac{f_2(X) f_{122}(X)}{f_{22}(X)} - \frac{f_2(X) f_{12}(X) f_{222}(X)}{f_{22}(X)^2} \right) \frac{f_{12}(X)}{f_{22}(X)}, \end{aligned}$$

which when simplified is

$$\begin{aligned}
F''(x) &= f_{11}(X) - \frac{f_{12}(X)^2}{f_{22}(X)} \\
&\quad - \left( \frac{f_{12}(X)^2}{f_{22}(X)} + \frac{f_2(X) f_{112}(X)}{f_{22}(X)} - \frac{f_2(X) f_{12}(X) f_{122}(X)}{f_{22}(X)^2} \right) \\
&\quad + \left( \frac{f_{12}(X)^2}{f_{22}(X)} + \frac{f_2(X) f_{12}(X) f_{122}(X)}{f_{22}(X)^2} - \frac{f_2(X) f_{12}(X)^2 f_{222}(X)}{f_{22}(X)^3} \right) \\
&= f_{11}(X) - \frac{f_{12}(X)^2}{f_{22}(X)} - \frac{f_2(X) f_{112}(X)}{f_{22}(X)} + 2 \frac{f_2(X) f_{12}(X) f_{122}(X)}{f_{22}(X)^2} - \frac{f_2(X) f_{12}(X)^2 f_{222}(X)}{f_{22}(X)^3}.
\end{aligned}$$

We can also write

$$(4.10) \quad F''(x) = f_{11}(X) - \frac{f_{12}(X)^2}{f_{22}(X)} - \frac{f_2(X)}{f_{22}(X)} \left( f_{112}(X) + 2 \frac{f_{12}(X) f_{122}(X)}{f_{22}(X)} - \frac{f_{12}(X)^2 f_{222}(X)}{f_{22}(X)^2} \right).$$

If we use that

$$(f_{22})^{\frac{1}{2+2\delta}} \approx \rho_f = \max \left\{ f^{\frac{1}{4+2\delta}}, \left( \sup_{\theta \in \mathbb{S}^{n-1}} [\partial_\theta^2 f]_+ \right)^{\frac{1}{2+2\delta}}, |\nabla^4 f|^{\frac{1}{2\delta}} \right\},$$

together with the estimates

$$|\nabla^\ell f(x)| \lesssim \rho^{4+2\delta-\ell}, \quad \text{for } \ell \leq 4,$$

we obtain

$$|F''(x)| \lesssim \rho^{2+2\delta} + \frac{\rho^{3+2\delta}}{\rho^{2+2\delta}} (\rho^{1+2\delta}) \approx \rho^{2+2\delta} \approx f_{22}(X).$$

So writing  $\rho_f$  to emphasize the dependence on  $f$ , we have both

$$(4.11) \quad F(x)^{\frac{1}{4+2\delta}} = f(X)^{\frac{1}{4+2\delta}} \leq \rho_f \text{ and } |F''(x)|^{\frac{1}{2+2\delta}} \lesssim \rho_f.$$

Suppose for the moment that we could show

$$(4.12) \quad |\nabla^4 F(x)|^{\frac{1}{2\delta}} \lesssim \rho_f$$

as well. Then we would conclude that since we are in Case II,

$$\begin{aligned}
\rho_F(x) &= \max \left\{ F(x)^{\frac{1}{4+2\delta}}, \left( \sup_{\theta \in \mathbb{S}^{n-2}} [\partial_\theta^2 F(x)]_+ \right)^{\frac{1}{2+2\delta}}, |\nabla^4 F(x)|^{\frac{1}{2\delta}} \right\} \\
&\approx \max \left\{ F(x)^{\frac{1}{4+2\delta}}, |\nabla^2 F(x)|^{\frac{1}{2+2\delta}}, |\nabla^4 F(x)|^{\frac{1}{2\delta}} \right\} \lesssim \rho_f(X) \approx f_{22}(X)^{\frac{1}{2+2\delta}},
\end{aligned}$$

which together with the second assumption in (4.5), namely  $f_{22}(X) \lesssim f(X)^{\frac{\delta}{2+\delta}(2+2\delta)}$ , would yield the crucial inductive hypotheses (4.5) for the function  $F$  in place of  $f$ , namely

$$\begin{aligned}
\rho_F(x) &\lesssim f_{22}(X)^{\frac{1}{2+2\delta}} \lesssim f(X)^{\frac{\delta}{2+\delta}} = F(x)^{\frac{\delta}{2+\delta}}, \\
\text{hence } |\nabla^2 F(x)| &\lesssim \rho_f(X)^{2+2\delta} \approx f_{22}(X) \lesssim f(X)^{\frac{\delta}{2+\delta}(2+2\delta)} = F(x)^{\frac{\delta}{2+\delta}(2+2\delta)}.
\end{aligned}$$

In higher dimensions  $n > 2$ , a straightforward exercise in extending notation shows that if  $F_1$  denotes the analogue of  $F$  above, then  $F_1(x)$  is a function of  $n-1$  variables and

$$\rho_{F_1}(x) \lesssim F_1(x)^{\frac{\delta}{2+\delta}}.$$

If we iterate this entire construction  $k$  times to obtain a function  $F_k(x)$  of  $n-k$  variables on a small ball at the  $k^{\text{th}}$  inductive step, then

$$|\nabla^4 F_{k+1}| \lesssim \rho_{F_k} \lesssim \rho_{F_{k-1}} \lesssim \dots \lesssim \rho_{F_1} \lesssim \rho_f = f_{22}^{\frac{1}{2+2\delta}} \lesssim f^{\frac{\delta}{2+\delta}} = F^{\frac{\delta}{2+\delta}},$$

and from (4.11) we conclude that

$$|\nabla^2 F_{k+1}| \lesssim \rho_{F_k}^{2+2\delta} \lesssim \rho_f^{2+2\delta} = f_{22} \lesssim f^{\frac{\delta}{2+\delta}(2+2\delta)} = F^{\frac{\delta}{2+\delta}(2+2\delta)}.$$

Thus it remains only to calculate and estimate  $\nabla^4 F(x)$ , for which we return to dimension  $n = 2$ , and to prove (4.12). From (4.10) we have

$$\begin{aligned} F''''(x) &= \frac{d^2}{dx^2} \left( f_{11}(X) - \frac{f_{12}(X)^2}{f_{22}(X)} \right) \\ &\quad - \left[ \frac{d^2}{dx^2} \frac{f_2(X)}{f_{22}(X)} \right] \left( f_{112}(X) + 2 \frac{f_{12}(X) f_{122}(X)}{f_{22}(X)} - \frac{f_{12}(X)^2 f_{222}(X)}{f_{22}(X)^2} \right) \\ &\quad - \left[ \frac{d}{dx} \frac{f_2(X)}{f_{22}(X)} \right] \frac{d}{dx} \left( f_{112}(X) + 2 \frac{f_{12}(X) f_{122}(X)}{f_{22}(X)} - \frac{f_{12}(X)^2 f_{222}(X)}{f_{22}(X)^2} \right) \\ &\quad - \frac{f_2(X)}{f_{22}(X)} \frac{d^2}{dx^2} \left( f_{112}(X) + 2 \frac{f_{12}(X) f_{122}(X)}{f_{22}(X)} - \frac{f_{12}(X)^2 f_{222}(X)}{f_{22}(X)^2} \right), \end{aligned}$$

and a lengthy and tedious calculation, using only the chain rule, the product rule, the estimates (4.9), and the equivalence  $\rho_f(X) \approx f_{22}(X)$  in force in Case II, shows that each of these terms is dominated in modulus by  $\rho_f(X)^{2\delta}$ . For example

$$\begin{aligned} &\frac{d^2}{dx^2} \left( f_{11}(X) - \frac{f_{12}(X)^2}{f_{22}(X)} \right) \\ &= \frac{d}{dx} \left( f_{111}(X) + f_{112}(X) h'(x) - \frac{f_{22}(X) 2f_{12}(X) [f_{112}(X) + f_{122}(X) h'(x)] - f_{12}(X)^2 [f_{122}(X) + f_{222}(X) h'(x)]}{f_{22}(X)^2} \right) \\ &= f_{1111}(X) - f_{1112}(X) \frac{f_{12}(X)}{f_{22}(X)} - \frac{d}{dx} \left[ \frac{f_{112}(X) f_{12}(X)}{f_{22}(X)} \right] \\ &\quad - \frac{d}{dx} \frac{f_{22}(X) 2f_{12}(X) \left[ f_{112}(X) - \frac{f_{122}(X) f_{12}(X)}{f_{22}(X)} \right] - f_{12}(X)^2 \left[ f_{122}(X) - \frac{f_{222}(X) f_{12}(X)}{f_{22}(X)} \right]}{f_{22}(X)^2}, \end{aligned}$$

which is dominated by  $C\rho_f(X)^{2\delta}$  upon using the estimates  $|\nabla^\ell f(x)| \lesssim \rho(x)^{4+2\delta-\ell}$ . By way of illustration, we compute that the third term satisfies

$$\begin{aligned} \frac{d}{dx} \left[ \frac{f_{112}(X) f_{12}(X)}{f_{22}(X)} \right] &= \frac{[f_{1112}(X) + f_{1122}(X) h'(x)] f_{12}(X)}{f_{22}(X)} \\ &\quad + \frac{f_{112}(X) [f_{112}(X) + f_{122}(X) h'(x)]}{f_{22}(X)} \\ &\quad - \frac{[f_{1112}(X) + f_{1122}(X) h'(x)] f_{12}(X)}{f_{22}(X)^2} [f_{122}(X) + f_{222}(X) h'(x)], \end{aligned}$$

and the estimates (4.9) then show both

$$\begin{aligned} \left| \frac{d}{dx} \left[ \frac{f_{112}(X) f_{12}(X)}{f_{22}(X)} \right] \right| &\leq C\rho(X)^{2\delta}, \\ \text{and } \left[ \frac{f_{112} f_{12}}{f_{22}} \right]_{1,2\delta}(X) &\leq C. \end{aligned}$$

The remaining calculations are similar, and the analogous derivative calculations in higher dimensions  $n > 2$  are a straightforward exercise in extending notation. ■

**4.2. A two dimensional SOS decomposition.** Here we sketch the proof of a decomposition into a sum of squares of  $C^{2,\delta}(\mathbb{R}^2)$  functions in the plane, in which the second differential inequality in (4.5) can be dropped. In dimension  $n \geq 5$ , this second inequality cannot be dropped as shown in Remark 4.10 below.

**Theorem 4.7.** *Suppose that  $f \in C^{4,2\delta}(\mathbb{R}^2)$  and satisfies the pointwise inequality*

$$(4.13) \quad |\nabla^4 f(x)| \leq f(x)^{\frac{\delta}{2+\delta}}.$$

Then  $f = \sum_{\ell=1}^N g_\ell^2$  can be decomposed as a finite sum of squares of functions  $g_\ell \in C^{2+\delta}(\mathbb{R}^2)$  where

$$\begin{aligned} |D^\alpha g_\ell(x)| &\leq C\rho(x)^{2+\delta-|\alpha|}, & 0 \leq |\alpha| \leq 2, \\ [g_\ell]_{\alpha,\delta}(x) &\leq C, & |\alpha| = 2. \end{aligned}$$

and

$$\begin{aligned} |D^\alpha g_\ell^2(x)| &\leq C\rho(x)^{4+2\delta-|\alpha|}, & 0 \leq |\alpha| \leq 4, \\ [g_\ell^2]_{\alpha,2\delta}(x) &\leq C, & |\alpha| = 4. \end{aligned}$$

*Proof.* The pointwise inequality on  $|\nabla^4 f|$  shows that  $|\nabla^4 f(x)|^{\frac{1}{2\delta}} \leq f(x)^{\frac{1}{4+2\delta}}$ , and hence

$$\begin{aligned} \rho(x) &\equiv \max \left\{ f(x)^{\frac{1}{4+2\delta}}, \left( \sup_{\theta \in \mathbb{S}^{n-1}} [\partial_\theta^2 f(x)]_+ \right)^{\frac{1}{2+2\delta}}, |\nabla^4 f(x)|^{\frac{1}{2\delta}} \right\} \\ &= \max \left\{ f(x)^{\frac{1}{4+2\delta}}, \left( \sup_{\theta \in \mathbb{S}^{n-1}} [\partial_\theta^2 f(x)]_+ \right)^{\frac{1}{2+2\delta}} \right\}, \end{aligned}$$

by (4.13). Now the Hölder argument of Bony [Bon, Subsection 5.1] proves the result since there is no Case III, and the function  $F(x') = f(x', X(x'))$  that arises in Case II of the argument is in  $C^{4,2\delta}(\mathbb{R})$ , and so Bony's one-dimensional result shows that  $F$  can be written as a sum of two squares of  $C^{2,\delta}(\mathbb{R})$  functions. Now we proceed with the Fefferman-Phong argument as modified by Bony, and along the lines of the argument used in the proof of the provisional Theorem 4.5 above. ■

**4.3. A higher dimensional SOS decomposition.** Here we prove our main decomposition of a smooth nonnegative function into a sum of squares of  $C^{2,\delta}(\mathbb{R}^n)$  functions in arbitrary dimension, but restricted to elliptical flat smooth functions that satisfy certain differential inequalities, that are in turn implied by assuming  $f$  is  $\omega_s$ -monotone for appropriate  $0 < s < 1$ .

**Theorem 4.8.** *Suppose  $0 < \delta < 1$ , that  $f$  is a  $C^{4,2\delta}$  function on  $\mathbb{R}^n$ , and that  $\rho(x)$  is as defined in the formula (4.4) above.*

(1) *If  $f$  satisfies both of the differential inequalities in (4.5), i.e.*

$$|\nabla^4 f(x)| \leq C f(x)^{\frac{\delta}{2+\delta}} \quad \text{and} \quad |\nabla^2 f(x)| \leq C f(x)^{\frac{2\delta(1+\delta)}{2+\delta}},$$

*then  $f = \sum_{\ell=1}^N g_\ell^2$  can be decomposed as a finite sum of squares of functions  $g_\ell \in C^{2+\delta}(\mathbb{R}^2)$  where*

$$\begin{aligned} |D^\alpha g_\ell(x)| &\leq C\rho(x)^{2+\delta-|\alpha|} \leq C f(x)^{\frac{\delta}{2+\delta}(2+\delta-|\alpha|)}, & 0 \leq |\alpha| \leq 2, \\ [g_\ell]_{\alpha,\delta}(x) &\leq C, & |\alpha| = 2. \end{aligned}$$

*and*

$$\begin{aligned} |D^\alpha g_\ell^2(x)| &\leq C\rho(x)^{4+2\delta-|\alpha|} \leq C f(x)^{\frac{\delta}{2+\delta}(4+2\delta-|\alpha|)}, & 0 \leq |\alpha| \leq 4, \\ [g_\ell^2]_{\alpha,2\delta}(x) &\leq C, & |\alpha| = 4. \end{aligned}$$

(2) *In particular, the inequalities (4.5) hold provided  $f$  is also flat, smooth and  $\omega_s$ -monotone for some  $s < 1$  satisfying*

$$(4.14) \quad s > \sqrt[4]{\delta} \sqrt[4]{\frac{2}{4+2\delta}}, \quad \text{and} \quad s \geq \sqrt{\delta} \sqrt{\frac{2+2\delta}{2+\delta}}.$$

**Remark 4.9.** *The inequalities (4.5) also hold if the smoothness assumption on  $f$  is relaxed to  $f \in C^k$ , provided that  $s$  is replaced by  $s - \frac{C}{k}$  in (4.14) for a sufficiently large constant  $C$ . See the proof of Theorem 3.7.*

*Proof.* For (1) use induction on dimension together with Theorem 4.5, and for (2) use inequality (3.5) in Theorem 3.7. ■

**Remark 4.10.** The counterexamples in [BoBrCoPe] show that the differential inequalities in (4.5) cannot both be dropped. The first differential inequality in (4.5) plays an important role in reducing matters to cases in the proof where the implicit function theorem can be applied decisively, and we now show that the second differential inequality in (4.5) cannot be dropped either, not even significantly weakened. Indeed, fix  $\delta > 0$ . If we set  $\beta = \delta$  in part (1) of Theorem 2.5, then inequality  $\delta > s$  implies that there is an elliptical flat smooth  $\omega_s$ -monotone function  $f$  that cannot be written as a finite sum of squares of  $C^{2,\beta}$  functions, contradicting Theorem 4.8, and thus showing that we cannot relax the second inequality in (4.14) to just  $s > \delta$ . On the other hand,  $\delta > s$  is compatible with the first inequality in (4.14) for  $\delta$  close to 1, since  $\delta > s \geq \sqrt[4]{\delta} \sqrt[4]{\frac{1\delta}{2+\delta}}$  holds for  $\delta = \frac{1}{\sqrt[3]{2}} = 0.7937$ .

The utility of this theorem for our purposes lies in part (2), where given any  $0 < s < 1$ , we can find  $\delta > 0$  so small that (4.14) is satisfied. We also have the following extension of this theorem to  $C^{2m,2\delta}$  functions  $f$  on  $\mathbb{R}^n$ , but where we no longer have the same utility. Here the control distance that is used in the proof is

$$(4.15) \quad \rho_f(x) \equiv \max \left\{ f(x)^{\frac{1}{2m+2\delta}}, \left( \sup_{\theta \in \mathbb{S}^{n-1}} [\partial_\theta^2 f(x)]_+ \right)^{\frac{1}{2m-2+2\delta}}, |\nabla^4 f(x)|^{\frac{1}{2m-4+2\delta}}, \dots, |\nabla^{2m} f(x)|^{\frac{1}{2\delta}} \right\},$$

and the differential inequalities we impose include  $|\nabla^{2(m-p)} f(x)| \leq C f(x)^{\frac{p+\delta}{m+\delta}}$  for  $0 \leq p \leq m-2$ . In particular this requires  $s > \sqrt[2(m-1)]{\frac{1}{m}}$ , which forces  $s$  closer and closer to 1 as  $m \rightarrow \infty$  since  $\lim_{m \rightarrow \infty} m^{\frac{1}{2m-1}} = 1$ . As a consequence, the following theorem will not be used for hypoellipticity in the paper [KoSa3], and we give only a sketch of the proof, leaving the details to the interested reader.

**Theorem 4.11.** Suppose  $0 < \delta < 1$  and that  $f$  is a  $C^{2m,2\delta}$  function on  $\mathbb{R}^n$ .

(1) If  $f$  satisfies the differential inequalities,

$$(4.16) \quad \begin{aligned} |\nabla^{2(m-p)} f(x)| &\leq C f(x)^{\frac{p+\delta}{m+\delta}}, \quad \text{for } 0 \leq p \leq m-2, \\ \text{and } |\nabla^2 f| &\lesssim f^{\frac{2\delta(m-1+\delta)}{m+\delta}}, \end{aligned}$$

then  $f$  can be decomposed as a finite sum of squares of  $C^{m,\delta}(\mathbb{R}^n)$  functions.

(2) In particular, the inequalities (4.16) hold provided  $f$  is also flat, smooth and  $\omega_s$ -monotone for some  $s < 1$  satisfying

$$(4.17) \quad \begin{aligned} s &> \sqrt[2m]{\frac{\delta}{m+\delta}}, \\ s &> \sqrt[2(m-p)]{\frac{p+\delta}{m+\delta}} \text{ for } 1 \leq p \leq m-1, \\ \text{and } s &> \sqrt[2]{2\delta} \sqrt[2]{\frac{m-1+\delta}{m+\delta}}. \end{aligned}$$

*Proof.* The proof of Theorem 4.11 is similar to that of Theorem 4.8 if we use  $\rho_f$  defined by (4.15). Just as in the proof of Theorem 4.8, the differential inequalities (4.16) show that

$$\rho_f(x) \approx \max \left\{ f(x)^{\frac{1}{2m+2\delta}}, \left( \sup_{\theta \in \mathbb{S}^{n-1}} [\partial_\theta^2 f(x)]_+ \right)^{\frac{1}{2m-2+2\delta}} \right\},$$

and so only Cases I and II arise in the arguments. The proof that  $\rho_F(x) \lesssim \rho_f(x)$  in Case II is the same, and the proof that  $F$  satisfies the differential inequalities (4.16), is simply a more complicated version of the analogous calculation in the proof of Theorem 4.8. Indeed, we modify the proof of assertion (ii) in the provisional Theorem 4.5 when  $n = 2$  as follows. In Case II, we have that

$$(f_{22})^{\frac{1}{2m-2+2\delta}} \approx \rho_f = \max \left\{ f^{\frac{1}{2m+2\delta}}, \left( \sup_{\theta \in \mathbb{S}^{n-1}} [\partial_\theta^2 f]_+ \right)^{\frac{1}{2m-2+2\delta}}, |\nabla^4 f|^{\frac{1}{2m-4+2\delta}}, \dots, |\nabla^{2m} f|^{\frac{1}{2\delta}} \right\},$$

and since  $f_{22} \lesssim f^{(s')^2}$  by Theorem 3.7, we then have

$$\rho_f \lesssim (f_{22})^{\frac{1}{2m-2+2\delta}} \lesssim f^{\frac{(s')^2}{2m-2+2\delta}} \leq f^{\frac{(\sqrt[2]{2\delta} \sqrt[2]{\frac{m-1+\delta}{m+\delta}})^2}{2m-2+2\delta}} = f^{\frac{\delta}{m+\delta}},$$

if we choose  $s > s' > \sqrt{2\delta} \sqrt{\frac{m-1+\delta}{m+\delta}}$ , which we can by the third line in (4.17). The proof now proceeds as before. ■

## 5. COUNTEREXAMPLES

Here we begin by constructing an example of an elliptical flat smooth function  $f$  on  $B_{\mathbb{R}^4}(0, 1) \times (-1, 1)$  that cannot be written as a finite sum of squares of  $C^{2,\beta}$  functions for  $\beta > 0$ . Even more, we prove the following result that answers a question in [BoBrCoPe, Remark 1.4].

**Theorem 5.1.** *Given any modulus of continuity  $\omega$ , there is an elliptical flat smooth function  $f$  on  $B_{\mathbb{R}^4}(0, 1) \times (-1, 1)$  that cannot be written as a finite sum of squares of  $C^{2,\omega}$  functions.*

Then we investigate the connection between  $\omega_s$ -monotonicity and these counterexamples. To construct our counterexample we modify the example in [BoBrCoPe, Theorem 1.2 (d)] by adding an additional term  $\eta(t, r)$ ,  $r = |(w, x, y, z)|$ , and to prepare for this we modify the construction of the function  $C(\varepsilon)$  appearing in their argument. But first recall the following lemma, where

$$L(w, x, y, z) \equiv w^4 + x^2y^2 + y^2z^2 + z^2x^2 - 2wxyz, \quad (w, x, y, z) \in \mathbb{R}^4.$$

and for a modulus of continuity  $\omega$  and  $h$  defined on the unit ball  $B_{\mathbb{R}^4}(0, 1)$  in  $\mathbb{R}^4$ ,

$$\|h\|_{C^{2,\omega}(B_{\mathbb{R}^4}(0,1))} \equiv \sum_{k=0}^2 \left\| \nabla^k h \right\|_{L^\infty(B_{\mathbb{R}^4}(0,1))} + \sup_{W, W' \in B_{\mathbb{R}^4}(0,1)} \frac{|\nabla^2 h(W) - \nabla^2 h(W')|}{\omega(|W - W'|)}.$$

**Lemma 5.2** ([BoBrCoPe, Theorem 1.2 (d)]). *Let  $\omega$  be a modulus of continuity. For every  $\nu \in \mathbb{N}$  there is a decreasing function  $\mathcal{C}_\nu : (0, 1) \rightarrow (0, \infty)$  such that*

$$\lim_{\tau \searrow 0} \mathcal{C}_\nu(\tau) = \infty,$$

$$\sum_{j=1}^{\nu} \|g_{j,\tau}\|_{C^{2,\omega}(B_{\mathbb{R}^4}(0,1))} \geq \mathcal{C}_\nu(\tau),$$

whenever  $\{g_{j,\tau}\}_{j=1}^{\nu} \subset C^{2,\omega}(B_{\mathbb{R}^4}(0, 1))$  satisfy

$$(5.1) \quad L(w, x, y, z) + \tau = \sum_{j=1}^{\nu} g_{j,\tau}(w, x, y, z)^2, \quad \text{for } (w, x, y, z) \in B_{\mathbb{R}^4}(0, 1).$$

*Proof.* Fix  $\nu \in \mathbb{N}$ . Suppose, in order to derive a contradiction, that for all  $0 < \tau < 1$ , there are  $\nu$  functions  $\{g_{j,\tau}\}_{j=1}^{\nu} \subset C^{2,\omega}(B_{\mathbb{R}^4}(0, 1))$  satisfying (5.1) and  $\|g_{j,\tau}\|_{C^{2,\omega}} \leq C$ , for a constant  $C$  independent of  $\tau$ . Then the collection of functions  $\{g_{j,\tau}\}_{1 \leq j \leq \nu, 0 < \tau < 1}$  is bounded in  $C^{2,\omega}(B_{\mathbb{R}^4}(0, 1))$ , and hence compact in  $C^2(B_{\mathbb{R}^4}(0, 1))$ . Thus there is a decreasing sequence  $\{\tau_n\}_{n=1}^{\infty} \subset (0, 1)$  and a set of  $\nu$  functions  $\{g_j\}_{j=1}^{\nu} \subset C^2(B_{\mathbb{R}^4}(0, 1))$  such that  $g_{j,\tau_n} \rightarrow g_j$  in  $C^2(B_{\mathbb{R}^4}(0, 1))$  for each  $1 \leq j \leq \nu$ , and it follows from (5.1) that

$$L(w, x, y, z) = \sum_{j=1}^{\nu} g_j(w, x, y, z)^2, \quad \text{for } (w, x, y, z) \in B_{\mathbb{R}^4}(0, 1),$$

contradicting [BoBrCoPe, Theorem 1.2 (c)]. ■

Now we construct a lower bound  $\mathcal{C}(\tau)$  for  $\{\mathcal{C}_\nu(\tau)\}_{\nu=1}^{\infty}$  as in [BoBrCoPe]. First, use Lemma 5.2 to choose a strictly decreasing sequence  $\{\tau_n\}_{n=1}^{\infty} \subset (0, 1)$  such that

$$\mathcal{C}_\nu(\tau) \geq \tau^2, \quad \text{for } 0 < \tau \leq \tau_n \text{ and } \nu \leq n,$$

and then

$$\mathcal{C}(\tau) \equiv \sum_{n=1}^{\infty} n \mathbf{1}_{[\tau_{n+1}, \tau_n)}(\tau),$$

so that we have

$$(5.2) \quad \lim_{\tau \searrow 0} \mathcal{C}(\tau) = \infty \text{ and } \lim_{\tau \searrow 0} \frac{\mathcal{C}_\nu(\tau)}{\mathcal{C}(\tau)} = \infty, \quad \text{for all } \nu \in \mathbb{N}.$$

It is clear that we can now modify  $\mathcal{C}$  to be strictly decreasing and still satisfy (5.2).

Now let  $\varphi : (0, 1) \rightarrow (0, 1)$  be a strictly increasing elliptical flat smooth function on  $(0, 1)$ , and with  $r = |(w, x, y, z)|$  define

$$(5.3) \quad f(w, x, y, z, t) \equiv \varphi(t) L(w, x, y, z) + \psi(t) + \eta(t, r)$$

where  $\psi(t)$  and  $\eta(t, r)$  are smooth nonnegative functions constructed as follows, in order that  $f$  is elliptical on  $B_{\mathbb{R}^4}(0, 1) \times (-1, 1)$ , yet cannot be written as a finite sum of squares of  $C^{2, \omega}$  functions.

The function  $\psi(t)$  is constructed similar to that in [BoBrCoPe] but incorporating an additional function as follows. First we fix a smooth strictly increasing function  $\lambda : (0, 1) \rightarrow (0, 1)$  with  $\lim_{r \searrow 0} \lambda(r) = 0$ , so that the inverse function  $\lambda^{-1}(t)$  is also strictly increasing with limit 0 at the origin. We will almost exclusively choose  $\lambda(r) = r$  for  $0 < r < 1$ . Next, we choose a nondecreasing flat elliptical function  $\psi_0$  on  $(-1, 1)$  such that  $\frac{\psi_0(t)}{\varphi(t)\lambda^{-1}(t)^4}$  is also nondecreasing and

$$(5.4) \quad \frac{1}{\sqrt{\varphi(t)\lambda^{-1}(t)^2}} \leq \mathcal{C} \left( \frac{\psi_0(t)}{\varphi(t)\lambda^{-1}(t)^4} \right), \quad 0 < t < 1,$$

e.g. using that  $\mathcal{C}$  may be assumed strictly decreasing, we may take

$$\psi_0(t) \equiv \varphi(t)\lambda^{-1}(t)^4 \mathcal{C}^{-1} \left( \frac{1}{\sqrt{\varphi(t)\lambda^{-1}(t)^2}} \right).$$

Then in order to obtain a *smooth* such function, set

$$\psi(t) \equiv \int_{\frac{t}{2}}^t \psi_0(s) g\left(\frac{t-s}{t}\right) \frac{ds}{t},$$

where  $g$  is smooth nonnegative function supported in  $(0, \frac{1}{2})$  with  $\int g = 1$ . Then  $\psi(t)$  is smooth and because  $\psi_0$  is nondecreasing, we conclude from the definition of  $\psi$ , that  $\psi$  is also nondecreasing, and moreover that  $\psi(t) \leq \psi_0(t)$  for  $0 < t < 1$ . Finally, since  $\mathcal{C}(\tau)$  is decreasing, we obtain from (5.4) that

$$(5.5) \quad \frac{1}{\sqrt{\varphi(t)\lambda^{-1}(t)^2}} \leq \mathcal{C} \left( \frac{\psi(t)}{\varphi(t)\lambda^{-1}(t)^4} \right), \quad 0 < t < 1.$$

The function  $\eta(t, r)$  is chosen to have the form  $\eta(t, r) = \sigma(r) h\left(\frac{t}{\lambda(r)}\right)$  where  $h$  is a smooth nonnegative function on  $(-1, 1)$  with  $h(0) = 1$ , and where  $\sigma(r)$  is an elliptical flat smooth function on  $(0, 1)$ , chosen so small that  $\eta(t, r)$  is a flat smooth function on  $(-1, 1) \times (0, 1)$ . More precisely we need only choose  $\sigma(r)$  small enough so that for all  $m, n \in \mathbb{N}$ ,

$$(5.6) \quad \begin{aligned} \frac{\partial^{m+n}}{\partial t^m \partial r^n} \eta(t, r) &= \frac{\partial^n}{\partial r^n} \left( \frac{\sigma(r)}{\lambda(r)^m} h^{(m)}\left(\frac{t}{\lambda(r)}\right) \right) \\ &= \sum_{k=0}^n \binom{n}{k} \frac{\partial^k}{\partial r^k} \left( \frac{\sigma(r)}{\lambda(r)^m} \right) \frac{\partial^{n-k}}{\partial r^{n-k}} h^{(m)}\left(\frac{t}{\lambda(r)}\right) \end{aligned}$$

tends to 0 as  $(t, r) \rightarrow (0, 0)$ . Thus we now have

$$(5.7) \quad f(w, x, y, z, t) \equiv \varphi(t) L(w, x, y, z) + \psi(t) + \sigma(r) h\left(\frac{t}{\lambda(r)}\right).$$

With these constructions completed, we see that  $f$  is an elliptical flat smooth function on  $B_{\mathbb{R}^4}(0, 1) \times (-1, 1)$ . Now suppose, in order to derive a contradiction, that  $f = \sum_{j=1}^{\nu} G_j^2$  where  $G_j \in C^{2, \omega}(B_{\mathbb{R}^4}(0, 1) \times (-1, 1))$ , i.e.

$$\begin{aligned} \varphi(t) L(w, x, y, z) + \psi(t) + \sigma(r) h\left(\frac{t}{\lambda(r)}\right) &= \sum_{j=1}^{\nu} G_j(w, x, y, z, t)^2, \\ \text{for } (w, x, y, z, t) &\in B_{\mathbb{R}^4}(0, 1) \times (-1, 1). \end{aligned}$$

Then since  $h\left(\frac{t}{\lambda(r)}\right)$  vanishes for  $\lambda(r) \leq |t|$ , i.e.  $r \leq \lambda^{-1}(|t|)$ , we have with  $W \equiv (w, x, y, z)$  and wlog  $t > 0$ , that

$$\varphi(t) L(W) + \psi(t) = \sum_{j=1}^{\nu} G_j(W, t)^2, \quad \text{for } r \leq \lambda^{-1}(|t|),$$

and rescaling  $W$  by  $\lambda^{-1}(t)$  we have,

$$\begin{aligned} \varphi(t) L(\lambda^{-1}(t) W) + \psi(t) &= \sum_{j=1}^{\nu} G_j(\lambda^{-1}(t) W, t)^2, \\ \text{for } r &= |W| < 1, t \in (0, 1). \end{aligned}$$

Multiplying by  $\frac{1}{\varphi(t)\lambda^{-1}(t)^4}$ , and using that  $L$  is homogeneous of degree four, we obtain

$$\begin{aligned} L(W) + \frac{\psi(t)}{\varphi(t)\lambda^{-1}(t)^4} &= \sum_{j=1}^{\nu} \left( \frac{G_j(\lambda^{-1}(t) W, t)}{\sqrt{\varphi(t)\lambda^{-1}(t)^2}} \right)^2, \\ \text{for } r &= |W| < 1, t \in (0, 1). \end{aligned}$$

Since  $G_j \in C^{2,\omega}(B_{\mathbb{R}^4}(0, 1) \times (-1, 1))$ , the functions  $W \rightarrow G_j(W, t)$  lie in a bounded set in  $C^{2,\omega}(B_{\mathbb{R}^4}(0, 1))$  independent of  $t$  and  $j$ , and hence also the collection of functions

$$H_j^t(W) \equiv G_j(\lambda^{-1}(t) W, t), \quad 1 \leq j \leq \nu, t \in (0, 1),$$

is bounded in  $C^{2,\omega}(B_{\mathbb{R}^4}(0, 1))$ , say  $\sum_{j=1}^{\nu} \|H_j^t\|_{C^{2,\omega}(B_{\mathbb{R}^4}(0, 1))} \leq \mathfrak{N}_{\nu}$ . Thus with  $\tau = \tau(t) \equiv \frac{\psi(t)}{\varphi(t)\lambda^{-1}(t)^4}$ , we have from Lemma 5.2 and (5.5) that

$$\begin{aligned} \frac{\mathfrak{N}_{\nu}}{\sqrt{\varphi(t)\lambda^{-1}(t)^2}} &\geq \sum_{j=1}^{\nu} \left\| \frac{H_j^t}{\sqrt{\varphi(t)\lambda^{-1}(t)^2}} \right\|_{C^{2,\omega}(B_{\mathbb{R}^4}(0, 1))} \geq \mathcal{C}_{\nu}(\tau(t)) \\ &= \frac{\mathcal{C}_{\nu}(\tau(t))}{\mathcal{C}(\tau(t))} \mathcal{C}(\tau(t)) \geq \frac{\mathcal{C}_{\nu}(\tau(t))}{\mathcal{C}(\tau(t))} \frac{1}{\sqrt{\varphi(t)\lambda^{-1}(t)^2}}, \end{aligned}$$

which contradicts  $\lim_{\tau \searrow 0} \frac{\mathcal{C}_{\nu}(\tau)}{\mathcal{C}(\tau)} = \infty$  in (5.2), provided that we choose  $\psi(t)$  to satisfy in addition that

$$(5.8) \quad \lim_{t \searrow 0} \frac{\psi(t)}{\varphi(t)\lambda^{-1}(t)^4} = \lim_{t \searrow 0} \tau(t) = 0.$$

This completes our construction of an elliptical flat smooth function  $f$  on  $B_{\mathbb{R}^4}(0, 1) \times (-1, 1)$  as in (5.3) that cannot be written as a finite sum of squares of  $C^{2,\omega}$  functions.

**Remark 5.3.** *In order to derive a contradiction in the above argument, it is enough to take  $t$  so small that*

$$\frac{\mathcal{C}_{\nu}(\tau(t))}{\mathcal{C}(\tau(t))} > \mathfrak{N}_{\nu} = \sum_{j=1}^{\nu} \|H_j^t\|_{C^{2,\omega}(B_{\mathbb{R}^4}(0, 1))}.$$

**5.1. Connection with weak monotonicity.** Here we investigate conditions on  $0 < s < 1$  under which the function  $f$  in (5.7) above, i.e.

$$f(w, x, y, z, t) \equiv \varphi(t) L(w, x, y, z) + \psi(t) + \sigma(r) h\left(\frac{t}{\lambda(r)}\right),$$

is  $\omega_s$ -monotone on  $B_{\mathbb{R}^4}(0, 1) \times (-1, 1)$ , resulting in the following theorem that connects the parameter  $s$  to the functions  $\varphi$  and  $\psi$  in the definition of the flat function  $f_{\varphi, \psi, \sigma}$ . We will assume the following further restrictions on  $f$ .

**Further restrictions:** We suppose that the functions  $\varphi, \psi, \sigma, \lambda$ , and  $h$  in the construction of  $f_{\varphi, \psi, \sigma}$  in (5.7) also satisfy

- (1)  $\psi(t) = o(\varphi(t)t^4)$  as  $t \searrow 0$ ,
- (2)  $\sigma(t) = \varphi(t)$ , for  $t > 0$ ,
- (3)  $\lambda(r) = r$ , for  $r > 0$ ,

- (4) there is a constant  $0 < \rho < 1$  such that the function  $h = h_\rho$  is a smooth nonnegative even function on  $\mathbb{R}$  that is decreasing on  $[0, \infty)$ , and satisfies

$$\begin{aligned} h_\rho(x) &= 1, \text{ for } 0 \leq x \leq \rho, \\ 0 &< h_\rho(x) < 1, \text{ for } \rho < x < 1, \\ \text{and } h_\rho(x) &= 0, \text{ for } x \geq 1. \end{aligned}$$

We denote such a function  $f$  by  $f_{\varphi, \psi, h_\rho}$  when we wish to emphasize the dependence on  $\varphi, \psi, h_\rho$ . Recall that for a modulus of continuity  $\omega$ , we defined in (1.1) the  $\omega$ -monotone functional of  $f$  by

$$\|f\|_{\omega\text{-mon}} \equiv \sup_{x \in B(0,1), y \in B(\frac{x}{2}, \frac{|x|}{2})} \frac{f(y)}{\omega(f(x))}.$$

Now for any functions  $\varphi, \psi$  as above, define the three functionals

$$\begin{aligned} \mathcal{R}_{(\varphi, \psi)}^\omega(\gamma) &\equiv \sup_{0 < t \leq 1} \frac{\psi(t)}{\varphi(t)} \frac{\varphi(\gamma t)}{\omega(\psi(t))}, \quad \text{for } 0 < \gamma < \infty, \\ \mathcal{S}_{(\varphi, \psi)}^\omega(\gamma) &\equiv \sup_{0 < t < 1} \frac{\varphi(\gamma t) t^4}{\omega(\psi(t))}, \quad \text{for } 0 < \gamma < \infty, \\ \mathcal{T}_\varphi^\omega(\gamma) &\equiv \sup_{0 < t < 1} \frac{\varphi(\gamma t) t^4}{\omega(\varphi(t) t^4)}, \quad \text{for } 0 < \gamma < \infty, \end{aligned}$$

where  $\mathcal{R}_{(\varphi, \psi)}^\omega(\gamma) \lesssim \mathcal{S}_{(\varphi, \psi)}^\omega(\gamma)$  for  $0 < \gamma < \infty$ , because  $\psi(t) = o(\varphi(t) t^4)$ .

**Theorem 5.4.** *Set*

$$f_{\varphi, \psi, h_\rho}(W, t) = \varphi(t) L(W) + \psi(t) + \varphi(r) h_\rho\left(\frac{t}{r}\right), \quad r = |W|, W \in \mathbb{R}^4, t > 0,$$

where  $\varphi, \psi, h_\rho$  satisfy the conditions listed above. Let  $\gamma_\alpha \equiv \frac{1+\sqrt{1+\alpha^2}}{2\alpha}$  for  $0 < \alpha < \infty$ .

- (1) Then for  $0 < \rho < 1$ , and every  $\delta > 0$ , the function  $f = f_{\varphi, \psi, h_\rho}$  on  $\mathbb{R}^n$  is elliptical flat and smooth, and there are positive constants  $c_{\rho, \delta}$  and  $C_{\rho, \delta}$ , such that

$$\begin{aligned} (5.9) \quad & c \mathcal{S}_{\varphi, \psi}^\omega\left(\frac{1}{2}\right) + c \mathcal{T}_\varphi^\omega(\gamma_1) \leq \|f_{\varphi, \psi, h_\rho}\|_{\omega\text{-mon}} \\ & \leq C_\delta \left[ \mathcal{R}_{(\varphi, \psi)}^\omega(1 + \delta) + \mathcal{S}_{\varphi, \psi}^\omega\left(\frac{1}{2} + \delta\right) \right] + C_{\rho, \delta} \mathcal{T}_\varphi^\omega(\gamma_\rho + \delta). \end{aligned}$$

- (2) Now take  $\varphi(t) = e^{-\frac{1}{t^2}}$  and suppose  $0 < s < s_0$  where

$$(5.10) \quad s_0 \equiv \left( \frac{1 + \sqrt{2}}{2} \right)^{-2} = 0.68629.$$

- (a) Then there are no functions  $g_\ell \in C^{2, \beta}$  with  $f = f_{\varphi, \psi, \sigma, h_\rho} = \sum_{\ell=1}^\nu g_\ell^2$  for any  $\nu \in \mathbb{N}$  if

$$\lim_{t \searrow 0} \frac{\varphi(t)^{\frac{4}{\beta}} t^{\frac{16}{\beta}}}{\psi(t)} = \infty.$$

- (b) If  $0 < s < \min\{\beta, s_0\}$ , then there is  $0 < \rho < 1$  and a function  $\psi(t)$  such that the elliptical flat smooth function  $f = f_{\varphi, \psi, h_\rho}$  is  $\omega_s$ -monotone but not  $SOS_{\omega_\beta}$ , i.e. there are no functions  $g_\ell \in C^{2, \beta}$  with  $f = f_{\varphi, \psi, \sigma, h_\rho} = \sum_{\ell=1}^\nu g_\ell^2$  for  $\nu \in \mathbb{N}$ .

**Remark 5.5.** The quantities  $\mathcal{R}_{(\varphi, \psi)}^\omega$  and  $\mathcal{S}_{(\varphi, \psi)}^\omega$  in (5.9) are the key functionals controlling the  $\omega$ -monotone functional of  $f_{\varphi, \psi, h_\rho}$ .

- (1) The estimate (5.9) is sharp in the sense that the lower bound ‘equals up to multiplicative constants’ the limit as  $\delta \rightarrow 0$  and  $\rho \rightarrow 1$  of the upper bound, namely

$$\begin{aligned} \mathcal{R}_{(\varphi, \psi)}^\omega(1 + \delta) + \mathcal{S}_{\varphi, \psi}^\omega\left(\frac{1}{2} + \delta\right) + \mathcal{T}_\varphi^\omega(\gamma_\rho + \delta) &\longrightarrow \mathcal{R}_{(\varphi, \psi)}^\omega(1) + \mathcal{S}_{\varphi, \psi}^\omega\left(\frac{1}{2}\right) + \mathcal{T}_\varphi^\omega(\gamma_1) \\ &\approx \mathcal{S}_{\varphi, \psi}^\omega\left(\frac{1}{2}\right) + \mathcal{T}_\varphi^\omega(\gamma_1) \end{aligned}$$

since  $\mathcal{R}_{(\varphi, \psi)}^\omega(1)$  is a constant.

- (2) Note also that the right hand side of (5.9) is dominated by a multiple of the single term  $\mathcal{S}_{\varphi, \psi}^\omega(\gamma_\rho + \delta)$ , but the smaller limiting term  $\mathcal{S}_{\varphi, \psi}^\omega(\gamma_1)$  is already far larger than the lower bound.
- (3) The functional  $\mathcal{T}_\varphi^\omega$  is an admissibility requirement for the function  $\varphi$ , and plays no other role in distinguishing which pairs of functions  $(\varphi, \psi)$  give rise to  $f_{\varphi, \psi, h_\rho}$  being  $\omega$ -monotone.

5.1.1. *Proof of necessity in part (1).* Here we prove the lower bound

$$c\mathcal{S}_{\varphi, \psi}^\omega\left(\frac{1}{2}\right) + c\mathcal{T}_\varphi^\omega(\gamma_1) \leq \|f_{\varphi, \psi, h_\rho}\|_{\omega\text{-mon}}.$$

Given points  $P, Q \in \mathbb{R}^5$  with  $Q \in \partial B\left(\frac{P}{2}, \frac{|P|}{2}\right)$ , we have from  $\omega$ -monotonicity that

$$\frac{f(Q)}{\omega(f(P))} \leq \|f\|_{\omega\text{-mon}}.$$

We now consider two specific pairs of points  $(P_1, Q_1)$  and  $(P_2, Q_2)$ , in order to derive the lower bounds above.

Let  $P_1 \equiv (0, t)$  and  $Q_1 \equiv (W, \frac{t}{2})$ , where  $W$  is any point in  $\mathbb{R}^4$  with  $r = |W| = \frac{t}{2}$ , so that  $Q_1 \in \partial B\left(\frac{P_1}{2}, \frac{|P_1|}{2}\right)$ , and

$$\begin{aligned} f(P_1) &= f(0, t) = \psi(t), \\ f(Q_1) &= f(W, |W|) = \varphi\left(\frac{t}{2}\right)L(W) + \psi\left(\frac{t}{2}\right) \approx \varphi\left(\frac{t}{2}\right)t^4. \end{aligned}$$

This gives

$$\|f\|_{\omega\text{-mon}} \geq \frac{f(Q)}{\omega(f(P))} \approx \frac{\varphi\left(\frac{t}{2}\right)t^4}{\omega(\psi(t))}, \quad \text{for all } 0 < t \leq 1,$$

and thus

$$\|f\|_{\omega\text{-mon}} \geq c\mathcal{S}_{\varphi, \psi}^\omega\left(\frac{1}{2}\right).$$

Next let  $P_2 \equiv (W, |W|)$  and  $Q_2 \equiv \left(\frac{W}{2}, \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right)|W|\right)$ , so that  $Q_2 \in \partial B\left(\frac{P_2}{2}, \frac{|P_2|}{2}\right)$ , and

$$\begin{aligned} f(P_2) &= f(W, |W|) = \varphi(r)L(W) + \psi(r) \approx \varphi(r)r^4, \\ f(Q_2) &= \varphi\left(\left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right)r\right)L\left(\frac{W}{2}\right) + \psi\left(\left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right)r\right) \approx \varphi(\gamma_1 r)r^4, \end{aligned}$$

where  $\gamma_1 = \frac{1}{2} + \frac{1}{\sqrt{2}}$ . Therefore,

$$\|f\|_{\omega\text{-mon}} \geq \frac{f(Q)}{\omega(f(P))} \approx \frac{\varphi(\gamma_1 r)r^4}{\omega(\varphi(r)r^4)}$$

for all  $r \in (0, 1)$  and thus

$$\|f\|_{\omega\text{-mon}} \geq C\mathcal{T}_\varphi^\omega(\gamma_1).$$

5.1.2. *Proof of sufficiency in part (1).* Fix a modulus of continuity and a function  $f = f_{\varphi, \psi, h_\rho}$  given by

$$f(W, t) = \varphi(t)L(W) + \psi(t) + \varphi(r)h_\rho\left(\frac{t}{|W|}\right),$$

as in the hypotheses of Theorem 5.4. We consider pairs of points  $(P, Q) \in \mathbb{R}^5 \times \mathbb{R}^5$  restricted by

$$\begin{aligned} (P, Q) &= ((W, t), (V, u)) \in \Omega, \\ \Omega &\equiv \left(\overline{B_{\mathbb{R}^4}(0, 1)} \times [0, 1]\right) \times \left(\overline{B_{\mathbb{R}^4}(0, 1)} \times [0, 1]\right), \end{aligned}$$

and will estimate the supremum,

$$\mathcal{M}f(\omega) \equiv \sup_{(P, Q) \in \Omega: Q \in \overline{B_P}} \frac{f(Q)}{\omega(f(P))},$$

where  $B_P = B\left(\frac{P}{2}, \frac{|P|}{2}\right)$  is the unique ball centered at  $\frac{P}{2}$  that includes both the origin and  $P$  in its boundary. This is a localized version of the functional  $\|f\|_{\omega_s-\text{mon}}$ .

Now the functions

$$z \rightarrow f(zW, zt) = \varphi(zt) z^4 L(W) + \psi(zt) + \varphi(zr) h\left(\frac{t}{|W|}\right)$$

are nondecreasing, which has the consequence that the supremum in  $\mathcal{M}_\omega$  is achieved for  $Q \in \partial B_P$ , so

$$\mathcal{M}f(\omega) = \sup_{(P,Q) \in \Omega: Q \in \partial B_P} \frac{f(Q)}{\omega(f(P))}.$$

We now claim that it further suffices to restrict the supremum to pairs  $(P, Q) = ((W, t), (V, u)) \in \Omega$  with  $Q \in \partial B_P$  and  $V \parallel W$ , where  $V$  and  $W$  are parallel if  $V = \lambda W$  or  $W = \lambda V$  for some  $\lambda \in \mathbb{R}$ .

**Claim 1.**

$$(5.11) \quad \mathcal{M}f(\omega) \approx \sup_{\substack{(P,Q)=((W,t),(V,u)) \in \Omega \\ Q \in \partial B_P \text{ and } V \parallel W}} \frac{f(Q)}{\omega(f(P))}$$

*Proof.* Denote the supremum on the right hand side of (5.11) by  $\mathcal{M}_\parallel f(\omega)$ , so that  $\mathcal{M}_\parallel f(\omega) \leq \mathcal{M}f(\omega)$ . We have

$$f(Q) \approx \varphi(u) |V|^4 + \psi(u) + \varphi(|V|) h\left(\frac{u}{|V|}\right).$$

Rotate the ball  $B_P$  about its vertical axis, namely the diameter of  $B_P$  that is parallel to the vector  $\mathbf{e}_t$ , so that  $Q = (V, u)$  is rotated to the point  $Q' = (V', u)$  in the plane spanned by  $\mathbf{e}_t$  and  $\mathbf{e}_W$ , for which  $|V'| \geq |V|$ . Then

$$f(Q) \lesssim \varphi(u) |V'|^4 + \psi(u) + \varphi(|V'|) h\left(\frac{u}{|V'|}\right) \approx f(Q') \leq \mathcal{M}_\parallel f(\omega) \omega(f(P)),$$

since  $Q' \in \partial B_P$  and  $V' \parallel W$ . Thus we have

$$\mathcal{M}f(\omega) = \sup_{\substack{(P,Q)=((W,t),(V,u)) \in \Omega \\ Q \in \partial B_P}} \frac{f(Q)}{\omega(f(P))} \lesssim \mathcal{M}_\parallel f(\omega).$$

■

Now  $P = (W, t)$  and  $Q = (V, u)$  satisfy  $Q \in \partial B_P$  if and only if

$$\begin{aligned} \left|V - \frac{W}{2}\right|^2 + \left(u - \frac{t}{2}\right)^2 &= \left|(V, u) - \left(\frac{W}{2}, \frac{t}{2}\right)\right|^2 = \left|Q - \frac{P}{2}\right|^2 \\ &= \left(\frac{|P|}{2}\right)^2 = \frac{|W|^2 + t^2}{4}. \end{aligned}$$

Set  $r = |W|$ ,  $z = |V|$  and suppose that  $V \parallel W$ , so that  $|V - \frac{W}{2}| = |\lambda - \frac{1}{2}|r = |z - \frac{r}{2}|$ . Under these conditions, we then have  $Q \in \partial B_P$  if and only if

$$(5.12) \quad \left(z - \frac{r}{2}\right)^2 + \left(u - \frac{t}{2}\right)^2 = \frac{r^2 + t^2}{4},$$

and

$$(5.13) \quad \begin{aligned} f(P) &\approx \varphi(t) r^4 + \psi(t) + \varphi(r) h\left(\frac{t}{r}\right), \\ f(Q) &\approx \varphi(u) z^4 + \psi(u) + \varphi(z) h\left(\frac{u}{z}\right). \end{aligned}$$

Here we prove the upper bound for  $\mathcal{M}f(\omega)$ , which is comparable to  $\|f\|_{\omega_s-\text{mon}}$ . To estimate the supremum in (5.11), we will consider different cases depending on the sizes of  $\frac{t}{r}$  and  $\frac{u}{z}$ , and depending on which of three terms dominates in the expression for  $f(Q)$  in (5.13). We will use the abbreviation  $\sup_{\text{restricted}}$  at various places in the proof to denote the supremum of the ratio  $\frac{f(P)}{\omega(f(Q))}$  subject to the restrictions in force at that time.

**The case**  $r = |W| \leq t$ : We will first prove that when  $r = |W| \leq t$ , we have the upper bound,

$$(5.14) \quad \mathcal{M}_1 f_{\varphi, \psi, h_\rho}(\omega) \equiv \sup_{\substack{(P, Q) = ((W, t), (V, u)) \in \Omega \\ Q \in \partial B_P \text{ and } V \parallel W \text{ and } |W| \leq t}} \frac{f(P)}{\omega(f(Q))} \\ \leq C_\delta \mathcal{R}_{(\varphi, \psi)}^\omega (1 + \delta) + C_\delta \mathcal{T}_\varphi^\omega (\gamma_1 + \delta) + C_\delta \mathcal{S}_{(\varphi, \psi)}^\omega \left( \frac{1}{2} + \delta \right),$$

where  $\gamma_\alpha = \frac{1+\sqrt{1+\alpha^2}}{2\alpha}$  and  $0 < \delta < 1$ . Note that  $h(\frac{t}{r}) = 0$  in this case, so from (5.13) we have

$$f(P) \approx \varphi(t) r^4 + \psi(t).$$

*Proof.* We consider further subcases depending on the size of  $\frac{u}{z}$ , and on which term dominates in the expression for  $f(Q)$  in (5.13).

**Case**  $u \geq z$ : Suppose first that the variables  $(V, u)$  satisfy  $\frac{u}{z} \geq 1$  and  $\varphi(u) z^4 \geq \psi(u)$ , so that

$$f(Q) \approx \varphi(u) z^4.$$

Using the restrictions  $z \leq u$  and  $r^4 \varphi(t) \geq \psi(t)$  we get

$$\sup_{\text{restricted}} \frac{f(Q)}{\omega(f(P))} \approx \sup_{\text{restricted}} \frac{\varphi(u) z^4}{\omega(\varphi(t) r^4 + \psi(t))},$$

and from (5.12),  $(z - \frac{r}{2})^2 + (u - \frac{t}{2})^2 = \frac{r^2 + t^2}{4}$ , together with the restriction  $r \leq t$  we obtain  $z = \frac{r}{2}$  and

$$u = \frac{t}{2} + \sqrt{\frac{r^2 + t^2}{4}},$$

since  $z = \frac{r}{2} \leq \frac{t}{2} + \sqrt{\frac{r^2 + t^2}{4}} = u$ . Now in the case where  $\psi(t)$  dominates in the denominator, we have  $r^2 \leq \sqrt{\frac{\psi(t)}{\varphi(t)}} = o(t^2)$  and so  $u = \frac{t}{2} + \sqrt{\frac{r^2 + t^2}{4}} = (1 + o(1))t$  as  $t \searrow 0$ , and so under all of these restrictions in  $\sup_{\text{restricted}}$  we have for any  $\delta > 0$ ,

$$\begin{aligned} \sup_{\text{restricted}} \frac{f(Q)}{\omega(f(P))} &\approx \sup_{\text{restricted}} \frac{\varphi(u) z^4}{\omega(\varphi(t) r^4 + \psi(t))} \\ &\lesssim \sup_{\text{restricted}} \frac{\varphi\left(\left(\frac{1}{2} + \sqrt{\frac{o(1)+1}{4}}\right)t\right) \frac{\psi(t)}{\varphi(t)}}{\omega(\psi(t))} \lesssim C_\delta \mathcal{R}_{(\varphi, \psi)}^\omega (1 + \delta). \end{aligned}$$

On the other hand, in the case when  $\varphi(t) r^4$  dominates in the denominator, we can use the inequality  $\omega(xy) \geq \omega(x)y$  with  $x = \varphi(t) t^4$  to obtain

$$\begin{aligned} \sup_{\text{restricted}} \frac{f(Q)}{\omega(f(P))} &\approx \sup_{\text{restricted}} \frac{\varphi(u) z^4}{\omega(\varphi(t) r^4)} \approx \sup_{0 < r \leq t \leq 1} \frac{\varphi\left(\frac{t}{2} + \sqrt{\frac{r^2 + t^2}{4}}\right) r^4}{\omega(\varphi(t) r^4)} \\ &\lesssim \sup_{0 < r \leq t \leq 1} \frac{\varphi\left(\frac{t}{2} + \sqrt{\frac{r^2 + t^2}{4}}\right) r^4}{\omega(\varphi(t) t^4) \left(\frac{r}{t}\right)^4} = \sup_{0 < t \leq 1} \frac{\varphi\left(\left(\frac{1}{2} + \sqrt{\frac{1}{4}}\right)t\right) t^4}{\omega(\varphi(t) t^4)} = \mathcal{T}_\varphi^\omega (\gamma_1). \end{aligned}$$

**Case**  $u \leq z$ : In this case  $f(Q) \lesssim \varphi(z)$  and

$$\sup_{\text{restricted}} \frac{f(Q)}{\omega(f(P))} \approx \sup_{\text{restricted}} \frac{\varphi(z)}{\omega(\varphi(t) r^4 + \psi(t))}.$$

The supremum on the right hand side is maximized for  $z$  as large as possible, which by (5.12) occurs when  $u = \frac{t}{2}$  and

$$z = \frac{r}{2} + \frac{1}{2} \sqrt{r^2 + t^2}.$$

We now consider two cases, where  $\frac{r}{t}$  is small and large. Note that with

$$\Theta(\delta) \equiv \frac{\left(\frac{1}{2} + \delta\right)^2 - \frac{1}{4}}{\frac{1}{2} + \delta} = \frac{\delta(1 + \delta)}{\frac{1}{2} + \delta},$$

we have

$$z = \frac{r}{2} + \frac{1}{2}\sqrt{r^2 + t^2} \leq \left(\frac{1}{2} + \delta'\right)t, \quad \text{for } 0 \leq r \leq \Theta(\delta')t,$$

as is easily seen by squaring the inequality  $\frac{1}{2}\sqrt{r^2 + t^2} \leq \left(\frac{1}{2} + \delta'\right)t - \frac{r}{2}$ . Thus for  $r \leq \Theta(\delta')t$  with  $\delta' > 0$ , we obtain

$$\begin{aligned} \sup_{\text{restricted}} \frac{f(Q)}{\omega(f(P))} &\approx \sup_{\text{restricted}} \frac{\varphi(z)}{\omega(\varphi(t)r^4 + \psi(t))} \lesssim \sup_{0 < t \leq 1} \frac{\varphi\left(\left(\frac{1}{2} + \delta'\right)t\right)}{\omega(\psi(t))} \\ &\lesssim C_\varepsilon \sup_{0 < t \leq 1} \frac{\varphi\left(\left(\frac{1}{2} + \delta' + \varepsilon\right)t\right)t^4}{\omega(\psi(t))}, \end{aligned}$$

since  $\lim_{t \searrow 0} \frac{\varphi\left(\left(\frac{1}{2} + \delta'\right)t\right)}{\varphi\left(\left(\frac{1}{2} + \delta' + \varepsilon\right)t\right)t^4} = \infty$  for  $\varepsilon > 0$  and any  $\varphi$  flat at the origin. We now choose  $\delta'$  and  $\varepsilon$  small enough that  $\delta' + \varepsilon \leq \delta$ , so as to conclude that

$$\sup_{\text{restricted}} \frac{f(Q)}{\omega(f(P))} \lesssim C_\varepsilon \sup_{0 < t \leq 1} \frac{\varphi\left(\left(\frac{1}{2} + \delta\right)t\right)t^4}{\omega(\psi(t))} = \mathcal{S}_{(\varphi, \psi)}^\omega\left(\frac{1}{2} + \delta\right).$$

On the other hand, if  $r > \Theta(\delta')t$ , then

$$\begin{aligned} \sup_{\text{restricted}} \frac{f(Q)}{\omega(f(P))} &\approx \sup_{\text{restricted}} \frac{\varphi(z)}{\omega(\varphi(t)r^4)} \approx \sup_{\Theta(\delta')t < r \leq t \leq 1} \frac{\varphi\left(\frac{r}{2} + \frac{1}{2}\sqrt{r^2 + t^2}\right)}{\omega(\varphi(t)r^4)} \\ &\lesssim \sup_{0 < t \leq 1} \frac{\varphi(\gamma_1 t)}{\omega(\varphi(t)\Theta(\delta')^4 t^4)} \lesssim \frac{1}{\Theta(\delta')^4} \sup_{0 < t \leq 1} \frac{\varphi(\gamma_1 t)}{\omega(\varphi(t)t^4)} \\ &\lesssim \frac{1}{\Theta(\delta')^4} \sup_{0 < t \leq 1} \frac{\varphi((\gamma_1 + \delta)t)t^4}{\omega(\varphi(t)t^4)} = C_\delta \mathcal{T}_\varphi^\omega(\gamma_1 + \delta), \end{aligned}$$

using the flatness of  $\varphi$  again. ■

**The case  $r = |W| > t$ :** We will now prove that when  $r = |W| > t$  we have the following upper bound,

$$\mathcal{M}_2 f_{\varphi, \psi, h_\rho}(\omega) \equiv \sup_{\substack{(P, Q) = ((W, t), (V, u)) \in \Omega \\ Q \in \partial B_P \text{ and } V \parallel W \text{ and } |W| > t}} \frac{f(P)}{\omega(f(Q))} \leq C_{\rho, \delta} \mathcal{T}_\varphi^\omega(\gamma_\rho + \delta).$$

*Proof.* We consider separately the cases  $t \leq \rho r$  when  $h_\rho(\frac{t}{r}) = 1$ , and  $r > t > \rho r$  when  $0 < h_\rho(\frac{t}{r}) < 1$ .

**Case  $t \leq \rho r$ :** We have  $h_\rho(\frac{t}{r}) = 1$  and so from (5.13) that

$$f(P) \approx \varphi(r).$$

In the case  $u \leq z$ , we have  $f(Q) \lesssim \varphi(z)$ , and by (5.12), i.e.  $\left(z - \frac{r}{2}\right)^2 + \left(u - \frac{t}{2}\right)^2 = \frac{r^2 + t^2}{4}$ , we then have  $\varphi(z) \approx \varphi\left(\frac{r}{2} + \frac{\sqrt{r^2 + t^2}}{2}\right)$  if we choose  $u = \frac{t}{2}$  and  $z = \frac{r}{2} + \frac{\sqrt{r^2 + t^2}}{2}$ , so that  $u = \frac{t}{2} \leq \frac{r}{2} + \frac{\sqrt{r^2 + t^2}}{2} = z$ . We conclude from the inequality  $\omega(xy) \geq \omega(x)y$  with  $x = \varphi(r)r^4$  and  $y = r^{-4}$  that

$$\begin{aligned} \sup_{\text{restricted}} \frac{f(Q)}{\omega(f(P))} &\lesssim \sup_{0 < t \leq \rho r < 1} \frac{\varphi\left(\frac{r}{2} + \frac{\sqrt{r^2 + t^2}}{2}\right)}{\omega(\varphi(r))} \leq \sup_{0 < r < 1} \frac{\varphi\left(\left(\frac{1}{2} + \frac{\sqrt{1 + \rho^2}}{2}\right)r\right)}{\omega(\varphi(r))} \\ &\lesssim \sup_{0 < r < 1} \frac{\varphi\left(\left(\frac{1}{2} + \frac{\sqrt{1 + \rho^2}}{2}\right)r\right)r^4}{\omega(\varphi(r)r^4)} = \mathcal{T}_\varphi^\omega\left(\frac{1}{2} + \frac{\sqrt{1 + \rho^2}}{2}\right) \leq \mathcal{T}_\varphi^\omega(\gamma_1). \end{aligned}$$

On the other hand, if  $u \geq z$ , then  $f(Q) \approx \varphi(u)z^4 + \psi(u)$ , and by (5.12) we then have that  $u$  is maximized when  $z = \frac{r}{2}$  and  $u = \frac{t}{2} + \frac{\sqrt{r^2 + t^2}}{2}$ , which we note satisfies the requirement  $u \geq z$ . Thus we have

$u \leq \frac{\rho r}{2} + \frac{\sqrt{r^2 + \rho^2 r^2}}{2} = \left( \frac{\rho}{2} + \frac{\sqrt{1 + \rho^2}}{2} \right) r$  and so

$$\begin{aligned}
 (5.15) \quad & \sup_{\text{restricted}} \frac{f(Q)}{\omega(f(P))} \approx \sup_{0 < r < 1} \frac{\varphi(u) r^4 + \psi(u)}{\omega(\varphi(r))} \\
 & \approx \sup_{0 < r < 1} \frac{\varphi\left(\left(\frac{\rho}{2} + \frac{\sqrt{1 + \rho^2}}{2}\right) r\right) r^4 + \psi\left(\left(\frac{\rho}{2} + \frac{\sqrt{1 + \rho^2}}{2}\right) r\right)}{\omega(\varphi(r))} \\
 & \approx \sup_{0 < r < 1} \frac{\varphi\left(\left(\frac{\rho}{2} + \frac{\sqrt{1 + \rho^2}}{2}\right) r\right) r^4}{\omega(\varphi(r))} \lesssim \mathcal{T}_\varphi^\omega\left(\left(\frac{\rho}{2} + \frac{\sqrt{1 + \rho^2}}{2}\right)\right) \lesssim \mathcal{T}_\varphi^\omega(\gamma_1).
 \end{aligned}$$

**Case  $r > t > \rho r$ :** Here we have  $f(P) \approx \varphi(t) t^4 + \varphi(r) h\left(\frac{t}{r}\right)$  since  $r \approx t$  and  $\psi(t) = o(\varphi(t) t^4)$ . If  $u \leq z$  we have  $f(Q) \lesssim \varphi(z)$ , and by (5.12), we have  $\varphi(z) \approx \varphi\left(\frac{r}{2} + \frac{\sqrt{r^2 + t^2}}{2}\right)$  if we maximize  $z$  by choosing  $u = \frac{t}{2}$  and  $z = \frac{r}{2} + \frac{\sqrt{r^2 + t^2}}{2}$ . From this and  $r \leq \frac{1}{\rho} t$ , we obtain

$$z = \frac{r}{2} + \frac{1}{2} \sqrt{r^2 + t^2} \leq \frac{t}{2\rho} + \frac{1}{2\rho} \sqrt{t^2 + \rho^2 t^2} = \frac{1 + \sqrt{1 + \rho^2}}{2\rho} t = \gamma_\rho t,$$

and so

$$\begin{aligned}
 \sup_{\text{restricted}} \frac{f(Q)}{\omega(f(P))} & \lesssim \sup_{\rho r < t < r} \frac{\varphi\left(\frac{r}{2} + \frac{\sqrt{r^2 + t^2}}{2}\right)}{\omega(\varphi(t) t^4 + \varphi(r) h\left(\frac{t}{r}\right))} \lesssim \sup_{\rho r < t < r} \frac{\varphi(\gamma_\rho t)}{\omega(\varphi(t) t^4)} \\
 & \lesssim \sup_{\rho r < t < r} \frac{\varphi((\gamma_\rho + \delta) t) t^4}{\omega(\varphi(t) t^4)} = \mathcal{T}_\varphi^\omega(\gamma_\rho + \delta),
 \end{aligned}$$

where we have used the flatness of  $\varphi$  as before.

Next, if  $u \geq z$  we have  $f(Q) \approx \varphi(u) z^4 + \psi(u) \lesssim \varphi(u) u^4$ , and so maximizing  $u$  with  $z = \frac{r}{2}$  gives  $u = \frac{t}{2} + \frac{\sqrt{r^2 + t^2}}{2}$ , and so

$$\begin{aligned}
 \sup_{\text{restricted}} \frac{f(Q)}{\omega(f(P))} & \lesssim \sup_{\rho r < t < r \text{ and } u \geq z} \frac{\varphi(u) u^4}{\omega(\varphi(t) t^4 + \varphi(r) h\left(\frac{t}{r}\right))} \\
 & \lesssim \sup_{\rho r < t < r \text{ and } u \geq z} \frac{\varphi(u) u^4}{\omega(\varphi(t) t^4)} \lesssim \sup_{\rho r < t < r} \frac{\varphi\left(\frac{t}{2} + \frac{\sqrt{r^2 + t^2}}{2}\right) t^4}{\omega(\varphi(t) t^4)} \\
 & \approx \sup_{0 < t < 1} \frac{\varphi\left(\left(\frac{1}{2} + \frac{\sqrt{\frac{1}{\rho^2} + 1}}{2}\right) t\right) t^4}{\omega(\varphi(t) t^4)} = \mathcal{T}_\varphi^\omega\left(\frac{1}{2} + \frac{\sqrt{\frac{1}{\rho^2} + 1}}{2}\right) \leq \mathcal{T}_\varphi^\omega(\gamma_\rho).
 \end{aligned}$$

■

Combining the estimates for  $\mathcal{M}_1 f(\omega)$  and  $\mathcal{M}_2 f(\omega)$  completes the proof of Part (1) of Theorem 5.4.

5.1.3. *Proof of part (2)(a).* Denote by  $\mathcal{C}_{2,\omega}^\nu$  the function.

$$\mathcal{C}_{2,\omega}^\nu(\tau) \equiv \inf \left\{ \|\mathbf{G}\|_{2,\omega} : \mathbf{G} = \{G_\ell\}_{\ell=1}^\nu \in \oplus^\nu C^{2,\omega}(B_{\mathbb{R}^4}(0,1)) \text{ and } L(W) + \tau = \sum_{\ell=1}^\nu G_\ell(W)^2 \right\}.$$

Note that by Lemma 5.2, we have  $\lim_{\tau \rightarrow 0} \mathcal{C}_{2,\omega}^\nu(\tau) = \infty$ , but we will require the sharper inequality given in (5.27) below. Suppose that

$$\begin{aligned}
 L(W) + \tau &= \sum_{\ell=1}^\nu G_\ell(W)^2, \\
 G_\ell(W) &= a_\ell + S_\ell(W) + Q_\ell(W) + R_\ell(W),
 \end{aligned}$$

where

$$S_\ell(W) = \sum_{|\alpha|=1} a_{\ell,\alpha} W^\alpha \text{ and } Q_\ell(W) = \sum_{|\alpha|=2} f_{\ell,\alpha} W^\alpha.$$

Then setting  $W = 0$  in the equation gives

$$\tau = \sum_{\ell=1}^{\nu} a_\ell^2,$$

and so

$$\begin{aligned} L(W) &= \sum_{\ell=1}^{\nu} [a_\ell + S_\ell(W) + Q_\ell(W) + R_\ell(W)]^2 - \tau \\ &= \left( \sum_{\ell=1}^{\nu} a_\ell^2 \right) - \tau + \sum_{\ell=1}^{\nu} 2a_\ell S_\ell(W) + \sum_{\ell=1}^{\nu} [S_\ell(W)^2 + 2a_\ell Q_\ell(W)] \\ &\quad + \sum_{\ell=1}^{\nu} 2a_\ell R_\ell(W) + \sum_{\ell=1}^{\nu} 2S_\ell(W) R_\ell(W) + \sum_{\ell=1}^{\nu} [Q_\ell(W) + R_\ell(W)]^2. \end{aligned}$$

Now the sum of terms in the middle line vanishes identically since it is a quadratic polynomial, and all of the remaining terms in the identity vanish to order greater than 2 at the origin (simply evaluate the identity at  $W = 0$ , then differentiate and evaluate at  $W = 0$ , and finally differentiate once more and evaluate at  $W = 0$ , using that  $R_\ell(0)$ ,  $\nabla R_\ell(0)$  and  $\nabla^2 R_\ell(0) = 0$  all vanish). Thus we conclude that

$$(5.16) \quad L(W) - \sum_{\ell=1}^{\nu} [Q_\ell(W) + R_\ell(W)]^2 = \sum_{\ell=1}^{\nu} 2a_\ell R_\ell(W) + \sum_{\ell=1}^{\nu} 2S_\ell(W) R_\ell(W).$$

Now define  $\delta_\nu > 0$  by

$$(5.17) \quad \delta_\nu^2 \equiv \inf_{\{Q_\ell\}_{\ell=1}^{\nu}} \inf_{W \in \mathbb{S}^3} \left( L(W) - \sum_{\ell=1}^{\nu} Q_\ell(W)^2 \right)^2,$$

where the infimum is taken over all collections  $\{Q_\ell\}_{\ell=1}^{\nu}$  of quadratic forms  $Q_\ell(W) = \sum_{|\alpha|=2} f_{\ell,\alpha} W^\alpha$ , with  $W \in \mathbb{S}^4$  and coefficients  $f_{\ell,\alpha}$  of modulus at most a constant  $C_0$ , which will be determined in (5.23) below. Since the infimum is taken over a compact set, it is achieved, and must then be positive since  $L$  cannot be written as a sum of squares of quadratic forms.

Now fix a modulus of continuity  $\omega$ , and given  $\tau > 0$ , suppose there are functions  $G_\ell \in C^{2,\omega}$  with  $\sum_{\ell=1}^{\nu} \|G_\ell\|_{C^{2,\omega}} = \|\mathbf{G}\|_{2,\omega} < \infty$  such that

$$(5.18) \quad L(W) + \tau = \sum_{\ell=1}^{\nu} G_\ell(W)^2, \quad \text{for all } |W| \leq 1.$$

Recall that we can write

$$G_\ell(W) = a_\ell + S_\ell(W) + Q_\ell(W) + R_\ell(W),$$

where

$$(5.19) \quad \sum_{\ell=1}^{\nu} a_\ell^2 = \tau,$$

$$(5.20) \quad \sum_{\ell=1}^{\nu} |S_\ell(W)| \leq \|\mathbf{G}\|_{2,\omega} |W|$$

$$(5.21) \quad \sum_{\ell=1}^{\nu} |Q_\ell(W)| \leq \|\mathbf{G}\|_{2,\omega} |W|^2$$

$$(5.22) \quad \sum_{\ell=1}^{\nu} |R_\ell(W)| \leq \|\mathbf{G}\|_{2,\omega} |W|^2 \omega(W).$$

Also note that from  $\sum_{\ell=1}^{\nu} |Q_\ell(W)| \leq C\sqrt{L(W) + \tau}$ , we obtain that for  $0 < \tau < 1$ , we have

$$(5.23) \quad |f_{\ell,\alpha}| \leq C_0 \equiv C\sqrt{L(W) + 1}.$$

From (5.16) we have

$$\begin{aligned}
 (5.24) \quad L(W) - \sum_{\ell=1}^{\nu} Q_{\ell}(W)^2 &= L(W) - \sum_{\ell=1}^{\nu} [Q_{\ell}(W) + R_{\ell}(W)]^2 + \sum_{\ell=1}^{\nu} [2Q_{\ell}(W) + R_{\ell}(W)] R_{\ell}(W) \\
 &= h_1(W) + h_2(W) \equiv h(W),
 \end{aligned}$$

where

$$(5.25) \quad h_1(W) \equiv \sum_{\ell=1}^{\nu} 2a_{\ell} R_{\ell}(W) + \sum_{\ell=1}^{\nu} 2S_{\ell}(W) R_{\ell}(W)$$

$$(5.26) \quad h_2(W) \equiv \sum_{\ell=1}^{\nu} [2Q_{\ell}(W) + R_{\ell}(W)] R_{\ell}(W).$$

Using the last line of (5.19) we obtain

$$\begin{aligned}
 |h_1(W)| &\leq C\sqrt{\tau} \|\mathbf{G}\|_{2,\omega} |W|^2 \omega(|W|) + C \|\mathbf{G}\|_{2,\omega}^2 |W|^3 \omega(|W|) = C \|\mathbf{G}\|_{2,\omega}^2 |W|^2 \omega(|W|) \left( \frac{\sqrt{\tau}}{\|\mathbf{G}\|_{2,\omega}} + |W| \right) \\
 |h_2(W)| &\leq C \|\mathbf{G}\|_{2,\omega} |W|^2 |R_{\ell}(W)| \leq C \|\mathbf{G}\|_{2,\omega}^2 |W|^4 \omega(|W|).
 \end{aligned}$$

So altogether we have

$$\begin{aligned}
 |h(W)| &\leq |h_1(W)| + |h_2(W)| \leq C \|\mathbf{G}\|_{2,\omega}^2 \omega(|W|) |W|^2 \left( \frac{\sqrt{\tau}}{\|\mathbf{G}\|_{2,\omega}} + |W| + |W|^2 \right) \\
 &\leq C \|\mathbf{G}\|_{2,\omega}^2 \omega(|W|) |W|^2 \left( \frac{\sqrt{\tau}}{\|\mathbf{G}\|_{2,\omega}} + |W| \right),
 \end{aligned}$$

provided  $|W| \leq 1$ . Note that we can assume without loss of generality that  $\frac{\sqrt{\tau}}{\|\mathbf{G}\|_{2,\omega}} \leq 1$ . Then if  $|W| = \frac{\sqrt{\tau}}{\|\mathbf{G}\|_{2,\omega}}$ , we have

$$|h(W)| \leq C \|\mathbf{G}\|_{2,\omega}^2 \omega(|W|) |W|^3.$$

However, this estimate is too weak, and we need an improved bound on  $|S_{\ell}(W)|$ .

We return to (5.18) to obtain

$$\begin{aligned}
 &L(W) + \lambda^{-4}\tau = \lambda^{-4}(L(\lambda W) + \tau) \\
 &= \sum_{\ell=1}^{\nu} \left[ \frac{S_{\ell}(W)}{\lambda} + \frac{a_{\ell}}{\lambda^2} + Q_{\ell}(W) + \lambda^{-2}R_{\ell}(\lambda W) \right]^2 \\
 &= \sum_{\ell=1}^{\nu} \left[ \frac{S_{\ell}(W)}{\lambda} \right]^2 + O \left( \sqrt{\sum_{\ell=1}^{\nu} \left[ \frac{S_{\ell}(W)}{\lambda} \right]^2} \left[ \frac{\sqrt{\tau}}{\lambda^2} + \|\mathbf{G}\|_{2,\omega} |W|^2 + \|\mathbf{G}\|_{2,\omega} |W|^2 \omega(\lambda |W|) \right] \right) \\
 &\quad + O \left( \left[ \frac{\sqrt{\tau}}{\lambda^2} + \|\mathbf{G}\|_{2,\omega} |W|^2 + \|\mathbf{G}\|_{2,\omega} |W|^2 \omega(\lambda |W|) \right]^2 \right),
 \end{aligned}$$

and hence

$$\begin{aligned}
 \sum_{\ell=1}^{\nu} \left[ \frac{S_{\ell}(W)}{\lambda} \right]^2 &\leq C |W|^4 + \lambda^{-4}\tau + C \left[ \frac{\sqrt{\tau}}{\lambda^2} + \|\mathbf{G}\|_{2,\omega} |W|^2 + \|\mathbf{G}\|_{2,\omega} |W|^2 \omega(\lambda |W|) \right]^2; \\
 \text{i.e. } \sum_{\ell=1}^{\nu} \left[ \frac{S_{\ell}(W)}{|W|\lambda} \right]^2 &\leq C |W|^2 + \frac{\tau}{|W|^2 \lambda^4} + C \left[ \frac{\sqrt{\tau}}{|W|\lambda^2} + \|\mathbf{G}\|_{2,\omega} |W| + \|\mathbf{G}\|_{2,\omega} |W| \omega(\lambda |W|) \right]^2 \\
 &\leq C \frac{\tau}{|W|^2 \lambda^4} + C \|\mathbf{G}\|_{2,\omega}^2 |W|^2, \quad \text{provided } \lambda |W| \text{ remains bounded.}
 \end{aligned}$$

But now we note that

$$\left\| \frac{S_{\ell}(W)}{|W|} \right\|_{\infty} = \left\| \sum_{|\alpha|=1} a_{\ell,\alpha} \left( \frac{W}{|W|} \right)^{\alpha} \right\|_{\infty} \approx \sum_{|\alpha|=1} |a_{\ell,\alpha}|,$$

and so we conclude that

$$\sum_{|\alpha|=1} |a_{\ell,\alpha}| \leq C \left( \frac{\tau}{|W|^2 \lambda^2} + \|\mathbf{G}\|_{2,\omega}^2 |W|^2 \lambda^2 \right)^{\frac{1}{2}} \leq 2C \|\mathbf{G}\|_{2,\omega}^{\frac{1}{2}} \tau^{\frac{1}{4}}$$

if we choose  $\lambda = \frac{\sqrt[4]{\tau}}{\sqrt{91}|W|}$ , and thus

$$|S_{\ell}(W)| \leq C \|\mathbf{G}\|_{2,\omega}^{\frac{1}{2}} \tau^{\frac{1}{4}} |W|.$$

Using this together with (5.19) in (5.25) we obtain

$$|h(W)| \leq |h_1(W)| + |h_2(W)| \leq C \|\mathbf{G}\|_{2,\omega}^2 \omega(|W|) |W|^2 \left( \frac{\sqrt{\tau}}{\|\mathbf{G}\|_{2,\omega}} + \frac{\sqrt[4]{\tau}}{\sqrt{\|\mathbf{G}\|_{2,\omega}}} |W| + |W|^2 \right).$$

If  $|W| = \frac{\sqrt[4]{\tau}}{\sqrt{91}}$  we have

$$|h(W)| \leq C \|\mathbf{G}\|_{2,\omega}^2 \omega(|W|) |W|^4,$$

and from (5.24) we obtain

$$L\left(\frac{W}{|W|}\right) - \sum_{\ell=1}^{\nu} Q_{\ell}\left(\frac{W}{|W|}\right)^2 = \left| \frac{L(W) - \sum_{\ell=1}^{\nu} Q_{\ell}(W)^2}{|W|^4} \right| \leq \frac{|h(W)|}{|W|^4} \leq C \|\mathbf{G}\|_{2,\omega}^2 \omega(|W|),$$

if  $|W| = \frac{\sqrt[4]{\tau}}{\sqrt{91}}$ . Using (5.17) and (5.23) we thus have the following estimate

$$\delta_{\nu} \leq C \|\mathbf{G}\|_{2,\omega}^2 \omega\left(\frac{\sqrt[4]{\tau}}{\sqrt{\|\mathbf{G}\|_{2,\omega}}}\right), \quad \text{for } C_0 \geq C \|\mathbf{G}\|_{2,\omega},$$

where  $C_0$  is the constant defined in (5.23). In the special case  $\omega(r) = r^{\beta}$  we have

$$\delta_{\nu} \leq C \|\mathbf{G}\|_{2,\omega}^{2-\frac{\beta}{2}} \tau^{\frac{\beta}{4}},$$

or equivalently

$$\|\mathbf{G}\|_{2,\omega} \geq \left(\frac{\delta_{\nu}}{C}\right)^{\frac{2}{4-\beta}} \left(\frac{1}{\tau}\right)^{\frac{\beta}{8-2\beta}}, \quad \text{provided } \|\mathbf{G}\|_{2,\omega} \leq \frac{C_0}{C}.$$

Altogether we have obtained thus far the crucial lower bound

$$(5.27) \quad \mathcal{C}_{2,\omega_{\beta}}^{\nu}(\tau) \geq \left(\frac{\delta_{\nu}}{C}\right)^{\frac{2}{4-\beta}} \tau^{-\frac{\beta}{8-2\beta}}.$$

The next lemma finishes the proof of part 2(a) of Theorem 5.4.

**Lemma 5.6.** *Suppose  $0 < \beta < 1$  and let  $f_{\varphi,\psi}(W, t)$  be as in (5.3). If*

$$(5.28) \quad \limsup_{t \rightarrow 0} \frac{\psi(t)}{\varphi(t)^{\frac{4}{\beta}} t^{\frac{16}{\beta}}} = 0,$$

*then  $f_{\varphi,\psi}$  fails to satisfy  $\mathcal{SOS}_{2,\omega_{\beta}}^{\nu}$  for any  $\nu \in \mathbb{N}$ . Note in particular we may even take both  $\varphi$  and  $\psi$  to be nearly monotone functions on  $(-1, 1)$ .*

*Proof.* Assume, in order to derive a contradiction, that  $f_{\varphi,\psi}(W, t)$  has the property  $\mathcal{SOS}_{2,\omega_{\beta}}^{\nu}$  for some  $\nu \in \mathbb{N}$ , i.e.  $f_{\varphi,\psi} = \sum_{\ell=1}^{\nu} G_{\ell}^2$  where  $G_{\ell} \in C^{2,\omega}(\Omega)$ , i.e.

$$\varphi(t) L(w, x, y, z, t) + \left[ \psi(t) + \varphi(r) h\left(\frac{t}{r}\right) \right] = \sum_{\ell=1}^{\nu} G_{\ell}(x, y, z, t)^2,$$

$$\text{for } (x, y, z, t) \in \Omega = B_{\mathbb{R}^3}(0, 1) \times (-1, 1).$$

Then since  $h\left(\frac{t}{r}\right)$  vanishes for  $r \leq |t|$ , we have with  $W \equiv (w, x, y, z)$ , and without loss of generality  $t > 0$ , that

$$\varphi(t) L(W) + \psi(t) = \sum_{\ell=1}^{\nu} G_{\ell}(W, t)^2, \quad \text{for } r \leq t,$$

and replacing  $W$  by  $tW$  we have,

$$\begin{aligned} \varphi(t) L(tW) + \psi(t) &= \sum_{\ell=1}^{\nu} G_{\ell}(tW, t)^2, \\ \text{for } |W| &\leq 1, t \in (0, 1). \end{aligned}$$

Multiplying by  $\frac{1}{\varphi(t)t^2}$ , and using that  $L$  is homogeneous of degree four, we obtain

$$\begin{aligned} \mathbf{L}(W) + \frac{\psi(t)}{\varphi(t)t^4} &= \sum_{\ell=1}^{\nu} \left( \frac{G_{\ell}(tW, t)}{\sqrt{\varphi(t)}t^2} \right)^2, \\ \text{for } |W| &\leq 1, t \in (0, 1). \end{aligned}$$

Since  $G_{\ell} \in C^{2,\omega}(B_{\mathbb{R}^4}(0, 1) \times (-1, 1))$ , the functions  $W \rightarrow G_{\ell}(W, t)$  lie in a bounded set in  $C^{2,\omega}(B_{\mathbb{R}^4}(0, 1))$  independent of  $t$  and  $j$ , and hence also the collection of functions

$$H_{\ell}^t(W) \equiv G_{\ell}(tW, t), \quad 1 \leq \ell \leq \nu, t \in (0, 1),$$

is bounded in  $C^{2,\omega}(B_{\mathbb{R}^4}(0, 1))$ , say

$$(5.29) \quad \sum_{\ell=1}^{\nu} \|H_{\ell}^t\|_{C^{2,\omega}(B_{\mathbb{R}^4}(0,1))} \leq \mathfrak{N}_{\nu}, \quad t \in (0, 1).$$

Thus with  $\tau = \tau(t) \equiv \frac{\psi(t)}{\varphi(t)t^4}$ , we have from (5.29) and (5.27) that

$$\begin{aligned} \frac{\mathfrak{N}_{\nu}}{\sqrt{\varphi(t)t^4}} &\geq \sum_{\ell=1}^{\nu} \left\| \frac{H_{\ell}^t}{\sqrt{\varphi(t)t^2}} \right\|_{C^{2,\omega}(B_{\mathbb{R}^4}(0,1))} \geq \mathcal{C}_{2,\omega}^{\nu}(\tau(t)) \\ &\geq \left( \frac{\delta_{\nu}}{C} \right)^{\frac{2}{4-\beta}} \tau(t)^{-\frac{\beta}{8-2\beta}} = \left( \frac{\delta_{\nu}}{C} \right)^{\frac{2}{4-\beta}} \left( \frac{\psi(t)}{\varphi(t)t^4} \right)^{-\frac{\beta}{8-2\beta}}, \end{aligned}$$

and hence

$$\left( \frac{\delta_{\nu}}{C} \right)^{\frac{2}{4-\beta}} \leq \liminf_{t \rightarrow 0} \frac{\mathfrak{N}_{\nu}}{\sqrt{\varphi(t)t^4}} \left( \frac{\psi(t)}{\varphi(t)t^4} \right)^{\frac{\beta}{8-2\beta}} = \mathfrak{N}_{\nu} \liminf_{t \rightarrow 0} \left( \frac{\psi(t)}{\varphi(t)^{\frac{4}{\beta}} t^{\frac{16}{\beta}}} \right)^{\frac{\beta}{8-2\beta}},$$

contradicting (5.28) as required. This completes the proof of Lemma 5.6. ■

5.1.4. *Proof of part (2)(b).* Choose  $s < s' < \beta$ . If we set  $\psi(t) = \varphi\left(\frac{t}{2}\right)^{\frac{1}{s'}} t^{\frac{4}{s'}}$ , then  $f$  is  $\omega_s$ -monotone by part (1), and we have

$$\lim_{t \searrow 0} \frac{\varphi(t)^{\frac{4}{\beta}} t^{\frac{16}{\beta}}}{\varphi\left(\frac{t}{2}\right)^{\frac{1}{s'}} t^{\frac{4}{s'}}} = \lim_{t \searrow 0} \frac{e^{-\frac{1}{t^2} \frac{4}{\beta} t^{\frac{16}{\beta}}}}{e^{-\frac{1}{t^2} \frac{4}{s'} t^{\frac{4}{s'}}}} = \lim_{t \searrow 0} e^{\frac{4}{t^2} \left(\frac{1}{s'} - \frac{1}{\beta}\right) t^4 \left(\frac{4}{\beta} - \frac{1}{s'}\right)} = \infty$$

since  $\beta > s'$ , and hence by part (2)(b), we cannot write  $f$  as a finite sum of squares of  $C^{2,\beta}$  functions.

This completes the proof of part (2), and hence that of Theorem 5.4.

**5.2. Extension to general moduli of continuity and proof of Theorem 2.5.** We first note that part (1) of Theorem 2.5 is implied by Theorem 5.4. To prove part (2) let

$$\psi(t) \equiv \omega^{-1}(\varphi(t)),$$

so that

$$f_{\varphi, \psi, h_{\rho}}(W, t) \approx \varphi(t) r^4 + \psi(t) + \varphi(r) h\left(\frac{t}{r}\right) \lesssim \varphi(r).$$

Then since  $\omega^{-1}$  vanishes to infinite order at the origin, we have

$$\lim_{t \searrow 0} \frac{\varphi(t)^{\frac{4}{\beta}} t^{\frac{16}{\beta}}}{\psi(t)} = \lim_{t \searrow 0} \frac{\varphi(t)^{\frac{4}{\beta}} t^{\frac{16}{\beta}}}{\omega^{-1}(\varphi(t))} \geq c_N \lim_{t \searrow 0} \frac{\varphi(t)^{\frac{4}{\beta}} t^{\frac{16}{\beta}}}{\varphi(t)^N} = c_N \lim_{t \searrow 0} \frac{e^{-\frac{1}{t^2} \frac{4}{\beta} t^{\frac{16}{\beta}}}}{e^{-\frac{1}{t^2} N}} = \infty$$

for  $N > \frac{4}{\beta}$ , Part (2)(a) of Theorem 5.4 shows that  $f_{\varphi, \psi, h_\rho}$  cannot be written as a finite sum of squares of  $C^{2,\beta}$  functions. On the other hand using  $\delta < 1/2$  we have for  $N > \max\{2, (\gamma_\rho + \delta)^2\}$

$$\begin{aligned} \mathcal{R}_{(\varphi, \psi)}^\omega(1 + \delta) &\equiv \sup_{0 < t \leq 1} \frac{\psi(t)}{\varphi(t)} \frac{\varphi((1 + \delta)t)}{\omega(\psi(t))} \lesssim \sup_{0 < t \leq 1} \frac{\varphi(t)^N \varphi((1 + \delta)t)}{\varphi(t)^2} = \sup_{0 < t < 1} \frac{e^{-\frac{N}{t^2} - \frac{1}{(1 + \delta)^2 t^2}}}{e^{-\frac{2}{t^2}}} \leq 1, \\ \mathcal{T}_\varphi^\omega(\gamma_\rho + \delta) &\equiv \sup_{0 < t \leq 1} \frac{\varphi((\gamma_\rho + \delta)t) t^4}{\omega(\varphi(t) t^4)} \lesssim \sup_{0 < t \leq 1} \frac{\varphi((\gamma_\rho + \delta)t)}{\varphi(t)^{\frac{1}{N}}} = \sup_{0 < t < 1} \frac{e^{-\frac{1}{(\gamma_\rho + \delta)^2 t^2}}}{e^{-\frac{1}{N t^2}}} \leq 1, \\ \mathcal{S}_{(\varphi, \psi)}^\omega\left(\frac{1}{2} + \delta\right) &\equiv \sup_{0 < t \leq 1} \frac{\varphi(\frac{t}{2} + \delta t) t^4}{\omega(\psi(t))} \lesssim \sup_{0 < t \leq 1} \frac{\varphi(\frac{t}{2} + \delta t)}{\varphi(t)} = \sup_{0 < t < 1} \frac{e^{-\frac{1}{((1/2 + \delta)t)^2}}}{e^{-\frac{1}{t^2}}} \leq 1. \end{aligned}$$

Thus, Part (1) of Theorem 5.4 shows that  $f_{\varphi, \psi, h_\rho}$  is  $\omega$ -monotone, which completes the proof of Theorem 2.5.

We end the paper by collecting the previous results into a somewhat sharp theorem in all dimensions, which can be summed up as *roughly saying* that an elliptical flat smooth function can be written as a finite sum of squares of regular functions ‘*if and only*’ if it is Hölder monotone.

**Theorem 5.7.** *Suppose that  $f$  is elliptical flat smooth and Hölder monotone, i.e.  $\omega_s$ -monotone on  $\mathbb{R}^n$  for some  $0 < s < 1$  and  $n \geq 1$ . Then there is  $\delta > 0$  such that  $f$  is a finite sum of squares of  $C^{2,\delta}$  functions. Conversely, for every modulus of continuity  $\omega$  satisfying  $\omega_s \ll \omega$  for all  $0 < s < 1$ , there is an elliptical flat smooth  $\omega$ -monotone function  $f$  on  $\mathbb{R}^5$  that cannot be written as a finite sum of squares of  $C^{2,\delta}$  functions for any  $0 < \delta < 1$ .*

*Proof.* The first assertion is a consequence of Theorem 4.8, while the converse assertion was proved in part (2) of Theorem 2.5. ■

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