

SUMS OF SQUARES II: MATRIX FUNCTIONS

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ABSTRACT. This paper is the second in a series of three papers devoted to sums of squares and hypoellipticity of infinitely degenerate operators. In the first paper we established a sharp ω -monotonicity criterion for writing a smooth nonnegative function f that is flat at, and positive away from, the origin, as a finite sum of squares of $C^{2,\delta}$ functions for some $\delta > 0$, namely that f is ω -monotone for some Hölder modulus of continuity ω . Counterexamples were provided for any larger modulus of continuity.

In this paper we consider the analogous sum of squares problem for smooth nonnegative matrix functions M that are flat at, and positive away from, the origin. We show that such a matrix function $M = [a_{kj}]_{k,j=1}^n$ can be written as a finite sum of squares of $C^{2,\delta}$ vector fields if the diagonal entries a_{kk} are ω -monotone for some Hölder modulus of continuity ω , and if the off diagonal entries satisfy certain differential bounds in terms of powers of the diagonal entries. Examples are given to show that in some cases at least, these differential inequalities cannot be relaxed.

Various refinements of this result are also given in which one or more of the diagonal entries need not be assumed to have any monotonicity properties at all. These sum of squares decompositions will be applied to hypoellipticity in the infinitely degenerate regime in the third paper in this series.

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1. INTRODUCTION

In the theory of partial differential operators, the ability to write a pure second order real differential operator $L(x) = \text{trace}[\mathbf{A}(x) \nabla \otimes \nabla]$ as a sum of squares of vector fields $X_j(x)$, i.e. $\mathbf{A}(x) = \sum_{j=1}^N X_j(x) X_j(x)^{\text{tr}}$, with some specified smoothness, has proven to be of immense value. See e. g. work of Hörmander [Hor], Rothschild and Stein [RoSt], Christ [Chr] and Sawyer, Rios and Wheeden [RiSaWh, RiSaWh] to mention just a few. If $\mathbf{A}(x)$ is nonnegative, then the spectral theorem shows that $\mathbf{A}(x) = \sum_{j=1}^n \left(\sqrt{\lambda_j(x)} \mathbf{v}_j(x) \right) \left(\sqrt{\lambda_j(x)} \mathbf{v}_j(x) \right)^{\text{tr}}$ where $\{\mathbf{v}_j(x)\}_{j=1}^n$ is an orthonormal set of eigenvectors for $\mathbf{A}(x)$, but little can be typically said regarding smoothness of $\sqrt{\lambda_j(x)} \mathbf{v}_j(x)$. Thus it becomes an important question as to whether or not a given matrix function $\mathbf{A}(x)$ can be represented as a finite sum of squares of vector functions with preassigned smoothness.

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We will refer to the rank one matrix $X_j(x) X_j(x)^{\text{tr}} = X_j(x) \otimes X_j(x)$ either as a square of a vector field, or as a positive dyad¹. We point out the obvious fact that a sum of squares $\sum_{j=1}^N X_j X_j^{\text{tr}}$ is positive semidefinite and equals $\mathbf{X} \mathbf{X}^{\text{tr}}$ where $\mathbf{X}$ is the $n \times N$ matrix with columns X_j .

In this paper we extend the sums of squares results for *scalar* functions obtained in [KoSa1] to the setting of matrix-valued functions, where these decompositions will be used in the subsequent and final paper of this series [KoSa3] to obtain hypoellipticity results in the infinitely degenerate regime.

In order for an $n \times n$ nonnegative matrix function $A(x) = [a_{ij}(x)]_{1 \leq i, j \leq n}$ to be a sum of squares $\sum_{\ell=1}^N \mathbf{v}^\ell(x) \otimes \mathbf{v}^\ell(x)$ of vector functions (with some specified smoothness such as $C^{2,\delta}$), it is of course necessary that the diagonal entries $a_{ii}(x)$ themselves be a sum of such squares, namely $\sum_{\ell=1}^N (\mathbf{v}^\ell(x))_i^2$. A well known and clever construction of Fefferman-Phong [FePh] has been used by Tataru [Tat] and Bony [Bon], see also Guan [Gua], to show that every $C^{3,1}$ scalar function $f(x)$ can be written as a sum of squares of $C^{1,1}$ functions. However, this degree of smoothness falls short of what is needed in the theory of hypoellipticity of sums of squares of infinitely degenerate vector fields, the appropriate level of smoothness being $C^{2,\delta}$ for some $\delta > 0$. The theorem proved in [KoSa1] shows that a nonnegative scalar function $f(x)$ can be written as a sum of squares of $C^{2,\delta}$ functions provided f is *strongly* $C^{4,2\delta}$, by which we mean f is $C^{4,2\delta}$, positive away from the origin, and satisfies $|\nabla^2 f(x)|, |\nabla^4 f(x)| \lesssim f(x)^{\delta'}$ for some $\delta' > 0$. A large class of examples of such functions is the set of Hölder monotone functions, those for which there is $C, s > 0$ such that

$$f(y) \leq C f(x)^s \text{ for } y \in B\left(\frac{x}{2}, \frac{|x|}{2}\right).$$

This class is sharp in the sense that the logarithmic version of this inequality fails to imply a decomposition into a sum of squares of $C^{2,\delta}$ functions, see part (2) of Theorem 6 below.

Nevertheless, this latter condition on the diagonal entries of $\mathbf{A}(x)$ is not sufficient for $\mathbf{A}(x)$ to be a sum of squares of $C^{2,\delta}$ vector fields, as our counterexamples below show. Instead, we assume in addition a collection of natural inequalities on derivatives up to order four of the off diagonal entries $a_{ij}(x)$ having the form:

$$|D^\alpha a_{k,j}(x)| \lesssim \left(\min_{1 \leq \ell \leq j} a_{\ell,j}(x) \right)^{\frac{1}{2} + (2 - |\mu|)\varepsilon}, \quad \text{for all } k < j, 0 \leq |\mu| \leq 4, \text{ and some } \varepsilon \geq \frac{1}{4}.$$

We also give examples showing that such differential inequalities on off diagonal entries of the matrix are sharp in some cases.

But first we recall the main results from [KoSa1] that will be used here.

Definition 1. A scalar or matrix function $A : \mathbb{R}^n \rightarrow \mathbb{R}^{N \times N}$ is elliptical if the associated quadratic form $Q_A(x, \xi) \equiv \xi^{\text{tr}} A(x) \xi$ is strictly positive away from the ‘axes’ $\{0\} \times \mathbb{R}^N$ and $\mathbb{R}^n \times \{0\}$ in $\mathbb{R}^n \times \mathbb{R}^N$.

Definition 2. We say that a scalar, vector or matrix function $g : \mathbb{R}^n \rightarrow [0, \infty)$ is regular if $g \in \bigcup_{\delta > 0} C^{2,\delta}(\mathbb{R}^n)$, i.e. g is $C^{2,\delta}$ for some $0 < \delta < 1$.

Now we recall various notions of monotonicity from [KoSa1].

Definition 3. Let ω be a modulus of continuity on $[0, 1]$, and let $f : \mathbb{R}^n \rightarrow [0, \infty)$. We say

- (1) f is ω -monotone if $f(y) \leq C\omega(f(x))$ for $y \in B\left(\frac{x}{2}, \frac{|x|}{2}\right)$ and some positive constant C ,
- (2) $\omega_s(t) = \begin{cases} t(1 + \ln \frac{1}{t}) & \text{if } s = 1 \\ t^s & \text{if } 0 < s < 1 \\ \frac{1}{1 + \ln \frac{1}{t}} & \text{if } s = 0 \end{cases}$, for $0 \leq s, t \leq 1$,
- (3) f is *nearly monotone* if f is ω_s -monotone for every $0 \leq s < 1$,
- (4) f is *Hölder monotone* if f is ω_s -monotone for some $0 < s \leq 1$.

¹A nontrivial dyad $a \otimes b$ is positive if and only if $a = b$. Indeed,

$$\xi^{\text{tr}} [a \otimes b] \xi = \xi^{\text{tr}} \langle a, \xi \rangle b = \langle \xi, b \rangle \langle a, \xi \rangle \geq 0$$

for all ξ if and only if $a = b$.

Finally we recall the following seminorm from [Bon],

$$(1.1) \quad [h]_{\alpha, \delta}(x) \equiv \limsup_{y, z \rightarrow x} \frac{|D^\alpha h(y) - D^\alpha h(z)|}{|y - z|^\delta},$$

We can also norm the space $C^{m, \alpha}(K)$ with $\|h\|_{m, \alpha} \equiv \|h\|_{C^m(K)} + \sum_{|\alpha|=m} \sup_{x \in K} [D^\alpha h]_{\alpha, \delta}(x)$ when K is compact.

Theorem 4 (Theorem 28 from [KoSa1]). *Suppose $0 < \delta < 1$ and that f is a $C^{4, 2\delta}$ function on $\mathbb{R}^n$. Let*

$$\rho(x) = \rho_f(x) \equiv \max \left\{ f(x)^{\frac{1}{4+2\delta}}, \left(\sup_{\Theta \in \mathbb{S}^{n-1}} [\partial_\Theta^2 f(x)]_+ \right)^{\frac{1}{2+2\delta}}, |\nabla^4 f(x)|^{\frac{1}{2\delta}} \right\}, \quad x \in \mathbb{R}^n.$$

(1) *If f satisfies the differential inequalities*

$$(1.2) \quad |\nabla^4 f(x)| \leq C f(x)^{\frac{\delta}{2+\delta}} \quad \text{and} \quad |\nabla^2 f(x)| \leq C f(x)^{\frac{2\delta(1+\delta)}{2+\delta}},$$

then $f = \sum_{\ell=1}^N g_\ell^2$ can be decomposed as a finite sum of squares of functions $g_\ell \in C^{2+\delta}(\mathbb{R}^2)$ where

$$(1.3) \quad \begin{aligned} |D^\alpha g_\ell(x)| &\leq C \rho(x)^{2+\delta-|\alpha|} \leq C f(x)^{\frac{\delta}{2+\delta}(2+\delta-|\alpha|)}, \quad 0 \leq |\alpha| \leq 2, \\ [g_\ell]_{\alpha, \delta}(x) &\leq C, \quad |\alpha| = 2. \end{aligned}$$

and

$$\begin{aligned} |D^\alpha g_\ell^2(x)| &\leq C \rho(x)^{4+2\delta-|\alpha|} \leq C f(x)^{\frac{\delta}{2+\delta}(4+2\delta-|\alpha|)}, \quad 0 \leq |\alpha| \leq 4, \\ [g_\ell^2]_{\alpha, 2\delta}(x) &\leq C, \quad |\alpha| = 4. \end{aligned}$$

(2) *In particular, the inequalities (1.2) hold provided f is flat, smooth and ω_s -monotone for some $s < 1$ satisfying*

$$(1.4) \quad s > \sqrt[4]{\delta} \sqrt{\frac{2}{4+2\delta}}, \quad \text{and} \quad s \geq \sqrt{\delta} \sqrt{\frac{2+2\delta}{2+\delta}}.$$

Remark 5. *The inequalities (1.2) also hold if the smoothness assumption on f is relaxed to $f \in C^k$, provided that s is replaced by $s - \frac{C}{k}$ in (1.4) for a sufficiently large constant C independent of k .*

The next result shows that the Hölder monotone class comes close to being sharp for a decomposition into a finite sum of squares of regular functions.

Theorem 6. *Let $n \geq 5$. There is an elliptical, flat, smooth ω_0 -monotone function f that cannot be written as a finite sum of squares of regular functions. Moreover, ω_0 can be replaced by any modulus of continuity ω with $\omega_s \ll \omega$, i.e. $\lim_{t \searrow 0} \frac{\omega_s(t)}{\omega(t)} = 0$, for all $0 < s < 1$.*

Corollary 7. *Suppose that $f : \mathbb{R}^n \rightarrow [0, \infty)$ is elliptical, flat and smooth.*

- (1) *Then f can be written as a finite sum of squares of regular functions if f is Hölder monotone.*
- (2) *Conversely, for any modulus of continuity ω satisfying $\omega_s \ll \omega$ for all $0 < s < 1$, there is an ω -monotone function f that cannot be written as a finite sum of squares of regular functions.*

Finally, we recall the following estimates for nearly monotone functions from [KoSa1].

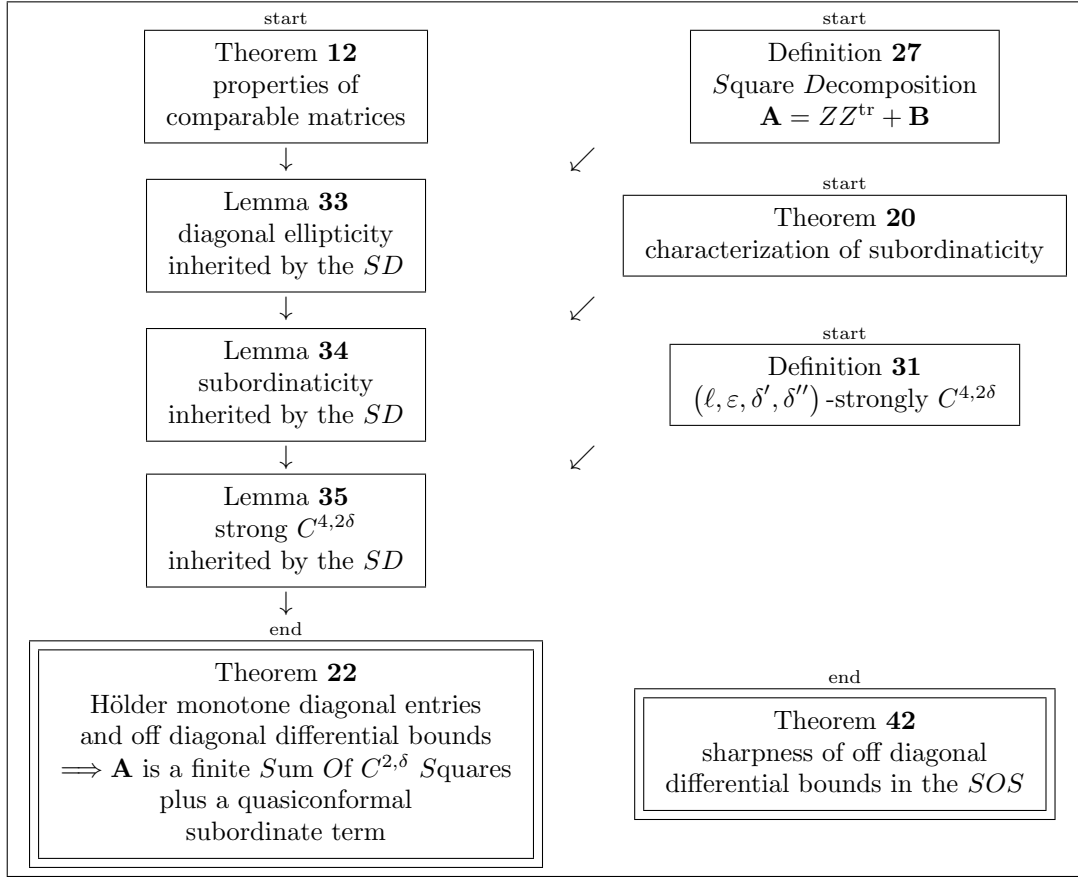
Theorem 8. *Let $n \geq 1$. Suppose that $f : B(0, a) \rightarrow [0, \infty)$ is an elliptical flat smooth function on $B(0, a) \subset \mathbb{R}^n$. Then the first three of the following four conditions are equivalent. Moreover, the fourth condition, which holds in particular if f is ω_1 monotone, implies the first three conditions, but not conversely. Finally, for any $0 < s < 1$, there is an ω_s -monotone function f such that $f^{\frac{1}{s}-1}$ is not smooth.*

- (1) *There is $\delta > 0$ such that $f(x)^\gamma$ is smooth on $B(0, a)$ for all $0 < \gamma < \delta$.*
- (2) *For every $m \geq 1$ and $0 < s < 1$, there is a positive constant $\Gamma_{n, m, s}$ such that*

$$|\nabla^m f(x)| \leq \Gamma_{n, m, s} f(x)^s, \quad \text{for } x \in B(0, a).$$

- (3) *The functions $f(x)^\gamma$ are flat smooth functions on $B(0, a)$ for all $\gamma > 0$.*
- (4) *The function f is nearly monotone.*

In this paper we will apply the above sums of squares representations for scalar functions to obtain representations of matrix functions as sums of squares of $C^{2,\delta}$ vector fields. For the reader's convenience we include a schematic diagram of connections between theorems. Results in a double box are logical ends.



2. STATEMENT OF MAIN MATRIX DECOMPOSITION THEOREMS

Definition 9. Let A and B be real symmetric positive semidefinite $n \times n$ matrices. We define $A \preceq B$ if $B - A$ is positive semidefinite. Let $\beta < \alpha$ be positive constants. A real symmetric positive semidefinite $n \times n$ matrix A is said to be (β, α) -comparable to a symmetric $n \times n$ matrix B , written $A \sim_{\beta, \alpha} B$, if $\beta B \preceq A \preceq \alpha B$, i.e.

$$(2.1) \quad \beta \xi^{\text{tr}} B \xi \leq \xi^{\text{tr}} A \xi \leq \alpha \xi^{\text{tr}} B \xi, \quad \text{for all } \xi \in \mathbb{R}^n.$$

Definition 10. Given two symmetric matrix-valued functions $\mathbf{A}(x)$ and $\mathbf{B}(x)$, we say $\mathbf{A}(x)$ is comparable to $\mathbf{B}(x)$ if there are positive constants $0 < \beta < \alpha$ independent of x , such that $\mathbf{A}(x)$ is (β, α) -comparable to $\mathbf{B}(x)$ for all $x \in \mathbb{R}^n$. In this case we write $\mathbf{A}(x) \sim \mathbf{B}(x)$.

Note that if A is comparable to B , then both A and B are positive semidefinite. Indeed, both $0 \leq (\alpha - \beta) \xi^{\text{tr}} B \xi$ and $0 \leq \left(\frac{1}{\beta} - \frac{1}{\alpha}\right) \xi^{\text{tr}} A \xi$ hold for all $\xi \in \mathbb{R}^n$. Moreover, for any x we have that $\mathbf{A}(x)$ is positive definite if and only if $\mathbf{B}(x)$ is positive definite.

We first give a simple characterization of when a symmetric positive semidefinite matrix-valued function is comparable to a diagonal matrix-valued function that is positive away from the origin. In order to state this result precisely, we first establish the general fact that if a matrix $\mathbf{A}$ is (β, α) -comparable to a diagonal

matrix $\mathbf{D}_\lambda$ as above, then $\mathbf{A}$ is $\left(\frac{\beta}{\alpha}, \frac{\alpha}{\beta}\right)$ -comparable to its associated diagonal matrix

$$\mathbf{A}_{\text{diag}} \equiv \mathbf{D}_\mathbf{F} = \begin{bmatrix} F_1 & 0 & \cdots & 0 \\ 0 & F_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F_n \end{bmatrix},$$

in short $\mathbf{A} \sim_{\beta, \alpha} \mathbf{D}_\lambda \implies \mathbf{A} \sim_{\frac{\beta}{\alpha}, \frac{\alpha}{\beta}} \mathbf{A}_{\text{diag}}$, equivalently

$$\beta \mathbf{D}_\lambda \preceq \mathbf{A} \preceq \alpha \mathbf{D}_\lambda \implies \frac{\beta}{\alpha} \mathbf{A}_{\text{diag}} \preceq \mathbf{A} \preceq \frac{\alpha}{\beta} \mathbf{A}_{\text{diag}}.$$

Indeed, it is an easy exercise to show that $\beta \mathbf{D}_\lambda \preceq \mathbf{A} \preceq \alpha \mathbf{D}_\lambda$ implies $\beta \lambda_k \leq F_k \leq \alpha \lambda_k$ implies $\beta \mathbf{D}_\lambda \preceq \mathbf{D}_\mathbf{F} \preceq \alpha \mathbf{D}_\lambda$, and since $\mathbf{D}_\mathbf{F} = \mathbf{A}_{\text{diag}}$ we then have

$$\frac{\beta}{\alpha} \mathbf{A}_{\text{diag}} \preceq \beta \mathbf{D}_\lambda \preceq \mathbf{A} \preceq \alpha \mathbf{D}_\lambda \preceq \frac{\alpha}{\beta} \mathbf{A}_{\text{diag}}.$$

This becomes important when we consider matrix-valued functions $\mathbf{A}(x)$ (of a variable $x \in \mathbb{R}^n$) that are (β, α) -comparable to a diagonal matrix-valued function $\mathbf{D}(x)$, and which are positive definite away from the origin. For example, the matrix function $\mathbf{A}(x) \equiv \begin{bmatrix} 1 & 1 - e^{-\frac{1}{x^2}} \\ 1 - e^{-\frac{1}{x^2}} & 1 \end{bmatrix}$ is positive definite away from the origin, yet is *not* (β, α) -comparable to any diagonal matrix-valued function $\mathbf{D}(x)$. Indeed, if it were, then $\mathbf{A}(x)$ would be (β, α) -comparable to its associated diagonal matrix, i.e. the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which clearly fails for x close to the origin. It turns out that for matrix functions $\mathbf{A}(x)$ that are both positive definite away from the origin and (β, α) -comparable to some diagonal matrix-valued function $\mathbf{D}(x)$, we can derive useful consequences on the off diagonal entries of $\mathbf{A}(x)$. Since such matrix functions play a major role in the sequel we make a definition to encompass them.

Definition 11. We say that a matrix-valued function $\mathbf{A}(x)$ is diagonally elliptical on $\mathbb{R}^n$ if $\mathbf{A}(x)$ is comparable to a diagonal matrix-valued function $\mathbf{D}(x)$ for all $x \in \mathbb{R}^n$, and if $\mathbf{A}(x)$ is positive definite away from the origin.

We see from the above discussion that, if $\mathbf{A}(x)$ is diagonally elliptical, then $\mathbf{A}(x)$ is comparable to its associated diagonal matrix-valued function $\mathbf{A}_{\text{diag}}(x)$, a fact which will play a key role in deriving useful properties of diagonally elliptical matrix functions. In the next two subsections we state and prove a relatively simple property of a diagonally elliptic matrix function, as well as an easy characterization of when a matrix function is subordinate. In the third subsection we state our sum of $C^{2,\delta}$ squares theorem, which is then proved in the third section, and finally we demonstrate in Section 4 some results on sharpness.

2.1. The comparability theorem. First note that if $\mathbf{A}(x) \sim \mathbf{B}(x)$, then $\widehat{\mathbf{A}}(x) \sim \widehat{\mathbf{B}}(x)$, with the same comparability constants $0 < \beta < \alpha < \infty$, where $\widehat{\mathbf{A}}(x)$ is any principal submatrix of $\mathbf{A}(x)$ and $\widehat{\mathbf{B}}(x)$ is the corresponding principal submatrix of $\mathbf{B}(x)$. Here is the only other consequence of comparability that we will need.

Theorem 12. A symmetric positive definite $n \times n$ matrix-valued function

$$\mathbf{A}(x) = [a_{k,j}(x)]_{k,j=1}^n = \begin{bmatrix} a_{11}(x) & b(x)^{\text{tr}} \\ b(x) & \mathbf{D}(x) \end{bmatrix}$$

is comparable to its associated diagonal matrix-valued function $\mathbf{A}_{\text{diag}}(x)$ only if there is $0 < \beta < 1$ such that

$$(2.2) \quad b(x)^{\text{tr}} [\mathbf{D}(x) - \beta \mathbf{D}_{\text{diag}}(x)]^{-1} b(x) < (1 - \beta) a_{11}(x), \quad \text{for all } x.$$

In particular,

$$(2.3) \quad |a_{k,j}(x)| < (1 - \beta) \sqrt{a_{k,k}(x) a_{j,j}(x)}, \quad \text{for all } 1 \leq k < j \leq n \text{ and all } x.$$

For this we will use the following three lemmas.

Lemma 13. *Let $\mathbf{A}$ be a real symmetric positive semidefinite $n \times n$ matrix. Then there is a real symmetric positive semidefinite $n \times n$ real matrix $\sqrt{\mathbf{A}}$ satisfying $\mathbf{A} = \sqrt{\mathbf{A}}\sqrt{\mathbf{A}}$.*

Proof. Let $\mathbf{P}$ be an orthogonal matrix such that $\mathbf{P}\mathbf{A}\mathbf{P}^{\text{tr}} = \mathbf{D}$ is a diagonal matrix with nonnegative entries $\{\lambda_i\}_{i=1}^n$ along the diagonal. If $\sqrt{\mathbf{D}}$ is the diagonal matrix with entries $\{\sqrt{\lambda_i}\}_{i=1}^n$ along the diagonal, then the matrix $\sqrt{\mathbf{A}} \equiv \mathbf{P}^{\text{tr}}\sqrt{\mathbf{D}}\mathbf{P}$ is symmetric and positive semidefinite, and satisfies

$$\sqrt{\mathbf{A}}\sqrt{\mathbf{A}} = \mathbf{P}^{\text{tr}}\sqrt{\mathbf{D}}\mathbf{P}\mathbf{P}^{\text{tr}}\sqrt{\mathbf{D}}\mathbf{P} = \mathbf{P}^{\text{tr}}\sqrt{\mathbf{D}}\sqrt{\mathbf{D}}\mathbf{P} = \mathbf{P}^{\text{tr}}\mathbf{D}\mathbf{P} = \mathbf{A}.$$

■

Lemma 14. *Let $n \geq 2$. For any number α , $(n-1)$ -dimensional vector v and invertible $(n-1) \times (n-1)$ matrix $\mathbf{M}$, we have the determinant formula*

$$\det \begin{bmatrix} \alpha & v^{\text{tr}} \\ v & \mathbf{M} \end{bmatrix} = \alpha \det \mathbf{M} - v^{\text{tr}} [\text{co } \mathbf{M}]^{\text{tr}} v = \{\alpha - v^{\text{tr}} \mathbf{M}^{-1} v\} \det \mathbf{M}.$$

Proof. We prove only the case $n = 3$, in which case we compute that with $\alpha = d$, $v = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{M} = \begin{bmatrix} e & c \\ c & f \end{bmatrix}$ we have

$$\begin{aligned} \det \begin{bmatrix} d & a & b \\ a & e & c \\ b & c & f \end{bmatrix} &= d \det \begin{bmatrix} e & c \\ c & f \end{bmatrix} - a \det \begin{bmatrix} a & b \\ c & f \end{bmatrix} + b \det \begin{bmatrix} a & b \\ e & c \end{bmatrix} \\ &= d \det \begin{bmatrix} e & c \\ c & f \end{bmatrix} - a(af - bc) + b(ac - be) \\ &= d \det \begin{bmatrix} e & c \\ c & f \end{bmatrix} - a^2 f + abc + abc - b^2 e \end{aligned}$$

and we continue with

$$\begin{aligned} \det \begin{bmatrix} d & a & b \\ a & e & c \\ b & c & f \end{bmatrix} &= d \det \begin{bmatrix} e & c \\ c & f \end{bmatrix} - \begin{pmatrix} a & b \end{pmatrix} \begin{bmatrix} f & -c \\ -c & e \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\ &= d \det \begin{bmatrix} e & c \\ c & f \end{bmatrix} - \begin{pmatrix} a & b \end{pmatrix} \text{co} \begin{bmatrix} e & c \\ c & f \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\ &= \det \begin{bmatrix} e & c \\ c & f \end{bmatrix} \left\{ d - \begin{pmatrix} a & b \end{pmatrix} \begin{bmatrix} e & c \\ c & f \end{bmatrix}^{-1} \begin{pmatrix} a \\ b \end{pmatrix} \right\}. \end{aligned}$$

■

The next lemma will use the following well known characterization of positive definite matrices.

Theorem 15. *A symmetric $n \times n$ matrix A is positive definite if and only if the determinant of every right lower principal submatrix of A is positive.*

Lemma 16. *Let $\mathbf{f}$ and $\mathbf{F}$ be real symmetric positive definite $(n-1) \times (n-1)$ matrices, and let v be a real $(n-1)$ -dimensional column vector. Then for β real,*

$$\beta \begin{bmatrix} H & 0^{\text{tr}} \\ 0 & \mathbf{f} \end{bmatrix} \prec \begin{bmatrix} h^2 & v^{\text{tr}} \\ v & \mathbf{F} \end{bmatrix},$$

if and only if

$$\begin{aligned} h^2 - \beta H &> 0, \\ \mathbf{G}_\beta &\equiv \mathbf{F} - \beta \mathbf{f} \text{ is real, symmetric, and positive definite,} \end{aligned}$$

and

$$v^{\text{tr}} \mathbf{G}_\beta^{-1} v \leq h^2 - \beta H.$$

Proof. Theorem 15 and Lemma 14 show that

$$0 \prec \begin{bmatrix} h^2 - \beta H & v^{\text{tr}} \\ v & \mathbf{F} - \beta \mathbf{f} \end{bmatrix} = \begin{bmatrix} h^2 - \beta H & v^{\text{tr}} \\ v & \mathbf{G}_\beta \end{bmatrix}$$

holds if and only if $h^2 - \beta H > 0$, $\mathbf{G}_\beta \succ 0$ and

$$0 < \det \begin{bmatrix} h^2 - \beta H & v^{\text{tr}} \\ v & \mathbf{G}_\beta \end{bmatrix} = (\det \mathbf{G}_\beta) \{h^2 - \beta H - v^{\text{tr}} \mathbf{G}_\beta^{-1} v\},$$

which in turn holds if and only if

$$v^{\text{tr}} \mathbf{G}_\beta^{-1} v < h^2 - \beta H.$$

■

Now we can prove Theorem 12.

Proof of Theorem 12. Suppose $\mathbf{A} \sim \mathbf{A}_{\text{diag}}$, say

$$\beta \mathbf{A}_{\text{diag}} \preceq \mathbf{A} \preceq \alpha \mathbf{A}_{\text{diag}}, \quad \text{with } 0 < \beta < \alpha < \infty,$$

where 1 must belong to $[\beta, \alpha]$ in this case. Now apply Lemma 16 with $h^2 = a_{11}$, $\mathbf{f} = \mathbf{D}_{\text{diag}}$ and $\mathbf{F} = \mathbf{D}$ to obtain

$$b^{\text{tr}} [\mathbf{D} - \beta \mathbf{D}_{\text{diag}}]^{-1} b \leq a_{11} - \beta a_{11} = (1 - \beta) a_{11},$$

which is (2.2).

To obtain (2.3), we use the fact noted just before Theorem 12, that every 2×2 principal submatrix $\hat{\mathbf{A}} = \begin{bmatrix} a_{k,k} & a_{k,j} \\ a_{k,j} & a_{j,j} \end{bmatrix}$ of $\mathbf{A}$ satisfies $\hat{\mathbf{A}} \sim \hat{\mathbf{A}}_{\text{diag}} = \widehat{\mathbf{A}_{\text{diag}}}$ with the *same* comparability constants $\beta < \alpha$, and hence by (2.2) we conclude that

$$\begin{aligned} a_{k,j} [a_{j,j} - \beta a_{j,j}]^{-1} a_{k,j} &\leq a_{k,k} - \beta a_{k,k} = (1 - \beta) a_{k,k}, \\ \text{i.e. } |a_{k,j}|^2 &\leq (a_{j,j} - \beta a_{j,j}) (1 - \beta) a_{k,k} = (1 - \beta)^2 a_{j,j} a_{k,k}, \end{aligned}$$

which is (2.3). ■

2.2. The subordinaticity theorem. There is a second concept that will play a significant role in hypoellipticity theorems, and which we now introduce.

Definition 17. A symmetric matrix function $A(x)$ defined for $x \in \mathbb{R}^M$ is said to be subordinate if there is $\Gamma > 0$ such that for every first order partial derivative $\frac{\partial}{\partial x_k}$, the matrix $\frac{\partial}{\partial x_k} A(x)$ exists and satisfies

$$(2.4) \quad \left| \frac{\partial}{\partial x_k} A(x) \xi \right|^2 \leq \Gamma^2 \xi^{\text{tr}} A(x) \xi, \quad 1 \leq k \leq M.$$

The inequality (2.4) holds for all smooth nonnegative *diagonal* matrices $D(x)$, by the classical Malgrange inequality for C^2 scalar functions. However, we note that the subordination property (2.4) **fails** miserably for nondiagonal matrices $A(x)$ in general, even for 2×2 matrices in one variable $x \in \mathbb{R}$ that are comparable to a smooth diagonally elliptical matrix $A(x)$ that is a sum of squares of smooth vector fields. For example

$$A(x) = \begin{bmatrix} 1 & \gamma f(x) \\ \gamma f(x) & f(x)^2 \end{bmatrix}, \quad \text{where } 0 < |\gamma| < 1,$$

fails to be subordinate since

$$\left| f'(x) \begin{bmatrix} 0 & \gamma \\ \gamma & 2f(x) \end{bmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \right|^2 \leq C \begin{pmatrix} \xi_1 & \xi_2 \end{pmatrix} \begin{bmatrix} 1 & \gamma f(x) \\ \gamma f(x) & f(x)^2 \end{bmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix},$$

implies with $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ that $f'(x)^2 \left| \begin{pmatrix} \gamma \\ 2f(x) \end{pmatrix} \right|^2 \leq C f(x)^2$, and hence $\left| \frac{d}{dx} \ln f(x) \right|^2 = \frac{f'(x)^2}{f(x)^2} \leq \frac{C}{\gamma^2}$.

But $\ln f(x)$ cannot be bounded near 0 for any smooth nonnegative $f(x)$ that vanishes at 0. However, in the case f is smooth, then $A(x)$ is a sum of smooth squares

$$A(x) = \begin{pmatrix} 1 \\ \gamma f(x_1) \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \gamma f(x_1) \end{pmatrix} + \begin{pmatrix} 0 \\ \sqrt{1 - \gamma^2} f(x_1) \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \sqrt{1 - \gamma^2} f(x_1) \end{pmatrix},$$

illustrating the fact that a sum of smooth squares need not be subordinate. Conversely, in the final section of this paper, we give an example in Theorem 37 of a diagonally elliptical 3×3 matrix function $\mathbf{Q}(x, y, z)$ of three variables that is subordinate, but not a finite sum of squares of $C^{1,\delta}$ vector fields for any $\delta > 0$.

Definition 18. A matrix function $\mathbf{M}$ is said to be $\mathcal{SOS}_{k,\delta}$ if it can be written as a finite sum of squares of $C^{k,\delta}$ vector fields.

Conclusion 19. The property that a matrix function $\mathbf{M}$ is $\mathcal{SOS}_{1,\delta}$ is in general incomparable with the property that $\mathbf{M}$ is subordinate.

Theorem 20. If $A = [a_{ij}]_{i,j=1}^n$ is an $n \times n$ positive semidefinite C^2 matrix function that is comparable to a diagonal matrix function, then A is subordinate if and only if

$$(2.5) \quad |\nabla a_{ij}|^2 \lesssim \min\{a_{ii}, a_{jj}\}, \quad 1 \leq i, j \leq n.$$

In particular, a matrix function $A = [a_{ij}]_{i,j=1}^n$ with $a_{ii} \approx \lambda$ for all $1 \leq i \leq n$, is subordinate if and only if

$$|\nabla a_{ij}|^2 \lesssim \lambda, \quad 1 \leq i, j \leq n.$$

Note that this latter set of inequalities amount to assuming an extension of Malgrange's classical inequality $|\nabla a_{ii}|^2 \lesssim a_{ii}$ (which requires $a_{ii} \in C^2$) to the off diagonal case $i \neq j$.

Proof. Let f' denote any of the partial derivatives $\frac{\partial f}{\partial x_k}$. We begin by noting that $A(x)$ is subordinate if and only if

$$\begin{aligned} \sum_{i=1}^n \left| \sum_{j=1}^n a'_{i,j}(x) \xi_j \right|^2 &= \left| \begin{bmatrix} a'_{11}(x) & \cdots & a'_{1n}(x) \\ \vdots & \ddots & \vdots \\ a'_{n1}(x) & \cdots & a'_{nn}(x) \end{bmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \right|^2 \\ &= |A'(x)\xi|^2 \lesssim \xi^{\text{tr}} A(x) \xi \approx \xi^{\text{tr}} \begin{bmatrix} a_{11}(x) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_{nn}(x) \end{bmatrix} \xi = \sum_{i=1}^n a_{i,i}(x) \xi_i^2. \end{aligned}$$

Taking $\xi = \mathbf{e}_1, \dots, \mathbf{e}_n$ respectively shows that the following n conditions are necessary and sufficient;

$$\begin{aligned} |a'_{1,1}(x)|^2 + \sum_{j: j \neq 1} |a'_{1,j}(x)|^2 &\lesssim a_{1,1}(x), \\ |a'_{2,2}(x)|^2 + \sum_{j: j \neq 2} |a'_{2,j}(x)|^2 &\lesssim a_{2,2}(x), \\ &\vdots \\ |a'_{n-1,n-1}(x)|^2 + \sum_{j: j \neq n-1} |a'_{n-1,j}(x)|^2 &\lesssim a_{n,n}(x), \\ |a'_{n,n}(x)|^2 + \sum_{j: j \neq n} |a'_{n,j}(x)|^2 &\lesssim a_{n,n}(x). \end{aligned}$$

The inequality of Malgrange already shows that the diagonal entries satisfy $|a'_{i,i}(x)|^2 \lesssim a_{i,i}(x)$, and since the derivative $\nabla a_{i,j}(x)$ of an off diagonal entry occurs in only two of the above lines, namely in the i^{th} and j^{th} lines, we see that the conditions in the display above hold if and only if (2.5) holds. ■

2.3. The sum of squares theorem. Here is the sum of squares decomposition for a Grushin type matrix function with a quasiformal block of order $(n-p+1) \times (n-p+1)$, where $1 < p \leq n$.

Definition 21. Suppose that $\mathbf{A}(x)$ is a diagonally elliptical $n \times n$ matrix function on $\mathbb{R}^n$.

- (1) We say that $\mathbf{A}(x)$ is quasiconformal if the eigenvalues $\lambda_i(x)$ of $\mathbf{A}(x)$ are nonnegative and comparable.
- (2) We say that $\mathbf{A}(x)$ is of Grushin type if $\mathbf{A}(x)$ is singular exactly on a vector subspace Λ of $\mathbb{R}^n$ having dimension $m < n$, and if the diagonal entries $a_{k,k}(x)$ of $\mathbf{A}(x)$ are each comparable to a function $\lambda_k(y)$ depending only on $y \in \Lambda$.

Roughly speaking, the next theorem says in particular that if a subordinate diagonally elliptical $C^{4,2\delta}$ matrix function $\mathbf{A}(x)$ of Grushin type has diagonal entries $a_{k,k}(x)$ that are comparable to the last entry $a_{n,n}$ for $p \leq k \leq n$ (but unrelated to the first $p-1$ diagonal entries), and finally if the off-diagonal entries satisfy suitable subordinate type inequalities, then $\mathbf{A}$ is a finite sum of squares of $C^{2,\delta}$ vector fields plus a $C^{4,2\delta}$ block matrix function $\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_p \end{bmatrix}$ where the $(n-p+1)$ -square matrix $\mathbf{Q}_p$ is both subordinate and quasiconformal.

Theorem 22. *Let*

$$1 < p \leq n, \quad \frac{1}{4} \leq \varepsilon < 1, \quad 0 < \delta < \delta', \delta'' < 1, \quad M \geq 1,$$

with

$$\delta' = \frac{2\delta(1+\delta)}{2+\delta}.$$

Suppose that $\mathbf{A}(x)$ is a $C^{4,2\delta}$ symmetric $n \times n$ matrix function of a variable $x \in \mathbb{R}^M$, which is comparable to a diagonal matrix function $\mathbf{D}(x)$, hence comparable to its associated diagonal matrix function $\mathbf{A}_{\text{diag}}(x)$.

- (1) Moreover, assume $a_{p,p}(x) \approx a_{p+1,p+1}(x) \approx \dots \approx a_{n,n}(x)$ and that the diagonal entries $a_{1,1}(x), \dots, a_{p-1,p-1}(x)$ satisfy the following differential estimates up to fourth order,

$$(2.6) \quad \begin{aligned} |D^\mu a_{k,k}(x)| &\lesssim a_{k,k}(x)^{[1-|\mu|\varepsilon]_+ + \delta'}, & 1 \leq |\mu| \leq 4 \text{ and } 1 \leq k \leq p-1, \\ [a_{k,k}]_{\mu, 2\delta}(x) &\lesssim 1, & |\mu| = 4 \text{ and } 1 \leq k \leq p-1. \end{aligned}$$

- (2) Furthermore, assume the off diagonal entries $a_{k,j}(x)$ satisfy the following differential estimates up to fourth order,

$$(2.7) \quad \begin{aligned} |D^\mu a_{k,j}| &\lesssim \left(\min_{1 \leq s \leq j} a_{s,s} \right)^{[\frac{1}{2} + (2-|\mu|)\varepsilon]_+ + \delta''}, & 0 \leq |\mu| \leq 4 \text{ and } 1 \leq k < j \leq p-1, \\ [a_{k,j}]_{\mu, 2\delta} &\lesssim 1, & |\mu| = 4 \text{ and } 1 \leq k < j \leq p-1, \\ |D^\mu a_{k,j}| &\lesssim \left(\min_{1 \leq s \leq k} a_{s,s} \right)^{[\frac{1}{2} + (2-|\mu|)\varepsilon]_+ + \delta''}, & 0 \leq |\mu| \leq 4 \text{ and } 1 \leq k \leq p-1 < j \leq n, \\ [a_{k,j}]_{\mu, 2\delta} &\lesssim 1, & |\mu| = 4 \text{ and } 1 \leq k \leq p-1 < j \leq n. \end{aligned}$$

- (3) Then there is a positive integer $I \in \mathbb{N}$ such that the matrix function $\mathbf{A}$ can be written as a finite sum of squares of $C^{2,\delta}$ vectors $X_{k,j}$, plus a matrix function $\mathbf{A}_p$,

$$\mathbf{A}(x) = \sum_{k=1}^{p-1} \sum_{i=1}^I X_{k,j}(x) X_{k,j}(x)^{\text{tr}} + \mathbf{A}_p(x), \quad x \in \mathbb{R}^M,$$

where the vectors $X_{k,i}(x)$, $1 \leq k \leq p-1$, $1 \leq i \leq I$ are $C^{2,\delta}(\mathbb{R}^M)$, $\mathbf{A}_p(x) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_p(x) \end{bmatrix}$, and $\mathbf{Q}_p(x) \in C^{4,2\delta}(\mathbb{R}^M)$ is quasiconformal. Moreover, $Z_k \equiv \sum_{i=1}^I X_{k,i} X_{k,i}^{\text{tr}} \in C^{4,2\delta}(\mathbb{R}^M)$ and

$$(2.8) \quad \begin{aligned} ca_{k,k} \mathbf{e}_k \otimes \mathbf{e}_k &\prec Z_k Z_k^{\text{tr}} + \sum_{m=k+1}^n a_{m,m} \mathbf{e}_m \otimes \mathbf{e}_m \prec C \sum_{m=k}^n a_{m,m} \mathbf{e}_m \otimes \mathbf{e}_m, & 1 \leq k \leq p-1, \\ \mathbf{Q}_p(x) &\sim a_{p,p}(x) \mathbb{I}_{n-p+1}. \end{aligned}$$

Finally, if in addition $A(x)$ is subordinate, then $\mathbf{Q}_p(x)$ is also subordinate.

Remark 23. If in addition $a_{k,k}(x) \approx 1$ for $1 \leq k \leq m < p$, then the conditions (2.6) and (2.7) in (1) and (2) are vacuous for $1 \leq k \leq m$, and moreover the proof shows that the vectors $X_{k,i}$ are actually in $C^{4,2\delta}(\mathbb{R}^M)$ for $1 \leq k \leq m$, $1 \leq i \leq I$.

This remark yields the following corollary in which conditions (2.6) and (2.7) in (1) and (2) play no role.

Corollary 24. Suppose $\mathbf{A}(x)$ is a $C^{4,2\delta}(\mathbb{R}^M)$ symmetric $n \times n$ matrix function that is comparable to a diagonal matrix function. In addition suppose that $a_{k,k}(x) \approx 1$ for $1 \leq k \leq p-1$ and $a_{k,k}(x) \approx a_{p,p}(x)$ for $p \leq k \leq n$. Then

$$\mathbf{A}(x) = \sum_{k=1}^{p-1} X_k(x) X_k(x)^{\text{tr}} + \mathbf{Q}_p(x), \quad x \in \mathbb{R}^M,$$

where $X_k, \mathbf{Q}_p \in C^{4,2\delta}(\mathbb{R}^M)$ and (2.8) holds for $1 \leq k \leq p-1$.

Remark 25. If the diagonal entry $a_{k,k}(x)$ is smooth and ω_s -montone on $\mathbb{R}^n$ for some $s > 1 - \varepsilon$, then the diagonal differential estimates (2.6) above hold for $a_{k,k}(x)$ since by [KoSa1, Theorem 18] we have $|D^\mu a_{k,k}(x)| \leq C_{s,s'} a_{k,k}(x)^{(s')^{|\mu|}}$ for any $0 < s' < s$. Indeed, we then have

$$a_{k,k}(x)^{(s')^{|\mu|}} \lesssim a_{k,k}(x)^{[1-|\mu|\varepsilon]_+ + \delta'},$$

since $(s')^{|\mu|} - [1-|\mu|\varepsilon]_+ > 0$ for $1 - \varepsilon < s' < s$, which in turn follows from the fact that $(1 - \varepsilon)^m + m\varepsilon$ has a strict minimum at $\varepsilon = 0$.

Remark 26. As ε increases from $\frac{1}{4}$ to 1, the diagonal assumptions (2.6) become more relaxed, while the off diagonal assumptions (2.7) become more stringent. Thus there is a tradeoff between assuming more on the diagonal entries or more on the off diagonal entries.

Remark 27. The positive number δ' in Theorem 22 was chosen in order to use Theorem 4 at various critical points in the proof.

Remark 28. Examples in the final section show that we cannot lower the exponent $[\frac{1}{2} + (2 - |\mu|)\varepsilon]_+$ on the right hand side of (2.7).

Remark 29. If in Theorem 22, we drop the hypothesis (2.6) that the diagonal entries satisfy the differential estimates, and even slightly weaken the off diagonal hypotheses (2.7), then using the Fefferman-Phong theorem for sums of squares of scalar functions, the proof of Theorem 22 shows that the operator $L = \nabla^{\text{tr}} \mathbf{A} \nabla$ can be written as $L = \sum_{j=1}^N X_j^{\text{tr}} X_j$ where the vector fields X_j are $C^{1,1}$ for $j = 1, 2, \dots, N$. However, unlike the situation for scalar functions, the examples in Theorems 37 and 42 show that we cannot dispense with the off diagonal hypotheses (2.7) in (1) (b). Moreover, the space $C^{1,1}$ seems to not be sufficient for gaining a positive degree δ of smoothness for solutions to the second order operators we consider, and so this result will neither be used nor proved here.

3. SQUARE DECOMPOSITIONS

Suppose $\mathbf{A} = [a_{k,j}]_{k,j=1}^n$ is a symmetric $n \times n$ matrix with top left entry $a_{11} > 0$. Then we can uniquely decompose $\mathbf{A}$ into a sum of a positive dyad ZZ^{tr} and a matrix $\mathbf{B}$ with zeroes in the first column and first row, namely

$$(3.1) \quad \begin{aligned} \mathbf{A} &= ZZ^{\text{tr}} + \mathbf{B} = \left[\left(\sum_{j=1}^n s_j \mathbf{e}_j \right) \otimes \left(\sum_{j=1}^n s_j \mathbf{e}_j \right) \right] + \mathbf{B}, \\ \mathbf{B} &= \begin{bmatrix} 0 & \mathbf{0}_{1 \times (n-1)} \\ \mathbf{0}_{(n-1) \times 1} & \mathbf{Q} \end{bmatrix}, \end{aligned}$$

where Z is $\frac{1}{\sqrt{a_{11}}}$ times the first column of $\mathbf{A}$, i.e. for $1 \leq k \leq j \leq n$,

$$\begin{aligned} a_{11} &= (s_1)^2 \text{ and } a_{1j} = s_1 s_j, \\ s_1 &= \sqrt{a_{11}} \text{ and } s_j = \frac{a_{1j}}{s_1}, \\ \mathbf{Q} &= [q_{k,j}]_{j,k=2}^n = [a_{k,j} - s_k s_j]_{j,k=2}^n = \left[a_{k,j} - \frac{a_{1k} a_{1j}}{a_{11}} \right]_{j,k=2}^n. \end{aligned}$$

As this particular decomposition $\mathbf{A} = ZZ^{\text{tr}} + \mathbf{B}$ will play a pivotal role in our inductive proof of Theorem 22, we give it a name.

Definition 30. The above decomposition $\mathbf{A} = ZZ^{\text{tr}} + \mathbf{B}$ is called the 1-Square Decomposition of $\mathbf{A}$.

Note that we are writing the $(n-1) \times (n-1)$ matrix $\mathbf{Q}$ as $[q_{k,j}]_{k,j=2}^n$, where the rows of $\mathbf{Q}$ are parameterized by k and the columns by j with $2 \leq k, j \leq n$.

Another main ingredient in our inductive proof is the well-known characterization of positive definiteness given in Theorem 15 above. We will also need the well-known determinant formula given in Lemma 14 above,

$$\det \begin{bmatrix} \alpha & v^{\text{tr}} \\ v & \mathbf{M} \end{bmatrix} = \alpha \det \mathbf{M} - v^{\text{tr}} \text{co} \mathbf{M} v = \{\alpha - v^{\text{tr}} \mathbf{M}^{-1} v\} \det \mathbf{M}.$$

We have already introduced the concepts of diagonally elliptical and subordinate matrix functions above, and we now introduce one more concept relevant to our decomposition, especially to the regularity of the vectors in the sum of squares. More precisely, we will need to show that the hypotheses of Theorem 22 propagate through the 1-Square Decomposition in an appropriate sense. The following definition encodes what is required.

Definition 31. Suppose $\mathbf{A}(x) \in C^{4,2\delta}(\mathbb{R}^M)$ is an $n \times n$ matrix function, $1 \leq \ell \leq n$, $\frac{1}{4} \leq \varepsilon < 1$ and $0 < \delta', \delta'' < 1$. Set

$$m_k(x) \equiv \min_{1 \leq s \leq k} \{a_{s,s}(x)\}, \quad 1 \leq k \leq n.$$

We say $\mathbf{A}(x)$ is $(\ell, \varepsilon, \delta', \delta'')$ -strongly $C^{4,2\delta}$ if

$$(3.2) \quad \begin{aligned} |D^\mu a_{k,k}| &\lesssim |a_{k,k}|^{[1-|\mu|\varepsilon]_+ + \delta'}, & 1 \leq |\mu| \leq 4 \text{ and } 1 \leq k \leq \ell, \\ [a_{k,k}]_{\mu, 2\delta} &\lesssim 1, & |\mu| = 4 \text{ and } 1 \leq k \leq \ell, \\ |D^\mu a_{k,j}| &\lesssim (m_j)^{[\frac{1}{2} + (2-|\mu|\varepsilon)_+ + \delta'']}, & 0 \leq |\mu| \leq 4 \text{ and } 1 \leq k < j \leq \ell, \\ [a_{k,j}]_{\mu, 2\delta} &\lesssim 1, & |\mu| = 4 \text{ and } 1 \leq k < j \leq \ell, \\ |D^\mu a_{k,j}| &\lesssim (m_k)^{[\frac{1}{2} + (2-|\mu|\varepsilon)_+ + \delta'']}, & 0 \leq |\mu| \leq 4 \text{ and } 1 \leq k \leq \ell < j \leq n, \\ [a_{k,j}]_{\mu, 2\delta} &\lesssim 1, & |\mu| = 4 \text{ and } 1 \leq k \leq \ell < j \leq n. \end{aligned}$$

Remark 32. When using this definition in the course of proving Theorem 22, the assumption that $\delta' > 0$ in the first line of (3.2) will be essential in order to apply our result on sums of squares of $C^{2,\delta}$ functions. The choices $\delta' = \frac{2\delta(1+\delta)}{2+\delta}$ and $\delta'' > \delta$ made in Theorem 22 will be required to show that the matrices being squared are $C^{2,\delta}$.

In the scalar case $n = 1$, the requirements (3.2) on the scalar function $a_{1,1}$ are equivalent to the inequalities

$$|D^\mu a_{1,1}| \lesssim |a_{1,1}|^{\delta'}, \quad |\mu| = 2, 4,$$

by the control of odd derivatives by even derivatives, see e.g. [Tat] and [KoSa1, Lemma 24]. In turn, these latter inequalities hold for some $\delta' > 0$ if and only if both $|\nabla^4 a_{1,1}| \leq C a_{1,1}^{\frac{\delta}{2+\delta}}$ and $|\nabla^2 a_{1,1}| \leq C a_{1,1}^{\frac{2\delta(1+\delta)}{2+\delta}}$ hold for some $\delta > 0$, and then Theorem 4 applies to show that $a_{1,1}$ is a finite sum of squares of $C^{2,\delta}$ functions.

We will now show in the next three lemmas that the properties of being diagonally elliptical, subordinate, and $(\ell, \varepsilon, \delta', \delta'')$ -strongly $C^{4,2\delta}$, are inherited by the $(n-1) \times (n-1)$ matrix $\mathbf{Q}(x)$ in the 1-Square Decomposition of $\mathbf{A}(x)$, with the proviso that ℓ is decreased by 1 in the definition of strongly $C^{4,2\delta}$.

3.1. The three lemmas.

Lemma 33. Suppose that $\mathbf{A}(x) = [a_{k,j}(x)]_{k,j=1}^n = \begin{bmatrix} a_{11}(x) & b(x)^{\text{tr}} \\ b(x) & \mathbf{D}(x) \end{bmatrix}$ is a diagonally elliptical $n \times n$ matrix function with 1-Square Decomposition

$$\mathbf{A}(x) = \mathbf{Z}(x) \mathbf{Z}(x)^{\text{tr}} + \mathbf{B}(x), \quad \text{where } \mathbf{B}(x) = \begin{bmatrix} 0 & 0^{\text{tr}} \\ 0 & \mathbf{Q}(x) \end{bmatrix}.$$

Then $\mathbf{Q}(x) = [q_{k,j}(x)]_{k,j=1}^{n-1}$ is a diagonally elliptical $(n-1) \times (n-1)$ matrix function, and moreover

$$(3.3) \quad q_{i,i}(x) \approx a_{i,i}(x), \quad 2 \leq i \leq n,$$

and

$$(3.4) \quad \begin{aligned} Z_1(x) Z_1(x)^{\text{tr}} &\prec C\mathbf{A}(x), \\ ca_{1,1}\mathbf{e}_1 \otimes \mathbf{e}_1 &\prec Z_1 Z_1^{\text{tr}} + \sum_{k=2}^n a_{k,k} \mathbf{e}_k \otimes \mathbf{e}_k \prec C\mathbf{A}. \end{aligned}$$

Proof. Note that

$$Z = \begin{pmatrix} \frac{\sqrt{a_{1,1}}}{\sqrt{a_{1,1}}} a_{1,2} \\ \vdots \\ \frac{1}{\sqrt{a_{1,1}}} a_{1,n} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{a_{1,1}}}{\sqrt{a_{1,1}}} \frac{b_2}{\sqrt{a_{1,1}}} \\ \vdots \\ \frac{b_n}{\sqrt{a_{1,1}}} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{a_{1,1}}}{\sqrt{a_{1,1}}} \\ \frac{b}{\sqrt{a_{1,1}}} \end{pmatrix}.$$

Since $\mathbf{A}(x)$ is comparable to $\mathbf{A}_{\text{diag}}(x)$, say

$$\beta_0 \mathbf{A}_{\text{diag}} \preccurlyeq \mathbf{A} \preccurlyeq \alpha_0 \mathbf{A}_{\text{diag}},$$

we also have

$$(3.5) \quad \beta_0 \mathbf{D}_{\text{diag}} \preccurlyeq \mathbf{D} \preccurlyeq \alpha_0 \mathbf{D}_{\text{diag}}$$

and Theorem 12 now shows that for $0 < \beta \leq \beta_0$ we have

$$b^{\text{tr}} \mathbf{D}_{\beta} b \leq (1 - \beta) a_{11},$$

where $\mathbf{D}_{\beta} \equiv [\mathbf{D} - \beta \mathbf{D}_{\text{diag}}]^{-1}$.

Since $\mathbf{D}_{\beta}(x)$ is positive semidefinite, it has a positive semidefinite square root $\mathbf{D}_{\beta}(x)^{\frac{1}{2}}$, and so we have

$$\left(\mathbf{D}_{\beta}(x)^{\frac{1}{2}} b(x) \right)^{\text{tr}} \mathbf{D}_{\beta}(x)^{\frac{1}{2}} b(x) = b(x)^{\text{tr}} \mathbf{D}_{\beta}(x) b(x) \leq (1 - \beta) a_{11}(x),$$

which implies that

$$\left| \mathbf{D}_{\beta}(x)^{\frac{1}{2}} b(x) \right| \leq \sqrt{1 - \beta} \sqrt{a_{11}(x)}.$$

But then we have that

$$\begin{aligned} \mathbf{D}_{\beta}(x)^{\frac{1}{2}} \mathbf{Q}(x) \mathbf{D}_{\beta}(x)^{\frac{1}{2}} &= \mathbf{D}_{\beta}(x)^{\frac{1}{2}} \left[\mathbf{D}(x) - \frac{1}{a_{11}(x)} b(x) b(x)^{\text{tr}} \right] \mathbf{D}_{\beta}(x)^{\frac{1}{2}} \\ &= \mathbf{D}_{\beta}(x)^{\frac{1}{2}} \mathbf{D}(x) \mathbf{D}_{\beta}(x)^{\frac{1}{2}} - \left(\frac{1}{\sqrt{a_{11}(x)}} \mathbf{D}_{\beta}(x)^{\frac{1}{2}} b(x) \right) \left(\frac{1}{\sqrt{a_{11}(x)}} \mathbf{D}_{\beta}(x)^{\frac{1}{2}} b(x) \right)^{\text{tr}} \end{aligned}$$

where

$$\begin{aligned} \mathbf{D}_{\beta}(x)^{\frac{1}{2}} \mathbf{D}(x) \mathbf{D}_{\beta}(x)^{\frac{1}{2}} &= \mathbf{D}_{\beta}(x)^{\frac{1}{2}} \left[\mathbf{D}_{\beta}(x)^{-1} + \beta \mathbf{D}_{\text{diag}}(x) \right] \mathbf{D}_{\beta}(x)^{\frac{1}{2}} \\ &= \mathbb{I}_{n-1} + \beta \mathbf{D}_{\beta}(x)^{\frac{1}{2}} \mathbf{D}_{\text{diag}}(x) \mathbf{D}_{\beta}(x)^{\frac{1}{2}}, \end{aligned}$$

and hence

$$\begin{aligned} \mathbf{D}_{\beta}(x)^{\frac{1}{2}} \mathbf{Q}(x) \mathbf{D}_{\beta}(x)^{\frac{1}{2}} &= \mathbb{I}_{n-1} + \beta \mathbf{D}_{\beta}(x)^{\frac{1}{2}} \mathbf{D}_{\text{diag}}(x) \mathbf{D}_{\beta}(x)^{\frac{1}{2}} \\ &\quad - \left(\frac{1}{\sqrt{a_{11}(x)}} \mathbf{D}_{\beta}(x)^{\frac{1}{2}} b(x) \right) \left(\frac{1}{\sqrt{a_{11}(x)}} \mathbf{D}_{\beta}(x)^{\frac{1}{2}} b(x) \right)^{\text{tr}}. \end{aligned}$$

Thus $\mathbf{D}_{\beta}(x)^{\frac{1}{2}} \mathbf{Q}(x) \mathbf{D}_{\beta}(x)^{\frac{1}{2}}$ is positive definite since $\mathbf{D}_{\text{diag}}(x)$ is and

$$\begin{aligned} \xi^{\text{tr}} \mathbf{D}_{\beta}(x)^{\frac{1}{2}} \mathbf{Q}(x) \mathbf{D}_{\beta}(x)^{\frac{1}{2}} \xi &= |\xi|^2 + \beta \left(\mathbf{D}_{\beta}(x)^{\frac{1}{2}} \xi \right)^{\text{tr}} \mathbf{D}_{\text{diag}}(x) \left(\mathbf{D}_{\beta}(x)^{\frac{1}{2}} \xi \right) \\ &\quad - \left| \frac{1}{\sqrt{a_{11}(x)}} \xi \cdot \mathbf{D}_{\beta}(x)^{\frac{1}{2}} b(x) \right|^2 \\ &\geq |\xi|^2 - |\xi|^2 \frac{1}{a_{11}(x)} \left| \mathbf{D}_{\beta}(x)^{\frac{1}{2}} b(x) \right|^2 \\ &\geq |\xi|^2 - |\xi|^2 (1 - \beta) = \beta |\xi|^2. \end{aligned}$$

Thus altogether we have

$$\begin{aligned} \mathbf{D}_\beta(x)^{\frac{1}{2}} \mathbf{Q}(x) \mathbf{D}_\beta(x)^{\frac{1}{2}} - \beta \mathbb{I}_{n-1} &\succcurlyeq 0, \\ \text{implies } \mathbf{Q}(x) - \beta \mathbf{D}_\beta(x)^{-1} &\succcurlyeq 0, \end{aligned}$$

and hence

$$\begin{aligned} \mathbf{Q}(x) &\succcurlyeq \beta \mathbf{D}_\beta(x)^{-1} = \beta (\mathbf{D}(x) - \beta \mathbf{D}_{\text{diag}}(x)) \\ &\succcurlyeq \beta \left(\mathbf{D}(x) - \frac{\beta}{\beta_0} \mathbf{D}(x) \right) = \beta \left(1 - \frac{\beta}{\beta_0} \right) \mathbf{D}(x), \end{aligned}$$

where β_0 is as in (3.5) above. If we now choose $\beta = \frac{\beta_0}{2}$ we obtain $\mathbf{Q}(x) \succcurlyeq \frac{\beta_0}{4} \mathbf{D}(x)$. Since we trivially have $\mathbf{Q}(x) \preccurlyeq \mathbf{D}(x)$, it follows that $\mathbf{Q}(x)$ is positive semidefinite and comparable to the matrix $\mathbf{D}(x)$, and hence also that $\mathbf{Q}(x)_{\text{diag}} \sim \mathbf{D}_{\text{diag}}(x)$. Since $\mathbf{D}(x)$ is comparable to its diagonal matrix $\mathbf{D}_{\text{diag}}(x)$ (because $\mathbf{A}(x)$ is comparable to $\mathbf{A}_{\text{diag}}(x)$), we now conclude that $\mathbf{Q}(x)$ is comparable to its associated diagonal matrix $\mathbf{Q}_{\text{diag}}(x)$. Thus $\mathbf{Q}(x)$ is diagonally elliptical by definition, and since $\mathbf{Q}_{\text{diag}}(x) \sim \mathbf{D}_{\text{diag}}(x)$ as mentioned above, we conclude that $q_{i,i}(x) \approx a_{i,i}(x)$ for $2 \leq i \leq n$, which is (3.3).

Finally,

$$\sum_{k=2}^n a_{k,k} \mathbf{e}_k \otimes \mathbf{e}_k \prec \mathbf{A}_{\text{diag}}(x) \prec C \mathbf{A}(x),$$

and

$$\begin{aligned} \xi^{\text{tr}} Z_1(x) Z_1(x)^{\text{tr}} \xi &= (Z_1(x) \cdot \xi)^2 = \left(\sqrt{a_{1,1}} \xi_1 + \sum_{k=2}^n \frac{a_{1,k}}{\sqrt{a_{1,1}}} \xi_k \right)^2 \\ &\leq \left(\sqrt{a_{1,1}} \xi_1 + \sum_{k=2}^n \frac{\gamma \sqrt{a_{1,1} a_{k,k}}}{\sqrt{a_{1,1}}} \xi_k \right)^2 = \left(\sqrt{a_{1,1}} \xi_1 + \sum_{k=2}^n \gamma \sqrt{a_{k,k}} \xi_k \right)^2 \\ &\leq C \left(a_{1,1} \xi_1^2 + \sum_{k=2}^n a_{k,k} \xi_k^2 \right) = C \xi^{\text{tr}} \mathbf{A}_{\text{diag}}(x) \xi \leq C \xi^{\text{tr}} \mathbf{A}(x) \xi, \end{aligned}$$

shows that $Z_1(x) Z_1(x)^{\text{tr}} \prec C \mathbf{A}(x)$. On the other hand, we have

$$\mathbf{Q}(x) \sim \mathbf{Q}_{\text{diag}}(x) \sim \sum_{k=2}^n \left(a_{k,k} - \frac{(a_{1,k})^2}{a_{1,1}} \right) \mathbf{e}_k \otimes \mathbf{e}_k \sim \sum_{k=2}^n a_{k,k} \mathbf{e}_k \otimes \mathbf{e}_k,$$

since $(a_{1,k})^2 \leq \gamma^2 a_{1,1} a_{k,k}$ for some $\gamma^2 < 1$ by diagonal positivity of $\mathbf{A}(x)$. Thus we have

$$\begin{aligned} a_{1,1} \mathbf{e}_1 \otimes \mathbf{e}_1 &\prec \sum_{k=1}^n a_{k,k} \mathbf{e}_k \otimes \mathbf{e}_k = \mathbf{A}_{\text{diag}}(x) \sim \mathbf{A}(x) \\ &= Z_1 Z_1^{\text{tr}} + \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{Q} \end{bmatrix} \sim Z_1 Z_1^{\text{tr}} + \sum_{k=2}^n a_{k,k} \mathbf{e}_k \otimes \mathbf{e}_k, \end{aligned}$$

which proves (3.4). ■

Lemma 34. *Suppose that the matrix function $\mathbf{A}(x)$ is diagonally elliptical and subordinate. Then with notation as in the previous lemma, $\mathbf{Q}(x)$ is subordinate.*

Proof. We have by definition,

$$\mathbf{Q} = \left[a_{k,j} - \frac{a_{1,k} a_{1,j}}{a_{1,1}} \right]_{k,j=2}^n,$$

and by the comparability and subordinaticity of the matrix $\mathbf{A}$, we have the estimates $q_{j,j} \approx a_{j,j}$ and

$$|a_{k,j}| \leq (a_{k,k} a_{j,j})^{\frac{1}{2}} \quad \text{and} \quad |\nabla a_{k,j}| \leq (a_{j,j})^{\frac{1}{2}}, \quad 1 \leq k \leq j \leq n.$$

Thus we compute

$$\begin{aligned}
\left| \nabla \left(a_{k,j} - \frac{a_{1,k} a_{1,j}}{a_{1,1}} \right) \right| &= \left| \nabla a_{k,j} - \frac{(\nabla a_{1,k}) a_{1,j}}{a_{1,1}} - \frac{a_{1,k} (\nabla a_{1,j})}{a_{1,1}} + \frac{a_{1,k} a_{1,j} \nabla E_1}{(a_{1,1})^2} \right| \\
&\leq (a_{j,j})^{\frac{1}{2}} + \frac{(a_{1,1} a_{k,k})^{\frac{1}{2}} (a_{j,j})^{\frac{1}{2}}}{a_{1,1}} + \frac{(a_{k,k})^{\frac{1}{2}} (a_{1,1} a_{j,j})^{\frac{1}{2}}}{a_{1,1}} + \frac{(a_{1,1} a_{k,k})^{\frac{1}{2}} (a_{1,1} a_{j,j})^{\frac{1}{2}} (a_{1,1})^{\frac{1}{2}}}{(a_{1,1})^2} \\
&\approx (q_{j,j})^{\frac{1}{2}} \left\{ 1 + \frac{(q_{1,1} q_{k,k})^{\frac{1}{2}}}{q_{1,1}} + \frac{(q_{1,1} q_{a_{k,k}})^{\frac{1}{2}}}{q_{1,1}} + \frac{(q_{k,k})^{\frac{1}{2}} (q_{j,j})^{\frac{1}{2}}}{(q_{1,1})^{\frac{1}{2}}} \right\} \lesssim (q_{j,j})^{\frac{1}{2}},
\end{aligned}$$

which shows that $\mathbf{Q}$ is subordinate by Theorem 20. ■

An induction argument using Lemma 33 shows in particular that we can iterate the 1-Square Decomposition starting with a diagonally elliptical matrix function $\mathbf{A}(x)$ to produce a $(p-1)$ -Square Decomposition

$$\mathbf{A}(x) = \sum_{k=1}^{p-1} X_k X_k^{\text{tr}} + \mathbf{A}_p(x)$$

where $\mathbf{A}_p(x)$ is an $(n-p+1) \times (n-p+1)$ diagonally elliptical matrix function. This iterated construction will be used to prove Theorem 22. The next lemma shows that the property of being $(\ell, \varepsilon, \delta', \delta'')$ -strongly $C^{4,2\delta}$ also propagates through the one step square decomposition 1-Square Decomposition upon decreasing ℓ by 1.

Lemma 35. *Suppose $\mathbf{A}(x) \in C^{4,2\delta}(\mathbb{R}^M)$ is $(\ell, \varepsilon, \delta', \delta'')$ -strongly $C^{4,2\delta}$ for some $1 \leq \ell \leq n$ and $\frac{1}{4} \leq \varepsilon < 1$. Then we have the corresponding inequalities for $\mathbf{Q}(x)$ but with $k \geq 2$:*

$$\begin{aligned}
(3.6) \quad |D^\mu q_{k,k}| &\lesssim |q_{k,k}|^{[1-|\mu|\varepsilon]_+ + \delta'}, \quad 1 \leq |\mu| \leq 4 \text{ and } 2 \leq k \leq \ell, \\
[q_{k,k}]_{\mu, 2\delta} &\lesssim 1, \quad |\mu| = 4 \text{ and } 2 \leq k \leq \ell, \\
|D^\mu q_{k,j}| &\lesssim \left(\min_{1 \leq s \leq j} q_{s,s} \right)^{\left[\frac{1}{2} + (2-|\mu|)\varepsilon \right]_+ + \delta''}, \quad 0 \leq |\mu| \leq 4 \text{ and } 2 \leq k \leq \ell < j \leq p-1, \\
[q_{k,j}]_{\mu, 2\delta} &\lesssim 1, \quad |\mu| = 4 \text{ and } 2 \leq k \leq \ell < j \leq p-1, \\
|D^\mu q_{k,j}| &\lesssim \left(\min_{1 \leq s \leq k} q_{s,s} \right)^{\left[\frac{1}{2} + (2-|\mu|)\varepsilon \right]_+ + \delta''}, \quad 0 \leq |\mu| \leq 4 \text{ and } 2 \leq k \leq \ell \leq j \leq p-1 < j \leq n, \\
[q_{k,j}]_{\mu, 2\delta} &\lesssim 1, \quad |\mu| = 4 \text{ and } 2 \leq k \leq \ell \leq p-1 < j \leq n.
\end{aligned}$$

In particular $\mathbf{Q} \in C^{4,2\delta}$, and if $\ell \geq 2$, then $\mathbf{Q}$ is $(\ell-1, \varepsilon, \delta', \delta'')$ -strongly $C^{4,2\delta}$.

In order to prove the lemma, we will use the trivial inequality $[a+b]_+ \leq [a]_+ + [b]_+$ together with (3.3), the product formula

$$(3.7) \quad D^\mu(fgh) = \sum_{\alpha+\beta+\gamma=\mu} c_{\alpha,\beta,\gamma}^\mu (D^\alpha f) (D^\beta g) (D^\gamma h),$$

and the composition formula,

$$(3.8) \quad \nabla^M(\psi \circ h) = \sum_{m=1}^M \left(\psi^{(m)} \circ h \right) \left(\sum_{\substack{\eta=(\eta_1, \dots, \eta_M) \in \mathbb{Z}_+^M \\ \eta_1 + \eta_2 + \dots + \eta_M = m \\ \eta_1 + 2\eta_2 + \dots + M\eta_M = M}} \begin{bmatrix} M \\ \eta \end{bmatrix} (\nabla h)^{\eta_1} \dots (\nabla^M h)^{\eta_M} \right),$$

where $\begin{bmatrix} M \\ \alpha \end{bmatrix}$ is defined for $\alpha = (\alpha_1, \dots, \alpha_M) \in \mathbb{Z}_+^M$ satisfying $\alpha_1 + 2\alpha_2 + \dots + M\alpha_M = M$. See e.g. [MaSaUrVu] or [KoSa1, (3.1)]. We will use the reciprocal function $\psi(t) = \frac{1}{t}$ in (3.8) so that $\psi^{(m)}(t) = (-1)^m m! t^{-m-1}$.

Proof. We first derive an auxiliary estimate that will be used throughout the proof. Namely, for any multi-index γ with $1 \leq |\gamma| \leq 4$ we have using (3.2) and (3.8),

$$\begin{aligned}
\left| D^\gamma \frac{1}{a_{1,1}} \right| &= \left| \sum_{m=1}^{|\gamma|} \left(\frac{1}{a_{11}} \right)^{m+1} \sum_{\substack{\eta=(\eta_1, \dots, \eta_M) \in \mathbb{Z}_+^{|\gamma|} \\ \eta_1 + \eta_2 + \dots + \eta_{|\gamma|} = m \\ \eta_1 + 2\eta_2 + \dots + |\gamma|\eta_{|\gamma|} = |\gamma|}} \begin{bmatrix} \gamma \\ \eta \end{bmatrix} (\nabla a_{11})^{\eta_1} \dots (\nabla^{|\gamma|} a_{11})^{\eta_{|\gamma|}} \right| \\
&\lesssim \sum_{m=1}^{|\gamma|} \left(\frac{1}{a_{11}} \right)^{m+1} \sum_{\substack{\eta=(\eta_1, \dots, \eta_M) \in \mathbb{Z}_+^{|\gamma|} \\ \eta_1 + \eta_2 + \dots + \eta_{|\gamma|} = m \\ \eta_1 + 2\eta_2 + \dots + |\gamma|\eta_{|\gamma|} = |\gamma|}} a_{11}^{([1-\varepsilon]_+ + \delta')\eta_1 + ([1-2\varepsilon]_+ + \delta')\eta_2 + \dots + ([1-|\gamma|\varepsilon]_+ + \delta')\eta_{|\gamma|}} \\
&\leq \left(\frac{1}{a_{11}} \right)^{m+1} \sum_{\substack{\eta=(\eta_1, \dots, \eta_M) \in \mathbb{Z}_+^{|\gamma|} \\ \eta_1 + \eta_2 + \dots + \eta_{|\gamma|} = m \\ \eta_1 + 2\eta_2 + \dots + |\gamma|\eta_{|\gamma|} = |\gamma|}} a_{11}^{(1-\varepsilon + \delta')\eta_1 + (1-2\varepsilon + \delta')\eta_2 + \dots + (1-|\gamma|\varepsilon + \delta')\eta_{|\gamma|}} \\
&= \sum_{m=1}^{|\gamma|} \left(\frac{1}{a_{11}} \right)^{m+1} \sum_{\substack{\eta=(\eta_1, \dots, \eta_M) \in \mathbb{Z}_+^{|\gamma|} \\ \eta_1 + \eta_2 + \dots + \eta_{|\gamma|} = m \\ \eta_1 + 2\eta_2 + \dots + |\gamma|\eta_{|\gamma|} = |\gamma|}} a_{11}^{m - |\gamma|\varepsilon + m\delta'}.
\end{aligned}$$

Note that this implies the estimate

$$(3.9) \quad \left| D^\gamma \frac{1}{a_{1,1}} \right| \leq C a_{11}^{-1-|\gamma|\varepsilon + \delta'}, \quad 1 \leq |\gamma| \leq 4,$$

The product formula (3.7), hypothesis (3.2) and the estimate (3.9) above imply that for $k \geq 2$ and $M = |\mu|$,

$$\begin{aligned}
|\nabla^M q_{k,k}| &= \left| \nabla^M \left(a_{k,k} - \frac{(a_{1,k})^2}{a_{1,1}} \right) \right| \leq |\nabla^M a_{k,k}| + \left| \nabla^M \frac{(a_{1,k})^2}{a_{1,1}} \right| \\
&\lesssim |a_{k,k}|^{[1-|\mu|\varepsilon]_+ + \delta'} \\
&\quad + \sum_{\alpha + \beta + \gamma = \mu} \left(\min_{1 \leq s \leq k} \{a_{s,s}(x)\} \right)^{\left[\frac{1}{2} + (2-|\alpha|)\varepsilon \right]_+ + \delta''} \left(\min_{1 \leq s \leq k} \{a_{s,s}(x)\} \right)^{\left[\frac{1}{2} + (2-|\beta|)\varepsilon \right]_+ + \delta''} \left| D^\gamma \frac{1}{a_{1,1}} \right| \\
&\lesssim |a_{k,k}|^{[1-|\mu|\varepsilon]_+ + \delta'} + \sum_{\alpha + \beta + \gamma = \mu} \left(\min_{1 \leq s \leq k} \{a_{s,s}(x)\} \right)^{[1 + (4-(|\alpha|+|\beta|))\varepsilon]_+ + 2\delta''} a_{11}^{-1-|\gamma|\varepsilon + \delta'} \\
&\lesssim |a_{k,k}|^{[1-|\mu|\varepsilon]_+ + \delta'} + \sum_{\alpha + \beta + \gamma = \mu} \left(\min_{1 \leq s \leq k} \{a_{s,s}(x)\} \right)^{[1 + (4-(|\alpha|+|\beta|))\varepsilon]_+ + 2\delta'' - 1 - |\gamma|\varepsilon + \delta'} \\
&\lesssim |a_{k,k}|^{[1-|\mu|\varepsilon]_+ + \delta'},
\end{aligned}$$

where the last line above follows from the fact that

$$\begin{aligned}
&[1 + (4 - (|\alpha| + |\beta|))\varepsilon]_+ + 2\delta'' - 1 - |\gamma|\varepsilon + \delta' \\
&\geq 1 + (4 - (|\alpha| + |\beta|))\varepsilon + 2\delta' - 1 - |\gamma|\varepsilon + \delta' \\
&= (4 - |\mu|)\varepsilon + 3\delta' \geq [1 - |\mu|\varepsilon]_+ + \delta',
\end{aligned}$$

if $|\mu| \leq 4$ and $\varepsilon \geq \frac{1}{4}$. Since $a_{k,k} \approx q_{k,k}$ by (3.3), we have established the first line in (3.6).

The second line is obtained similarly using the subproduct and subchain rules for the seminorm (1.1) as in [Bon] and [KoSa1, (3.7) and (3.8) and subsequent proofs]. Indeed, if $|\mu| = 4$, then

$$\begin{aligned} \left[\frac{(a_{1,k})^2}{a_{1,1}} \right]_{\mu, 2\delta} &= \left| \sum_{\alpha+\beta+\gamma=\mu} c_{\alpha,\beta,\gamma}^\mu (D^\alpha a_{1,k}) (D^\beta a_{1,k}) \left(D^\gamma \frac{1}{a_{1,1}} \right) \right|^{2\delta} \\ &\lesssim \sum_{\alpha+\beta+\gamma=\mu} [a_{1,k}]_{\alpha, 2\delta} |D^\beta a_{1,k}| \left| D^\gamma \frac{1}{a_{1,1}} \right| \\ &\quad + \sum_{\alpha+\beta+\gamma=\mu} |D^\alpha a_{1,k}| [a_{1,k}]_{\beta, 2\delta} \left| D^\gamma \frac{1}{a_{1,1}} \right| \\ &\quad + \sum_{\alpha+\beta+\gamma=\mu} |D^\alpha a_{1,k}| |D^\beta a_{1,k}| \left[\frac{1}{a_{1,1}} \right]_{\gamma, 2\delta}. \end{aligned}$$

Now use that if $|\alpha| < 4$, then

$$\begin{aligned} [a_{1,k}]_{\alpha, 2\delta} &= \limsup_{y, z \rightarrow x} \frac{|D^\alpha a_{1,k}(y) - D^\alpha a_{1,k}(z)|}{|y - z|^{2\delta}} \\ &\leq \limsup_{y, z \rightarrow x} \frac{|D^\alpha a_{1,k}(y) - D^\alpha a_{1,k}(z)|}{|y - z|} |y - z|^{1-2\delta} = 0, \end{aligned}$$

while if $|\alpha| = 4$, then

$$[a_{1,k}]_{\alpha, 2\delta} \leq \|a_{1,k}\|_{C^{4, 2\delta}} \leq C.$$

Now we turn to proving the third line in (3.6). In order to simplify notation, set

$$m_k(x) \equiv \min_{1 \leq s \leq k} \{q_{s,s}(x)\}, \quad 1 \leq k \leq n,$$

and note that by (3.3) we have $m_k(x) \approx \min_{1 \leq s \leq k} \{a_{s,s}(x)\}$. The case $|\mu| = 0$ is immediate from the identity $q_{k,j} = a_{k,j} - \frac{a_{1,k}a_{1,j}}{a_{1,1}}$, and the estimate

$$\left| \frac{a_{1,k}a_{1,j}}{a_{1,1}} \right| \lesssim \frac{(m_k)^{\frac{1}{2}+2\varepsilon+\delta''} (m_j)^{\frac{1}{2}+2\varepsilon+\delta''}}{a_{1,1}} \leq (a_{1,1})^{-\frac{1}{2}+2\varepsilon+\delta''} (m_j)^{\frac{1}{2}+2\varepsilon+\delta''} \leq (m_j)^{\left[\frac{1}{2}+(2-|\mu|)\varepsilon\right]_+ + \delta''},$$

since $-\frac{1}{2} + 2\varepsilon + \delta'' \geq 0$.

To prove the cases $|\mu| \geq 1$, we continue to use (3.2), (3.9) and (3.3), to obtain the estimate

$$\begin{aligned} \left| D^\mu \frac{a_{1,k}a_{1,j}}{a_{1,1}} \right| &= \left| \sum_{\alpha+\beta+\gamma=\mu} c_{\alpha,\beta,\gamma}^\mu (D^\alpha a_{1,k}) (D^\beta a_{1,j}) \left(D^\gamma \frac{1}{a_{1,1}} \right) \right| \\ &\lesssim \sum_{\alpha+\beta+\gamma=\mu} (m_k)^{\left[\frac{1}{2}+(2-|\alpha|)\varepsilon\right]_+ + \delta''} (m_j)^{\left[\frac{1}{2}+(2-|\beta|)\varepsilon\right]_+ + \delta''} (a_{1,1})^{-1-|\gamma|\varepsilon+\delta'} \\ &\lesssim (q_{1,1})^{-1-|\gamma|\varepsilon+\delta'} (m_k)^{\left[\frac{1}{2}+(2-|\alpha|)\varepsilon\right]_+ + \delta''} (m_j)^{\left[\frac{1}{2}+(2-|\beta|)\varepsilon\right]_+ + \delta''} \\ &\lesssim (m_k)^{\left[\frac{1}{2}+(2-|\alpha|)\varepsilon\right]_+ + \delta'' - 1 - |\gamma|\varepsilon+\delta'} (m_j)^{\left[\frac{1}{2}+(2-|\beta|)\varepsilon\right]_+ + \delta''}. \end{aligned}$$

Note that placing a derivative in position β in the last line above causes the most damage up to order 2 because $m_j \leq m_k \leq m_1 = q_{1,1}$. After that, placing a third derivative in position α equivalently γ is the next most damaging, or depending on the precise relation between m_j and m_k , repeating the position β once more might be worse. Indeed, placing a third derivative in position α or γ loses $m_k^{-\varepsilon}$, while placing a third derivative in position β loses $m_j^{-\frac{1}{2}}$, and since $\varepsilon \geq \frac{1}{4}$, either of these could be largest under the restriction $m_j \leq m_k$. However after three derivatives have been assigned as above, the fourth derivative does the most damage when applied to position α equivalently γ . As a consequence we need only consider the cases where β is filled up to order 2, and then α or β is filled thereafter. This will become clear as the reader progresses through the proof.

In the case $|\mu| = 1$, $\left| D^\mu \frac{a_{1,k} a_{1,j}}{a_{1,1}} \right|$ is bounded by

$$\begin{aligned} & (m_k)^{\left[\frac{1}{2} + (2-|\alpha|)\varepsilon\right]_+ + \delta'' - 1 - |\gamma|\varepsilon + \delta'} (m_j)^{\left[\frac{1}{2} + (2-|\beta|)\varepsilon\right]_+ + \delta''} \\ & \leq (m_k)^{\left[\frac{1}{2} + 2\varepsilon\right]_+ + \delta'' - 1 + \delta'} (m_j)^{\left[\frac{1}{2} + \varepsilon\right]_+ + \delta''} \\ & = (m_k)^{\frac{1}{2} + 2\varepsilon + \delta' + \delta'' - 1} (m_j)^{\frac{1}{2} + \varepsilon + \delta''} \leq (m_j)^{\frac{1}{2} + \varepsilon + \delta''} = (m_j)^{\left[\frac{1}{2} + (2-|\mu|)\varepsilon\right]_+ + \delta''}, \end{aligned}$$

since $\frac{1}{2} + 2\varepsilon + \delta' + \delta'' - 1 \geq \delta' + \delta'' > 0$.

In the case $|\mu| = 2$, $\left| D^\mu \frac{a_{1,k} a_{1,j}}{a_{1,1}} \right|$ is bounded by

$$\begin{aligned} & (m_k)^{\left[\frac{1}{2} + (2-|\alpha|)\varepsilon\right]_+ + \delta'' - 1 - |\gamma|\varepsilon + \delta'} (m_j)^{\left[\frac{1}{2} + (2-|\beta|)\varepsilon\right]_+ + \delta''} \\ & \leq (m_k)^{\left[\frac{1}{2} + 2\varepsilon\right]_+ + \delta'' - 1 + \delta'} (m_j)^{\frac{1}{2} + \delta''} = (m_k)^{\frac{1}{2} + 2\varepsilon - 1 + \delta' + \delta''} (m_j)^{\frac{1}{2} + \delta''} \\ & \leq (m_j)^{\frac{1}{2} + \delta''} = (m_j)^{\left[\frac{1}{2} + (2-|\mu|)\varepsilon\right]_+ + \delta''}, \end{aligned}$$

since $\frac{1}{2} + 2\varepsilon - 1 + \delta' + \delta'' \geq \delta' + \delta'' > 0$.

In the case $|\mu| = 3$, the most damage is caused either when $|\beta| = 2$ and $|\alpha| = 1$, or when $|\beta| = 3$. In the former instance, $\left| D^\mu \frac{a_{1,k} a_{1,j}}{a_{1,1}} \right|$ is bounded by

$$(m_k)^{\left[\frac{1}{2} + (2-|\alpha|)\varepsilon\right]_+ + \delta'' - 1 - |\gamma|\varepsilon + \delta'} (m_j)^{\left[\frac{1}{2} + (2-|\beta|)\varepsilon\right]_+ + \delta''} \leq (m_k)^{-\frac{1}{2} + \varepsilon + \delta' + \delta''} (m_j)^{\frac{1}{2} + \delta''}.$$

In the case $\varepsilon \geq \frac{1}{2}$, this is bounded by

$$(m_k)^{\delta' + \delta''} (m_j)^{\frac{1}{2} + \delta''} \leq (m_j)^{\frac{1}{2} + \delta''} = (m_j)^{\left[\frac{1}{2} + (2-|\mu|)\varepsilon\right]_+ + \delta''}.$$

If $\frac{1}{4} \leq \varepsilon < \frac{1}{2}$, then this is instead bounded by

$$(m_k)^{-\frac{1}{2} + 2\varepsilon + \delta' + \delta''} (m_j)^{\frac{1}{2} - \varepsilon + \delta''} \leq (m_j)^{\frac{1}{2} - \varepsilon + \delta''} = (m_j)^{\left[\frac{1}{2} + (2-|\mu|)\varepsilon\right]_+ + \delta' + \delta''}.$$

since $-\frac{1}{2} + 2\varepsilon + \delta' + \delta'' \geq \delta' + \delta'' > 0$. In the latter instance, when $|\beta| = 3$, $\left| D^\mu \frac{a_{1,k} a_{1,j}}{a_{1,1}} \right|$ is bounded by

$$\begin{aligned} & (m_k)^{\left[\frac{1}{2} + (2-|\alpha|)\varepsilon\right]_+ + \delta'' - 1 - |\gamma|\varepsilon + \delta'} (m_j)^{\left[\frac{1}{2} + (2-|\beta|)\varepsilon\right]_+ + \delta''} \\ & \leq (m_k)^{\left[\frac{1}{2} + 2\varepsilon\right]_+ + \delta'' - 1 + \delta'} (m_j)^{\left[\frac{1}{2} - \varepsilon\right]_+ + \delta''} \leq (m_j)^{\left[\frac{1}{2} - \varepsilon\right]_+ + \delta''} = (m_j)^{\left[\frac{1}{2} + (2-|\mu|)\varepsilon\right]_+ + \delta''}, \end{aligned}$$

since $2\varepsilon - \frac{1}{2} + \delta' + \delta'' \geq \delta' + \delta'' > 0$.

In the case $|\mu| = 4$, we again consider the two most damaging cases. When $|\beta| = 3$ and $|\alpha| = 1$, we have that $\left| D^\mu \frac{a_{1,k} a_{1,j}}{a_{1,1}} \right|$ is bounded by

$$\begin{aligned} & (m_k)^{\left[\frac{1}{2} + (2-|\alpha|)\varepsilon\right]_+ + \delta'' - 1 - |\gamma|\varepsilon + \delta'} (m_j)^{\left[\frac{1}{2} + (2-|\beta|)\varepsilon\right]_+ + \delta''} \\ & = (m_k)^{-\frac{1}{2} + \varepsilon + \delta' + \delta''} (m_j)^{\left[\frac{1}{2} - \varepsilon\right]_+ + \delta''}. \end{aligned}$$

In the case $\varepsilon \geq \frac{1}{2}$ this is bounded by $m_j^{\delta''} = (m_j)^{\left[\frac{1}{2} + (2-|\mu|)\varepsilon\right]_+ + \delta''}$, while in the case $\frac{1}{4} \leq \varepsilon < \frac{1}{2}$, this is bounded by

$$(m_k)^{-\frac{1}{2} + \varepsilon + \delta' + \delta''} (m_j)^{\frac{1}{2} - \varepsilon + \delta''} \leq (m_k)^{\delta' + \delta''} m_j^{\delta''} \leq m_j^{\delta''} = (m_j)^{\left[\frac{1}{2} + (2-|\mu|)\varepsilon\right]_+ + \delta''},$$

since $\delta' + \delta'' > 0$. When $|\alpha| = |\beta| = 2$, then $\left| D^\mu \frac{a_{1,k} a_{1,j}}{a_{1,1}} \right|$ is bounded by

$$\begin{aligned} & (m_k)^{\left[\frac{1}{2} + (2-|\alpha|)\varepsilon\right]_+ + \delta'' - 1 - |\gamma|\varepsilon + \delta'} (m_j)^{\left[\frac{1}{2} + (2-|\beta|)\varepsilon\right]_+ + \delta''} \\ & \leq (m_k)^{\left[\frac{1}{2}\right]_+ + \delta'' - 1 + \delta'} (m_j)^{\left[\frac{1}{2}\right]_+ + \delta''} \leq (m_j)^{\delta''} = (m_j)^{\left[\frac{1}{2} + (2-|\mu|)\varepsilon\right]_+ + \delta''}, \end{aligned}$$

since $\delta' + \delta'' > 0$.

This completes the proof of the third line in (3.6), and just as before, the fourth line is obtained similarly using the subproduct and subchain rules for the Hölder expression $[\cdot]_{\mu, 2\delta}$. Indeed, if $|\mu| = 4$, then

$$\begin{aligned} \left[\frac{a_{1,k} a_{1,j}}{a_{1,1}} \right]_{\mu, 2\delta} &\lesssim \sum_{\alpha+\beta+\gamma=\mu} [a_{1,k}]_{\alpha, 2\delta} |D^\beta a_{1,j}| \left| D^\gamma \frac{1}{a_{1,1}} \right| \\ &\quad + \sum_{\alpha+\beta+\gamma=\mu} |D^\alpha a_{1,k}| [a_{1,j}]_{\beta, 2\delta} \left| D^\gamma \frac{1}{a_{1,1}} \right| \\ &\quad + \sum_{\alpha+\beta+\gamma=\mu} |D^\alpha a_{1,k}| |D^\beta a_{1,j}| \left[\frac{1}{a_{1,1}} \right]_{\gamma, 2\delta}, \end{aligned}$$

and if $|\alpha| < 4$, then

$$\begin{aligned} [a_{1,k}]_{\alpha, 2\delta} &= \limsup_{y, z \rightarrow x} \frac{|D^\alpha a_{1,k}(y) - D^\alpha a_{1,k}(z)|}{|y - z|^{2\delta}} \\ &\leq \limsup_{y, z \rightarrow x} \frac{|D^\alpha a_{1,k}(y) - D^\alpha a_{1,k}(z)|}{|y - z|} |y - z|^{1-2\delta} = 0, \end{aligned}$$

while if $|\alpha| = 4$, then

$$[a_{1,k}]_{\alpha, 2\delta} \leq \|a_{1,k}\|_{C^{4, 2\delta}} \leq C.$$

Finally, we note that it is an easy matter to check that when $\ell < j \leq n$, we can replace $q_{j,j}$ with the larger quantity $q_{k,k}$ on the right hand side of the estimates in the fifth and sixth lines. This completes the proof of Lemma 35. ■

3.2. Proof of the main decomposition theorem. At this point we can apply induction together with Lemmas 33, 34 and 35, and Theorem 20, to prove Theorem 22.

Proof of Theorem 22. Set $\mathbf{Q}_1(x) \equiv \mathbf{A}(x)$ and suppose that $\mathbf{Q}_1(x)$ is diagonally elliptical and $(p-1, \varepsilon)$ -strongly $C^{4, 2\delta}$. Let $\mathbf{Q}_1(x) = Y_1(x) Y_1(x)^{\text{tr}} + \mathbf{B}_1(x)$ be the 1-Square Decomposition of $\mathbf{Q}_1(x)$ with $\mathbf{B}_1(x) = \begin{bmatrix} 0 & \mathbf{0}_{1 \times (n-1)} \\ \mathbf{0}_{(n-1) \times 1} & \mathbf{Q}_2(x) \end{bmatrix}$. Then Lemmas 33 and 35 show that $\mathbf{Q}_2(x)$ is diagonally elliptical and satisfies (3.4), and is $(p-2)$ -strongly $C^{4, 2\delta}$ so long as $p \geq 3$. Now we apply the above reasoning to the $(n-1) \times (n-1)$ matrix $\mathbf{Q}_2(x)$ to obtain the 1-Square Decomposition of $\mathbf{Q}_2(x) = Y_2(x) Y_2(x)^{\text{tr}} + \mathbf{B}_2(x)$ with $\mathbf{B}_2(x) = \begin{bmatrix} 0 & \mathbf{0}_{1 \times (n-2)} \\ \mathbf{0}_{(n-2) \times 1} & \mathbf{Q}_3(x) \end{bmatrix}$, and we see that $\mathbf{Q}_3(x)$ is diagonally elliptical and satisfies (3.4) relative to $\mathbf{Q}_2(x)$, and is $(p-3)$ -strongly $C^{4, 2}$ so long as $p \geq 4$. By induction we obtain that

$$\mathbf{Q}_{p-2}(x) = Y_{p-2}(x) Y_{p-2}(x)^{\text{tr}} + \mathbf{B}_{p-2}(x)$$

where $\mathbf{Q}_{p-1}(x)$ is diagonally elliptical and satisfies (3.4) relative to $\mathbf{Q}_{p-2}(x)$, and is 1-strongly $C^{4, 2\delta}$. One more application of Lemmas 33 and 35 yields that

$$\mathbf{Q}_{p-1}(x) = Y_{p-1}(x) Y_{p-1}(x)^{\text{tr}} + \mathbf{B}_{p-1}(x)$$

where $\mathbf{Q}_p(x)$ is diagonally elliptical. While we cannot now assert that $\mathbf{Q}_p(x)$ is 1-strongly $C^{4, 2\delta}$, we do have that $\mathbf{Q}_{p-1} = [q_{k,j}]_{k,j=1}^{n-p+1}$ is 1-strongly $C^{4, 2\delta}$, and hence

$$\mathbf{Q}_p = \left[q_{k,j} - \frac{q_{1k} q_{1j}}{q_{11}} \right]_{j,k=2}^n \in C^{4, 2\delta},$$

upon estimating derivatives of $\frac{q_{1k} q_{1j}}{q_{11}}$ as above.

Now let Z_k be the n -vector whose final $n-k+1$ entries are the entries of Y_k , with zeroes elsewhere, and similarly let $\mathbf{A}_p$ be the $n \times n$ matrix whose bottom right $(n-p+1) \times (n-p+1)$ block is $\mathbf{Q}_p$, with zeroes elsewhere. Then we have

$$\mathbf{A}(x) = \sum_{k=1}^{p-1} Z_k Z_k^{\text{tr}} + \mathbf{A}_p(x),$$

which is the claimed formula. Moreover we have the following extension of (3.4):

$$(3.10) \quad ca_{k,k} \mathbf{e}_k \otimes \mathbf{e}_k \prec Z_k(x) Z_k(x)^{\text{tr}} + \sum_{m=k+1}^n a_{m,m} \mathbf{e}_m \otimes \mathbf{e}_m \prec C \sum_{m=k}^n a_{m,m} \mathbf{e}_m \otimes \mathbf{e}_m, \quad 1 \leq k \leq p-1.$$

It remains now to prove that we can further decompose each of the positive dyads $Z_k(x) Z_k(x)^{\text{tr}}$ into a finite sum of squares of $C^{2,\delta}$ vector fields. Let $\mathbf{Q}_\ell(x) = [q_{k,j}]_{k,j=1}^n$ and $E_\ell \equiv q_{\ell,\ell}$. To obtain the sum of squares of $C^{2,\delta}$ vector fields decomposition of the positive dyad $Z_k(x) Z_k(x)^{\text{tr}}$, we will begin with the conclusion of Lemma 35, namely that $E_k = q_{k,k} \in C^{4,2\delta}$ and satisfies

$$|D^\mu E_k| \lesssim (E_k)^{[1-|\mu|\varepsilon]_+ + \delta'} \leq (E_k)^{\delta'} = (E_k)^{\frac{2\delta(1+\delta)}{2+\delta}}, \quad 1 \leq |\mu| \leq 4, \quad 2 \leq k \leq \ell,$$

together with the scalar sum of squares Theorem 4, and we will show that there exists a vector function

$$\mathbf{t}_k^k(x) = \left\{ t_{k,i}^k(x) \right\}_{i=1}^I \in C^{2,\delta} \text{ such that}$$

$$(3.11) \quad \begin{aligned} E_k(x) &= |\mathbf{t}_k^k(x)|^2, \\ |t_{k,i}^k(x)| &\leq (E_k)^{\frac{1}{2}}, \quad |\nabla t_{k,i}^k(x)| \lesssim (E_k)^{\frac{1}{2}([1-2\varepsilon]_+ + \delta'')}, \quad |\nabla^2 t_{k,i}^k(x)| \lesssim (E_k)^{\frac{\delta^2}{2+\delta}}, \end{aligned}$$

where $\delta''' = \frac{\delta}{4+2\delta}$.

Now for $1 \leq k \leq p-1$, define vector functions $\mathbf{t}_j^k(x) = \{t_{j,i}^k(x)\}_{i=1}^I$ by

$$t_{j,i}^k(x) \equiv t_{k,i}^k(x) \frac{a_{k,j}}{E_k}, \quad k < j \leq n,$$

where the functions $t_{k,i}^k(x)$ are given by applying Theorem 4 to $E_k(x) = |\mathbf{t}_k^k(x)|^2$, and then set

$$X_{k,i}(x) \equiv \sum_{j=k}^n t_{j,i}^k(x) \mathbf{e}_j$$

so that

$$\begin{aligned} \sum_{i=1}^I t_{k,i}^k(x) t_{j,i}^k(x) &= \sum_{i=1}^I t_{k,i}^k(x) t_{k,i}^k(x) \frac{a_{k,j}}{E_k} = \frac{|\mathbf{t}_k^k(x)|^2}{E_k} a_{k,j} = \frac{a_{k,k} a_{k,j}}{E_k}, \\ \text{i.e. } \sum_{i=1}^I X_{k,i}(x) \otimes X_{k,i}(x) &= \left(\sum_{j=k}^n \frac{a_{k,j}}{\sqrt{E_k}} \mathbf{e}_j \right) \otimes \left(\sum_{j=k}^n \frac{a_{k,j}}{\sqrt{E_k}} \mathbf{e}_j \right) = Z_k \otimes Z_k. \end{aligned}$$

At this point we have obtained our decomposition

$$\mathbf{A} = \sum_{k=1}^{p-1} \sum_{i=1}^I X_{k,i} X_{k,i}^{\text{tr}} + \mathbf{A}_p$$

where $\mathbf{A}_p = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_p \end{bmatrix}$ and $\mathbf{Q}_p \in C^{4,2\delta}(\mathbb{R}^M)$ is quasiconformal and $\mathbf{Q}_p \sim a_{p,p} \mathbb{I}_{(n-p+1) \times (n-p+1)}$. We also have

$$\begin{aligned} Z_k \otimes Z_k &= \sum_{i=1}^I X_{k,i} X_{k,i}^{\text{tr}} \in C^{4,2\delta}(\mathbb{R}^M), \quad 1 \leq k \leq p-1 \\ ca_{k,k} \mathbf{e}_k \otimes \mathbf{e}_k &\prec Z_k Z_k^{\text{tr}} + \sum_{m=k}^n a_{m,m} \mathbf{e}_m \otimes \mathbf{e}_m \prec C., \quad 1 \leq k \leq p-1. \end{aligned}$$

Thus it remains only to show the inequalities in the second line of (3.11), and then that $X_{k,i} \in C^{2,\delta}(\mathbb{R}^M)$.

First, $|t_{k,i}^k(x)| \leq (E_k)^{\frac{1}{2}}$ follows from the definition of $t_{k,i}^k(x)$. Second, from part (1) of Theorem 4 with $|\alpha| = 2$ we have

$$|\nabla^2 t_{k,i}^k(x)| \lesssim (E_k)^{\frac{\delta^2}{2+\delta}}.$$

Third, from (2.6) we have $|\nabla^2 E_k| \lesssim (E_k)^{[1-2\varepsilon]_+ + \delta'}$ and so from (1.3) we conclude

$$\begin{aligned} |\nabla t_{k,i}^k(x)| &\lesssim \rho_{E_k}^{1+\delta} \leq \max \left\{ (E_k)^{\frac{1+\delta}{4+2\delta}}, |\nabla^2 E_k|^{\frac{1+\delta}{2+2\delta}} \right\} \\ &\leq \max \left\{ (E_k)^{\frac{1+\delta}{4+2\delta}}, (E_k)^{([1-2\varepsilon]_+ + \delta') \frac{1+\delta}{2+2\delta}} \right\} \leq (E_k)^{\frac{1}{2}([1-2\varepsilon]_+ + \delta''')}, \end{aligned}$$

where $\delta''' = \frac{\delta}{4+2\delta}$, and this completes the proof of the second line in (3.11).

Now we can show that $X_{k,i} \in C^{2,\delta}(\mathbb{R}^M)$ using the product formula (3.7) together with the inequalities in the second line of (3.11) that we just proved. Indeed, we have

$$D^\mu t_{j,i}^k = D^\mu \left(t_{k,i}^k a_{k,j} \frac{1}{E_k} \right) = \sum_{\alpha+\beta+\gamma=\mu} c_{\alpha,\beta,\gamma}^\mu (D^\alpha t_{k,i}^k) (D^\beta a_{k,j}) \left(D^\gamma \frac{1}{E_k} \right),$$

where

$$\begin{aligned} |t_{k,i}^k(x)| &\leq (E_k)^{\frac{1}{2}}, \quad |\nabla t_{k,i}^k(x)| \lesssim (E_k)^{\frac{1}{2}([1-2\varepsilon]_+ + \delta''')}, \quad |\nabla^2 t_{k,i}^k(x)| \lesssim (E_k)^{\frac{\delta^2}{2+\delta}} \\ |D^\beta a_{k,j}| &\lesssim (E_j)^{[\frac{1}{2} + (2-|\beta|)\varepsilon]_+ + \delta''}, \quad 1 \leq k < j \leq p-1, \\ |D^\beta a_{k,j}| &\lesssim (E_k)^{[\frac{1}{2} + (2-|\beta|)\varepsilon]_+ + \delta''}, \quad 1 \leq k \leq p-1 < j, \\ \left| D^\gamma \frac{1}{E_k} \right| &\lesssim |E_k|^{-1-|\gamma|\varepsilon+|\gamma|\delta'}, \end{aligned}$$

for $0 \leq |\alpha|, |\beta|, |\gamma| \leq 2$. We now have the following estimates when $1 \leq k < j \leq p-1$, where we treat the cases $|\alpha| = 0, 1, 2$ separately due to the unorthodox form of the estimates for $|\nabla^2 t_{k,i}^k(x)|$. The presence of $\delta'' > 0$ will now play a crucial role.

When $|\alpha| = 0$ we have

$$\begin{aligned} \left| (D^\alpha t_{k,i}^k) (D^\beta a_{k,j}) \left(D^\gamma \frac{1}{E_k} \right) \right| &\lesssim (E_k)^{\frac{1}{2}} (E_j)^{[\frac{1}{2} + (2-|\beta|)\varepsilon]_+ + \delta''} |E_k|^{-1-|\gamma|\varepsilon+|\gamma|\delta'} \\ &\lesssim (E_k)^{\frac{1}{2} + [\frac{1}{2} + (2-|\beta|)\varepsilon]_+ + \delta'' - 1 - |\gamma|\varepsilon + |\gamma|\delta'}, \end{aligned}$$

which is bounded because the exponent of E_k is

$$\begin{aligned} &\frac{1}{2} + \frac{1}{2} + (2-|\beta|)\varepsilon + \delta'' - 1 - |\gamma|\varepsilon + |\gamma|\delta' \\ &= (2-|\beta|-|\gamma|)\varepsilon + |\gamma|\delta' + \delta'' = \delta'' + |\gamma|\delta' \geq \delta''. \end{aligned}$$

When $|\alpha| = 1$ we have the estimate

$$\left| (D^\alpha t_{k,i}^k) (D^\beta a_{k,j}) \left(D^\gamma \frac{1}{E_k} \right) \right| \lesssim (E_k)^{([1-2\varepsilon]_+ + \delta''') \frac{1+\delta}{2+2\delta}} (E_k)^{[\frac{1}{2} + \varepsilon]_+ + \delta''} (E_k)^{-1},$$

since the worst case is when $|\beta| = 1$. But this is bounded since

$$([1-2\varepsilon]_+ + \delta''') \frac{1}{2} + \left[\frac{1}{2} + \varepsilon \right]_+ + \delta'' - 1 \geq \delta'' + \frac{\delta'''}{2}, \quad \text{for } \frac{1}{4} \leq \varepsilon < 1.$$

Indeed, this is clear when $\varepsilon \geq \frac{1}{2}$, and when $\frac{1}{4} \leq \varepsilon < \frac{1}{2}$, we have

$$\begin{aligned} &([1-2\varepsilon]_+ + \delta''') \frac{1}{2} + \left[\frac{1}{2} + \varepsilon \right]_+ + \delta'' - 1 \\ &= \delta'' + (1-2\varepsilon + \delta''') \frac{1}{2} + \frac{1}{2} + \varepsilon - 1 = \delta'' + \frac{\delta'''}{2}. \end{aligned}$$

When $|\alpha| = 2$ we have the estimate

$$\begin{aligned} \left| (D^\alpha t_{k,i}^k) (D^\beta a_{k,j}) \left(D^\gamma \frac{1}{E_k} \right) \right| &\lesssim (E_k)^{\frac{\delta^2}{2+\delta}} (E_k)^{[\frac{1}{2} + 2\varepsilon]_+ + \delta''} (E_k)^{-1} \\ &= (E_k)^{\frac{\delta^2}{2+\delta} + \frac{1}{2} + 2\varepsilon + \delta'' - 1} \leq (E_k)^{\delta'' + \frac{\delta^2}{2+\delta}}, \end{aligned}$$

when $\varepsilon \geq \frac{1}{4}$.

Using the subproduct and subchain rules for the functional $[h]_{\mu, \delta''}$ in $\mathbb{R}^M$ as in (1.1) above, we claim that

$$[t_{j,i}^k]_{\mu, \delta''} \lesssim 1 \quad \text{for } |\mu| = 2.$$

To see this, consider the "worst" expression above,

$$H \equiv (t_{k,i}^k) (D^\beta a_{k,j}) \frac{1}{E_k}, \quad \text{where } |\beta| = 2,$$

in which the exponent of E_k in the estimate for H vanishes if $\delta'' = 0$.

Case $E_k(y) \geq E_k(z)$: In this case we write

$$\begin{aligned} H(y) - H(z) &= \frac{t_{k,i}^k(y)}{E_k(y)} D^\beta a_{k,j}(y) - \frac{t_{k,i}^k(z)}{E_k(z)} D^\beta a_{k,j}(z) \\ &= \frac{t_{k,i}^k(y)}{E_k(y)} (D^\beta a_{k,j}(y) - D^\beta a_{k,j}(z)) + \left(\frac{t_{k,i}^k(y)}{E_k(y)} - \frac{t_{k,i}^k(z)}{E_k(z)} \right) D^\beta a_{k,j}(z) \\ &\equiv I(y, z) + II(y, z). \end{aligned}$$

We estimate term $I(y, z)$ by considering two cases separately depending on the separation between y and z .

Subcase $|y - z|^\delta \geq E_k(y)^{\delta''}$: We estimate crudely in this case to obtain

$$\frac{|I(y, z)|}{|y - z|^\delta} \lesssim \frac{E_k(y)^{\frac{1}{2}} E_k(y)^{\frac{1}{2} + \delta''} + E_k(z)^{\frac{1}{2} + \delta''}}{E_k(y)^{\delta''}} \lesssim \frac{E_k(y)^{\delta''}}{E_k(y)^{\delta''}} = 1.$$

On the other hand,

$$\begin{aligned} \frac{|II(y, z)|}{|y - z|^\delta} &\lesssim \left(\left| \frac{t_{k,i}^k(y)}{E_k(y)} \right| + \left| \frac{t_{k,i}^k(z)}{E_k(z)} \right| \right) \frac{|D^\beta a_{k,j}(z)|}{E_k(y)^{\delta''}} \\ &\lesssim \left(\frac{1}{E_k(y)^{\frac{1}{2}}} + \frac{1}{E_k(z)^{\frac{1}{2}}} \right) \frac{E_k(z)^{\frac{1}{2} + \delta''}}{E_k(y)^{\delta''}} \\ &\lesssim \left(\frac{1}{E_k(z)^{\frac{1}{2}}} \right) \frac{E_k(z)^{\frac{1}{2} + \delta''}}{E_k(y)^{\delta''}} \leq 1. \end{aligned}$$

Subcase $|y - z|^\delta < E_k(y)^{\delta''}$: In this case we apply the submean value theorem to the difference $D^\beta a_{k,j}(y) - D^\beta a_{k,j}(z)$ to obtain the estimate

$$|D^\beta a_{k,j}(y) - D^\beta a_{k,j}(z)| \leq |D^\gamma a_{k,j}((1 - \theta)y + \theta z)| |y - z| \lesssim |y - z|,$$

since $|\gamma| = 3$ implies $|D^\gamma a_{k,j}|$ is bounded. Moreover, the submean value theorem applied to E_k yields

$$\begin{aligned} |E_k(y) - E_k(z)| &\leq |DE_k((1 - \theta)y + \theta z)| |y - z| \lesssim |y - z| \\ &\leq E_k(y)^{\frac{\delta''}{\delta}}, \end{aligned}$$

and since $\delta'' > \delta$, we conclude that $E_k(y) \approx E_k(z)$ for y and z sufficiently close to the origin, depending on the ratio $\frac{\delta''}{\delta} > 1$. Plugging all of this into the estimate for $\frac{|I(y, z)|}{|y - z|^\delta}$ gives

$$\frac{|I(y, z)|}{|y - z|^\delta} \lesssim \frac{E_k(y)^{\frac{1}{2}}}{E_k(y)} |y - z|^{1 - \delta} \lesssim \frac{E_k(y)^{\frac{1}{2}}}{E_k(y)} (E_k(y)^{\delta''})^{\frac{1 - \delta}{\delta}} \lesssim E_k(y)^{\delta'' \frac{1 - \delta}{\delta} - \frac{1}{2}},$$

which is bounded because $\delta'' > \delta$ implies $\delta'' \frac{1-\delta}{\delta} - \frac{1}{2} > 1 - \delta - \frac{1}{2} \geq 0$ if $0 < \delta \leq \frac{1}{2}$. On the other hand,

$$\begin{aligned} \frac{|II(y, z)|}{|y - z|^\delta} &\lesssim \left| \frac{t_{k,i}^k(y)}{E_k(y)} - \frac{t_{k,i}^k(z)}{E_k(z)} \right| \frac{|D^\beta a_{k,j}(z)|}{|y - z|^\delta} \\ &\lesssim \left| \frac{E_k(z) t_{k,i}^k(y) \pm E_k(z) t_{k,i}^k(z) - t_{k,i}^k(z) E_k(y)}{E_k(y) E_k(z)} \right| \frac{E_k(z)^{\frac{1}{2} + \delta''}}{|y - z|^\delta} \\ &\lesssim \left(\frac{E_k(z) |t_{k,i}^k(y) - t_{k,i}^k(z)| + |E_k(z) - E_k(y)| |t_{k,i}^k(z)|}{E_k(y) E_k(z)} \right) \frac{E_k(z)^{\frac{1}{2} + \delta''}}{|y - z|^\delta} \end{aligned}$$

and so using the mean value theorem,

$$\begin{aligned} \frac{|II(y, z)|}{|y - z|^\delta} &\lesssim \left(\frac{E_k(z) |y - z| + |y - z| E_k(z)^{\frac{1}{2}}}{E_k(y) E_k(z)} \right) \frac{E_k(z)^{\frac{1}{2} + \delta''}}{|y - z|^\delta} \\ &\lesssim \left(\frac{|y - z| E_k(z)^{\frac{1}{2}}}{E_k(y) E_k(z)} \right) \frac{E_k(z)^{\frac{1}{2} + \delta''}}{|y - z|^\delta} \approx |y - z|^{1-\delta} E_k(y)^{\delta''-1} \\ &= |y - z|^{\delta(\frac{1-\delta}{\delta})} E_k(y)^{\delta''-1} < E_k(y)^{\delta''(\frac{1-\delta}{\delta})} E_k(y)^{\delta''-1} \lesssim 1, \end{aligned}$$

since $|y - z|^\delta < E_k(y)^{\delta''}$ and $\delta'' > \delta$.

Case $E_k(z)^{\delta''} \geq E_k(y)^{\delta''}$: This case is similar to the previous case.

Finally, if in addition **A** is subordinate, then **Q_p** is subordinate by Lemma 34. This completes the proof of Theorem 22. ■

4. COUNTEREXAMPLES FOR SUMS OF SQUARES OF MATRIX FUNCTIONS

Consider the 3×3 matrix of quadratic homogeneous polynomials in three variables as in [HiNi, Example 6],

$$\mathbf{Q}(x, y, z) \equiv \begin{bmatrix} x^2 + 2z^2 & -xy & -xz \\ -xy & y^2 + 2x^2 & -yz \\ -xz & -yz & z^2 + 2y^2 \end{bmatrix}.$$

The dehomogenization of this form is

$$\mathbf{Q}(x, y) \equiv \begin{bmatrix} x^2 + 2 & -xy & -x \\ -xy & y^2 + 2x^2 & -y \\ -x & -y & 1 + 2y^2 \end{bmatrix}.$$

It was shown in [Cho] that **Q** is not a sum of squares of polynomials. However, the quadratic matrix form $\mathbf{Q}(x, y, z)$ is not elliptical since its determinant vanishes on the union of the three coordinate axes,

$$\det \mathbf{Q}(x, y, z) = 4(x^4 y^2 + y^4 z^2 + z^4 x^2 + x^2 y^2 z^2).$$

Here we modify $\mathbf{Q}(x, y, z)$ so that it is diagonally elliptical, i.e. its determinant vanishes only at the origin, yet still cannot be written as a sum of squares.

4.1. A positive quadratic matrix form that is not a sum of squares of forms. We prove the following theorem.

Theorem 36. *If $0 < \lambda < \frac{2}{81}$, then the quadratic matrix form*

$$\mathbf{Q}_\lambda(x, y, z) \equiv \begin{bmatrix} x^2 + \lambda y^2 + 2z^2 & -xy & -xz \\ -xy & y^2 + \lambda z^2 + 2x^2 & -yz \\ -xz & -yz & z^2 + \lambda x^2 + 2y^2 \end{bmatrix}$$

is both positive definite away from the origin, and not a sum of squares of linear matrix polynomials.

Proof. Suppose $0 < \lambda < \frac{2}{81}$. The quadratic matrix form $\mathbf{Q}_\lambda(x, y, z)$ is positive definite for all $(x, y, z) \neq (0, 0, 0)$. Indeed, for $\lambda > 0$, the top left entry of $\mathbf{Q}_\lambda(x, y, z)$ satisfies

$$x^2 + \lambda y^2 + 2z^2 \geq \min\{\lambda, 1\} (x^2 + y^2 + z^2),$$

the top left 2×2 minor of $\mathbf{Q}_\lambda(x, y, z)$ satisfies

$$\begin{aligned} & \det \begin{bmatrix} x^2 + \lambda y^2 + 2z^2 & -xy \\ -xy & y^2 + \lambda z^2 + 2x^2 \end{bmatrix} \\ &= \lambda^2 \{y^2 + z^2\} + \lambda \{z^2 (x^2 + 2z^2) + y^2 (y^2 + 2x^2)\} \\ & \quad + (x^2 + 2z^2) (y^2 + 2x^2) - x^2 y^2 \\ &\geq 2x^4 + \lambda (y^4 + 2z^4) \geq \min\{\lambda, 2\} (x^4 + y^4 + z^4), \end{aligned}$$

and the determinant of $\mathbf{Q}_\lambda(x, y, z)$ satisfies

$$\begin{aligned} \det \mathbf{Q}_\lambda(x, y, z) &= \lambda^3 (x^2 y^2 z^2) + \lambda^2 \{2 (x^2 z^4 + z^2 y^4 + y^2 x^4) + x^2 y^4 + y^2 z^4 + z^2 x^4\} \\ & \quad + 2\lambda \{x^6 + y^6 + z^6 + 2 (x^2 y^4 + y^2 z^4 + z^2 x^4) + 3x^2 y^2 z^2\} \\ & \quad + 4 (x^2 z^4 + z^2 y^4 + y^2 x^4 + x^2 y^2 z^2) \\ &\geq 2\lambda (x^6 + y^6 + z^6). \end{aligned}$$

Now assume, in order to derive a contradiction, that the dehomogenization

$$\mathbf{Q}_\lambda(x, y) \equiv \mathbf{Q}_\lambda(x, y, 1) = \begin{bmatrix} x^2 + \lambda y^2 + 2 & -xy & -x \\ -xy & y^2 + \lambda + 2x^2 & -y \\ -x & -y & 1 + \lambda x^2 + 2y^2 \end{bmatrix}$$

is a sum of squares of dyads of degree one, i.e.

$$\mathbf{Q}_\lambda(x, y) = \sum_{\ell=1}^L \mathbf{v}^\ell(x, y) \otimes \mathbf{v}^\ell(x, y) = \sum_{\ell=1}^L \begin{bmatrix} m_{11}^\ell & m_{12}^\ell & m_{13}^\ell \\ m_{21}^\ell & m_{22}^\ell & m_{23}^\ell \\ m_{31}^\ell & m_{32}^\ell & m_{33}^\ell \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{bmatrix} m_{11}^\ell & m_{21}^\ell & m_{31}^\ell \\ m_{12}^\ell & m_{22}^\ell & m_{32}^\ell \\ m_{13}^\ell & m_{23}^\ell & m_{33}^\ell \end{bmatrix}.$$

Then with $\mathbf{m}_{ij} \equiv (m_{ij}^\ell)_{\ell=1}^L$, we have upon equating the two formulas for $\mathbf{Q}_\lambda(x, y)$ that

$$\begin{aligned} |\mathbf{m}_{11}|^2 &= |\mathbf{m}_{22}|^2 = |\mathbf{m}_{33}|^2 = 1, \\ |\mathbf{m}_{13}|^2 &= |\mathbf{m}_{21}|^2 = |\mathbf{m}_{32}|^2 = 2, \\ |\mathbf{m}_{12}|^2 &= |\mathbf{m}_{23}|^2 = |\mathbf{m}_{31}|^2 = \lambda, \end{aligned}$$

and

$$\mathbf{m}_{22} \cdot \mathbf{m}_{11} + \mathbf{m}_{21} \cdot \mathbf{m}_{12} = \mathbf{m}_{33} \cdot \mathbf{m}_{11} + \mathbf{m}_{31} \cdot \mathbf{m}_{13} = \mathbf{m}_{33} \cdot \mathbf{m}_{22} + \mathbf{m}_{32} \cdot \mathbf{m}_{23} = -1.$$

Thus we conclude that

$$\begin{aligned} |\mathbf{m}_{11} + \mathbf{m}_{22}|^2 &= |\mathbf{m}_{11}|^2 + 2\mathbf{m}_{11} \cdot \mathbf{m}_{22} + |\mathbf{m}_{22}|^2 = 2 + 2(-1 - \mathbf{m}_{21} \cdot \mathbf{m}_{12}) = -2\mathbf{m}_{21} \cdot \mathbf{m}_{12}, \\ |\mathbf{m}_{22} + \mathbf{m}_{33}|^2 &= |\mathbf{m}_{22}|^2 + 2\mathbf{m}_{22} \cdot \mathbf{m}_{33} + |\mathbf{m}_{33}|^2 = 2 + 2(-1 - \mathbf{m}_{32} \cdot \mathbf{m}_{23}) = -2\mathbf{m}_{32} \cdot \mathbf{m}_{23}, \\ |\mathbf{m}_{33} + \mathbf{m}_{11}|^2 &= |\mathbf{m}_{33}|^2 + 2\mathbf{m}_{33} \cdot \mathbf{m}_{11} + |\mathbf{m}_{11}|^2 = 2 + 2(-1 - \mathbf{m}_{31} \cdot \mathbf{m}_{13}) = -2\mathbf{m}_{31} \cdot \mathbf{m}_{13}, \end{aligned}$$

and hence

$$\begin{aligned} 4 &= |2\mathbf{m}_{11}|^2 = |(\mathbf{m}_{11} + \mathbf{m}_{22}) - (\mathbf{m}_{22} + \mathbf{m}_{33}) + (\mathbf{m}_{33} + \mathbf{m}_{11})|^2 \\ &\leq 3 \left(|\mathbf{m}_{11} + \mathbf{m}_{22}|^2 + |\mathbf{m}_{22} + \mathbf{m}_{33}|^2 + |\mathbf{m}_{33} + \mathbf{m}_{11}|^2 \right) \\ &= -6(\mathbf{m}_{21} \cdot \mathbf{m}_{12} + \mathbf{m}_{32} \cdot \mathbf{m}_{23} + \mathbf{m}_{31} \cdot \mathbf{m}_{13}) \\ &\leq 6 \left(\sqrt{2}\sqrt{\lambda} + \sqrt{2}\sqrt{\lambda} + \sqrt{\lambda}\sqrt{2} \right) = 18\sqrt{2\lambda} < 4, \end{aligned}$$

if $0 < \lambda < \frac{2}{81}$, which is the desired contradiction. ■

4.2. A matrix-valued smooth function not a finite sum of vector $C^{1,\alpha}$ squares. Now suppose that $\mathbf{P}_\lambda(x, y, z) = \mathbf{Q}_\lambda(x, y, z) + O(r^{2+\alpha})$, where $r = \sqrt{x^2 + y^2 + z^2}$. Then if $\mathbf{P}_\lambda$ is a sum of $C^{1,\alpha}$ squares,

$$\mathbf{P}_\lambda(x, y, z) = \sum_{\ell=1}^L \mathbf{u}^\ell(x, y, z) \otimes \mathbf{u}^\ell(x, y, z), \quad \text{where } \mathbf{u}^\ell(x, y, z) \in C^{1,\alpha},$$

Taylor's theorem shows that

$$\mathbf{u}^\ell(x, y, z) = \mathbf{v}^\ell(x, y, z) + O(r^{1+\alpha}), \quad \text{where } \mathbf{v}^\ell \text{ is a linear form,}$$

and so

$$\begin{aligned} \mathbf{Q}_\lambda(x, y, z) + O(r^{2+\alpha}) &= \mathbf{P}_\lambda(x, y, z) \\ &= \sum_{\ell=1}^L (\mathbf{v}^\ell(x, y, z) + O(r^{1+\alpha})) \otimes (\mathbf{v}^\ell(x, y, z) + O(r^{1+\alpha})) \\ &= \sum_{\ell=1}^L \mathbf{v}^\ell(x, y, z) \otimes \mathbf{v}^\ell(x, y, z) + O(r^{2+\alpha}), \quad \text{since } \mathbf{v}^\ell(x, y, z) = O(r), \end{aligned}$$

implies that

$$\mathbf{Q}_\lambda(x, y, z) = \sum_{\ell=1}^L \mathbf{v}^\ell(x, y, z) \otimes \mathbf{v}^\ell(x, y, z),$$

the desired contradiction. Since $\mathbf{Q}_\lambda(x, y, z)$ is obviously subordinate, we have established the following.

Theorem 37. *There is a subordinate, diagonally elliptical 3×3 matrix-valued quadratic polynomial function of three variables, e.g. $\mathbf{Q}_\lambda(x, y, z)$ with $0 < \lambda < \frac{2}{81}$, that cannot be written as a finite sum of squares of $C^{1,\delta}$ vector functions for any $\delta > 0$.*

This conclusion shows in particular that a smooth matrix-valued function, comparable to the identity matrix, need not have even a $C^{1,\delta}$ sum of squares representation, in stark contrast to the scalar case where the Fefferman-Phong theorem shows that a $C^{1,1}$ sum of squares representation always holds.

However, the hypotheses $|D^\mu a_{k,k}(w)| \lesssim a_{k,k}(w)^{[1-|\mu|\varepsilon]_+ + \delta'}$ for some $\delta' > 0$, on the *diagonal* entries $a_{k,k}(w)$ in Theorem 22 with $n = 3$ and $p = 4$, are *not* satisfied by the matrix function $\mathbf{Q}_\lambda(x, y, z) = \mathbf{Q}_\lambda(w)$ with $w = (x, y, z)$, since $\frac{\partial^2}{\partial w_k^2} a_{k,k}(w) = 2$ and $a_{k,k}(w)^{[1-|\mu|\varepsilon]_+ + \delta'} \lesssim |w|^{\delta'}$. We now turn to constructing a counterexample in Theorem 42 below, where only the off diagonal inequalities fail to hold, which will show that in order to conclude that there is a representation as a sum of squares of $C^{2,\delta}$ vectors, it is necessary to impose additional conditions on the *off-diagonal* entries, such as we have done in Theorem 22.

4.3. The flat elliptical case. We will prove in Lemma 39 below that there is a positive constant C_β such that if

$$\psi(t) \leq C_\beta \varphi(t)^{\frac{2}{\beta}} t^{\frac{4}{\beta}}, \quad \text{for all sufficiently small } |t|,$$

then $\mathbf{F}_{\varphi,\psi}$ cannot be written as a finite sum of squares of $C^{1,\beta}$ vector fields.

To match notation with that used in the paper [KoSa1], we fix $0 < \lambda < \frac{2}{81}$, and for $W = (x, y, z) \in \mathbb{R}^3$, set

$$\mathbf{L}(W) = \mathbf{L}(x, y, z) \equiv \mathbf{Q}_\lambda(x, y, z) = \begin{bmatrix} x^2 + \lambda y^2 + 2z^2 & -xy & -xz \\ -xy & y^2 + \lambda z^2 + 2x^2 & -yz \\ -xz & -yz & z^2 + \lambda x^2 + 2y^2 \end{bmatrix},$$

so that $\mathbf{L}(W) \sim |W|^2 \mathbb{I}_3$. We now recall some of the constructions in [KoSa1], but in the context of $\mathbb{R}^3$ here, rather than $\mathbb{R}^4$ as was done in [KoSa1]. For a modulus of continuity ω , and h defined on the unit ball $B_{\mathbb{R}^3}(0, 1)$ in $\mathbb{R}^4$,

$$\|h\|_{C^{1,\omega}(B_{\mathbb{R}^3}(0,1))} \equiv \|h\|_{C^0(B_{\mathbb{R}^3}(0,1))} + \|\nabla h\|_{C^0(B_{\mathbb{R}^3}(0,1))} + \sup_{W, W' \in B_{\mathbb{R}^3}(0,1)} \frac{|\nabla h(W) - \nabla h(W')|}{\omega(|W - W'|)}.$$

Given $\tau > 0$, define

$$C_{1,\omega}^\nu(\tau) \equiv \inf \left\{ \|\mathcal{G}\|_{1,\omega} : \mathcal{G} \in \oplus^\nu C^{1,\omega}(B_{\mathbb{R}^3}(0,1)) \text{ and } \mathbf{L}(W) + \tau = \sum_{\ell=1}^\nu G_\ell(W) G_\ell(W)^{\text{tr}} \right\}.$$

Note that this expression differs from that in [KoSa1] by using the smoothness space $C^{1,\omega}$ in place of $C^{2,\omega}$, as was used in [KoSa1]. The reason is that the matrix form $\mathbf{L}(x, y, z)$ is homogenous of degree 2 here, whereas the function form $L(w, x, y, z)$ was homogenous of degree 4 in [KoSa1]. We will need a crucial lower bound in the case $\omega(s) = \omega_\beta(s) = s^\beta$. For this we define

$$(4.1) \quad \delta_\nu^2 \equiv \inf_{\{S_\ell\}_{\ell=1}^\nu} \inf_{W \in \mathbb{S}^2} \left(\mathbf{L}(W) - \sum_{\ell=1}^\nu S_\ell(W)^2 \right)^2.$$

Here the infimum is taken over all collections $\{S_\ell\}_{\ell=1}^\nu$ of linear forms $S_\ell(W) = f_{\ell,1}x + f_{\ell,2}y + f_{\ell,3}z$ with $W \in \mathbb{S}^2$ and coefficients $f_{\ell,j}$ of modulus at most C_0 , which will be determined in (4.6) below. Since the infimum is taken over a compact set, it is achieved, and must then be positive since $\mathbf{L}$ cannot be written as a sum of squares of linear forms by Theorem 36.

Lemma 38. *With notation as above, there is a positive constant C such that*

$$(4.2) \quad \mathcal{C}_{1,\omega_\beta}^\nu(\tau) \geq \left(\frac{\delta_\nu}{2C\tau^{\frac{\beta}{2}}} \right)^{\frac{1}{2-\beta}} = \left(\frac{\delta_\nu}{2C} \right)^{\frac{1}{2-\beta}} \tau^{-\frac{\beta}{4-2\beta}}.$$

Proof. We claim the inequalities

$$(4.3) \quad \begin{aligned} \delta_\nu &\leq \mathbf{L}\left(\frac{W}{|W|}\right) - \sum_{\ell=1}^\nu S_\ell\left(\frac{W}{|W|}\right)^2 = \left| \frac{\mathbf{L}(W) - \sum_{\ell=1}^\nu S_\ell(W)^2}{|W|^2} \right| \\ &\leq 2C \|\mathcal{G}\|_{1,\omega} \omega(|W|) = 2C \|\mathcal{G}\|_{1,\omega}^2 \omega\left(\frac{\sqrt{\tau}}{\|\mathcal{G}\|_{1,\omega}}\right), \end{aligned}$$

which then lead directly to (4.2) with $\omega = \omega_\beta$ upon using the definition of $\mathcal{C}_{1,\omega}^\nu(\tau)$. To see this claim we write

$$\begin{aligned} \mathbf{L}(W) + \tau &= \sum_{\ell=1}^\nu G_\ell(W)^2, \\ \text{where } G_\ell(W) &= a_\ell + S_\ell(W) + R_\ell(W), \\ a_\ell &\text{ is a constant, } S_\ell(W) \text{ is linear and } R_\ell(W) \text{ is } o(|W|). \end{aligned}$$

Then setting $W = 0$ in the equation gives

$$\tau = \sum_{\ell=1}^\nu a_\ell^2,$$

and so

$$\begin{aligned} \mathbf{L}(W) &= \sum_{\ell=1}^\nu [a_\ell + S_\ell(W) + R_\ell(W)]^2 - \tau \\ &= \left(\sum_{\ell=1}^\nu a_\ell^2 \right) - \tau + \sum_{\ell=1}^\nu 2a_\ell S_\ell(W) \\ &\quad + \sum_{\ell=1}^\nu 2a_\ell R_\ell(W) + \sum_{\ell=1}^\nu [S_\ell(W) + R_\ell(W)]^2. \end{aligned}$$

Now the sum of terms in the middle line vanishes identically since it is a linear polynomial, and all of the remaining terms in the final line of the identity vanish to order greater than 1 at the origin (simply differentiate this identity and then evaluate at $W = 0$, using that $\nabla \mathbf{L}(0)$ and $\nabla R_\ell(0)$ vanish). Thus we conclude that

$$(4.4) \quad \mathbf{L}(W) - \sum_{\ell=1}^\nu [S_\ell(W) + R_\ell(W)]^2 = \sum_{\ell=1}^\nu 2a_\ell R_\ell(W),$$

where

$$(4.5) \quad \begin{aligned} \sum_{\ell=1}^{\nu} a_{\ell}^2 &= \tau, \\ \sum_{\ell=1}^{\nu} |S_{\ell}(W)| &\leq \|\mathbf{G}\|_{1,\omega} |W|, \\ \sum_{\ell=1}^{\nu} |R_{\ell}(W)| &\leq \|\mathbf{G}\|_{1,\omega} |W| \omega(W). \end{aligned}$$

From (4.4) we have

$$\begin{aligned} \mathbf{L}(W) - \sum_{\ell=1}^{\nu} S_{\ell}(W)^2 &= \mathbf{L}(W) - \sum_{\ell=1}^{\nu} [S_{\ell}(W) + R_{\ell}(W)]^2 + \sum_{\ell=1}^{\nu} [2S_{\ell}(W) + R_{\ell}(W)] R_{\ell}(W) \\ &= h_1(W) + h_2(W) \equiv h(W), \end{aligned}$$

where

$$\begin{aligned} h_1(W) &\equiv \sum_{\ell=1}^{\nu} 2a_{\ell} R_{\ell}(W), \\ h_2(W) &\equiv \sum_{\ell=1}^{\nu} [2S_{\ell}(W) + R_{\ell}(W)] R_{\ell}(W). \end{aligned}$$

Using (4.5) we obtain

$$\begin{aligned} |h_1(W)| &\leq C\sqrt{\tau} \|\mathcal{G}\|_{1,\omega} |W| \omega(|W|) \\ |h_2(W)| &\leq C \|\mathcal{G}\|_{1,\omega} |W| |R_{\ell}(W)| \leq C \|\mathcal{G}\|_{1,\omega} |W|^2 \omega(|W|). \end{aligned}$$

So altogether we have

$$|h(W)| \leq |h_1(W)| + |h_2(W)| \leq C\sqrt{\tau} \|\mathcal{G}\|_{1,\omega} |W| \omega(|W|) + C \|\mathcal{G}\|_{1,\omega}^2 |W|^2 \omega(|W|),$$

provided $|W| \leq 1$.

Now note that we may assume without loss of generality that $\frac{\sqrt{\tau}}{\|\mathcal{G}\|_{1,\omega}} \leq 1$, since otherwise (4.2) holds trivially. Thus if $|W| = \frac{\sqrt{\tau}}{\|\mathcal{G}\|_{1,\omega}}$, then we have

$$|h(W)| \leq C\tau\omega\left(\frac{\sqrt{\tau}}{\|\mathcal{G}\|_{1,\omega}}\right) + C\tau\omega\left(\frac{\sqrt{\tau}}{\|\mathcal{G}\|_{1,\omega}}\right).$$

Consequently we conclude

$$\frac{\tau}{\|\mathcal{G}\|_{1,\omega}^2} \left| \mathbf{L}\left(\frac{W}{|W|}\right) - \sum_{\ell=1}^{\nu} S_{\ell}\left(\frac{W}{|W|}\right)^2 \right| = \left| \mathbf{L}(W) - \sum_{\ell=1}^{\nu} S_{\ell}(W)^2 \right| = |h(W)| \leq C\tau\omega\left(\frac{\sqrt{\tau}}{\|\mathcal{G}\|_{1,\omega}}\right).$$

Also note that from $\sum_{\ell=1}^{\nu} |S_{\ell}(W)| \leq C\sqrt{\mathbf{L}(W) + \tau}$, we obtain for $0 < \tau < 1$ that

$$(4.6) \quad |f_{\ell,\alpha}| \leq C_0 \equiv C\sqrt{\mathbf{L}(W) + 1}.$$

This completes the proof of our claimed inequality (4.3), and hence also that of Lemma 38. ■

For a positive integer $\nu \in \mathbb{N}$ and a modulus of continuity ω , we say that a smooth nonnegative matrix function $\mathbf{F}(W, t)$ has the property $\mathcal{SOS}_{1,\omega}^{\nu}$ if there exists a finite collection $\mathcal{G} = \{G_{\ell}(W, t)\}_{\ell=1}^{\nu} \in \oplus^{\nu} C^{1,\omega}(\Omega)$ of vector fields $G_{\ell}(W, t) \in C^{1,\omega}(\Omega)$ such that

$$\mathbf{F}(W, t) = \sum_{\ell=1}^{\nu} G_{\ell}(W, t) G_{\ell}(W, t)^{\text{tr}}, \quad (W, t) \in \Omega.$$

Now let $\varphi : (0, 1) \rightarrow (0, 1)$ be a strictly increasing elliptical flat smooth function on $(0, 1)$, and define the matrix function

$$(4.7) \quad \begin{aligned} \mathbf{F}_{\varphi, \psi}(W, t) &\equiv \varphi(t) \mathbf{L}(W) + (\psi(t) + \eta(t, r)) \mathbb{I}_3, \\ \text{for } (W, t) &\in \Omega \equiv B_{\mathbb{R}^3}(0, 1) \times (-1, 1), \end{aligned}$$

where $\mathbb{I}_3$ is the 3×3 identity matrix, $r = |W| = \sqrt{x^2 + y^2 + z^2}$, and $\psi(t)$ and $\eta(t, r)$ are smooth nonnegative functions constructed as follows. The function $\eta(t, r)$ is chosen to have the form $\eta(t, r) = \varphi(r) h\left(\frac{t}{r}\right)$ where h is a smooth nonnegative function supported in $(-1, 1)$ with $h(0) = 1$. With these constructions completed, we see that $\mathbf{F}_{\varphi, \psi}$ is a diagonally elliptical flat smooth 3×3 matrix function on $B_{\mathbb{R}^3}(0, 1) \times (-1, 1)$.

Lemma 39. *Suppose $0 < \beta < 1$ and let $\mathbf{F}_{\varphi, \psi}(W, t)$ be as in (4.7). If*

$$(4.8) \quad \lim_{t \rightarrow 0} \frac{\psi(t)}{\varphi(t)^{\frac{2}{\beta}} t^{\frac{4}{\beta}}} = 0,$$

then $\mathbf{F}_{\varphi, \psi}$ fails to satisfy $\mathcal{SOS}_{1, \omega_\beta}^\nu$ for any $\nu \in \mathbb{N}$. Note in particular we may even take both φ and ψ to be nearly monotone functions on $(-1, 1)$.

Proof. Assume that $\mathbf{F}_{\varphi, \psi}(W, t)$ has the property $\mathcal{SOS}_{1, \omega_\beta}^\nu$ for some $\nu \in \mathbb{N}$, i.e. $\mathbf{F}_{\varphi, \psi} = \sum_{\ell=1}^\nu G_\ell^2$ where $G_\ell \in C^{1, \omega}(\Omega)$, i.e.

$$\begin{aligned} \varphi(t) \mathbf{L}(x, y, z, t) + \left[\psi(t) + \sigma(r) h\left(\frac{t}{r}\right) \right] \mathbb{I}_3 &= \sum_{\ell=1}^\nu G_\ell(x, y, z, t)^2, \\ \text{for } (x, y, z, t) &\in \Omega = B_{\mathbb{R}^3}(0, 1) \times (-1, 1). \end{aligned}$$

Then since $h\left(\frac{t}{r}\right)$ vanishes for $r \leq |t|$, we have with $W \equiv (x, y, z)$, and without loss of generality $t > 0$, that

$$\varphi(t) \mathbf{L}(W) + \psi(t) \mathbb{I}_3 = \sum_{\ell=1}^\nu G_\ell(W, t)^2, \quad \text{for } r \leq t,$$

and replacing W by tW we have,

$$\begin{aligned} \varphi(t) \mathbf{L}(tW) + \psi(t) \mathbb{I}_3 &= \sum_{\ell=1}^\nu G_\ell(tW, t)^2, \\ \text{for } |W| &\leq 1, t \in (0, 1). \end{aligned}$$

Multiplying by $\frac{1}{\varphi(t)t^2}$, and using that $\mathbf{L}$ is homogeneous of degree two, we obtain

$$\begin{aligned} \mathbf{L}(W) + \frac{\psi(t)}{\varphi(t)t^2} \mathbb{I}_3 &= \sum_{\ell=1}^\nu \left(\frac{G_\ell(tW, t)}{\sqrt{\varphi(t)t}} \right)^2, \\ \text{for } |W| &\leq 1, t \in (0, 1). \end{aligned}$$

Since $G_\ell \in C^{1, \omega}(B_{\mathbb{R}^4}(0, 1) \times (-1, 1))$, the functions $W \rightarrow G_\ell(W, t)$ lie in a bounded set in $C^{1, \omega}(B_{\mathbb{R}^3}(0, 1))$ independent of t and j , and hence also the collection of functions

$$H_\ell^t(W) \equiv G_\ell(tW, t), \quad 1 \leq \ell \leq \nu, t \in (0, 1),$$

is bounded in $C^{1, \omega}(B_{\mathbb{R}^4}(0, 1))$, say

$$(4.9) \quad \sum_{\ell=1}^\nu \|H_\ell^t\|_{C^{1, \omega}(B_{\mathbb{R}^3}(0, 1))} \leq \mathfrak{N}_\nu, \quad t \in (0, 1).$$

Thus with $\tau = \tau(t) \equiv \frac{\psi(t)}{\varphi(t)t^2}$, we have from (4.9) and (4.2) that

$$\begin{aligned} \frac{\mathfrak{N}_\nu}{\sqrt{\varphi(t)t^2}} &\geq \sum_{\ell=1}^\nu \left\| \frac{H_\ell^t}{\sqrt{\varphi(t)t^2}} \right\|_{C^{1, \omega}(B_{\mathbb{R}^4}(0, 1))} \geq \mathcal{C}_\nu^\omega(\tau(t)) \\ &\geq \left(\frac{\delta_\nu}{C} \right)^{\frac{1}{2-\beta}} \tau(t)^{-\frac{\beta}{4-2\beta}} = \left(\frac{\delta_\nu}{C} \right)^{\frac{1}{2-\beta}} \left(\frac{\psi(t)}{\varphi(t)t^2} \right)^{-\frac{\beta}{4-2\beta}}, \end{aligned}$$

and hence

$$\left(\frac{\delta_\nu}{C}\right)^{\frac{1}{2-\beta}} \leq \liminf_{t \rightarrow 0} \frac{\mathfrak{N}_\nu}{\sqrt{\varphi(t)} t^2} \left(\frac{\psi(t)}{\varphi(t) t^2}\right)^{\frac{\beta}{4-2\beta}} = \mathfrak{N}_\nu \liminf_{t \rightarrow 0} \left(\frac{\psi(t)}{\varphi(t)^{\frac{2}{\beta}} t^{\frac{4}{\beta}}}\right)^{\frac{\beta}{4-2\beta}},$$

contradicting (4.8) as required. This completes the proof of Lemma 39. ■

4.3.1. *Sharpness.* In this subsection we take

$$\psi(t) \leq C \varphi(t)^{\frac{2}{\beta}} t^{\frac{4}{\beta}}, \quad \text{for some } \beta < 1,$$

where both φ and ψ are nearly monotone on $(-1, 1)$. Then by Lemma 39, the matrix function $\mathbf{F}_{\varphi, \psi}$ as in (4.7) fails to be a finite sum of squares of $C^{1, \beta}$ vector fields. On the other hand, we now show that (2.7) holds with $\varepsilon < \frac{1}{4}$.

Lemma 40. *Let φ be nearly monotone on $(-1, 1)$. The off diagonal entries of $\mathbf{F}_{\varphi, \psi}$ satisfy (2.7) for some $\delta, \delta', \delta'' > 0$ if $0 < \varepsilon < \frac{1}{4}$.*

Proof. The three off diagonal entries of $\mathbf{F}_{\varphi, \psi_s}(x, y, z, t) = [a_{k,j}]_{k,j=1}^3$ are

$$\begin{aligned} a_{1,2}(x, y, z, t) &= xy\varphi(t), \\ a_{2,3}(x, y, z, t) &= yz\varphi(t), \\ a_{1,3}(x, y, z, t) &= zx\varphi(t), \end{aligned}$$

and for $|\mu| \leq 4$, we have

$$|D^\mu(xy\varphi(t))| \lesssim |W|^2 |D^\alpha \varphi(t)| + |W| |D^\beta \varphi(t)| + |D^\gamma \varphi(t)|,$$

where $|\alpha| \leq 4$, $|\beta| \leq 3$ and $|\gamma| \leq 2$. Since $\varphi(t)$ nearly monotone implies $|D^\nu \varphi(t)| \leq C_{\nu, \eta} \varphi(t)^{1-\eta}$ for any $\eta > 0$, we have

$$|D^\mu(xy\varphi(t))| \lesssim (1 + |W|^2) \varphi(t)^{1-\eta}, \quad \text{for } |\mu| \leq 4.$$

Now the diagonal entries are all comparable to $|W|^2 \varphi(t)$, and thus we obtain

$$|D^\mu(xy\varphi(t))| \lesssim (1 + |W|^2) \varphi(t)^{1-\eta} \lesssim (|W|^2 \varphi(t))^{\left[\frac{1}{2} + (2-|\mu|)\varepsilon\right]_+ + \delta''},$$

since

$$1 - \eta > \left[\frac{1}{2} + (2 - |\mu|)\varepsilon\right]_+ + \delta'' \quad \text{for } |\mu| \leq 4 \text{ and } \varepsilon < \frac{1}{4},$$

provided $\delta'' > 0$ is sufficiently small. ■

Lemma 41. *Let φ and ψ be nearly monotone on $(-1, 1)$. The diagonal entries of $\mathbf{F}_{\varphi, \psi} = [a_{k,j}]_{k,j=1}^3$ satisfy (2.6) for $\delta > 0$ and $\varepsilon > \frac{1}{4}$.*

Proof. We note that the diagonal entries are

$$\begin{aligned} a_{1,1}(x, y, z, t) &= \varphi(t)(x^2 + \lambda y^2 + 2z^2) + \psi(t) + \varphi(r)h\left(\frac{t}{r}\right), \\ a_{2,2}(x, y, z, t) &= \varphi(t)(y^2 + \lambda z^2 + 2x^2) + \psi(t) + \varphi(r)h\left(\frac{t}{r}\right), \\ a_{3,3}(x, y, z, t) &= \varphi(t)(z^2 + \lambda x^2 + 2y^2) + \psi(t) + \varphi(r)h\left(\frac{t}{r}\right), \end{aligned}$$

which are each comparable to

$$\varphi(t)|W|^2 + \psi(t) + \varphi(r)h\left(\frac{t}{r}\right).$$

Recall that for any $0 < \eta < 1$, we have $|D_t^\mu \varphi(t)| \leq C_\eta \varphi(t)^{1-\eta}$. Thus for $|\mu| = 1$ we have

$$\begin{aligned} |D^\mu a_{1,1}(x, y, z, t)| &\lesssim \varphi(t)^{1-\eta} |W|^2 + \varphi(t)|W| + |\psi'(t)| + |\varphi'(r)|h\left(\frac{t}{r}\right) + \varphi(r)\frac{1}{r}\left(1 + \frac{t}{r}\right) \\ &\lesssim \left(\varphi(t)r^2 + \psi(t) + \varphi(r)h\left(\frac{t}{r}\right)\right)^{1-\varepsilon+\delta'} \approx a_{1,1}(x, y, z, t)^{[1-|\mu|\varepsilon]_+ + \delta'} \end{aligned}$$

provided we choose η so that $1 - \eta > 1 - \varepsilon + \delta'$, and provided

$$(4.10) \quad \varphi(t)r \lesssim \left(\varphi(t)r^2 + \varphi(t)^2 t^4 + \varphi(r)h\left(\frac{t}{r}\right) \right)^{1-\varepsilon+\delta'}.$$

But this latter inequality holds since

(i) if $t \leq \frac{3}{4}r$, then

$$\varphi(t)r = \varphi(t)^{1-\varepsilon+\delta'} \varphi(t)^{\varepsilon-\delta'} r \lesssim \varphi(t)^{1-\varepsilon+\delta'} \varphi\left(\frac{3}{4}r\right)^{\varepsilon-\delta'} r \lesssim (\varphi(t)r^2)^{1-\varepsilon+\delta'},$$

since φ is flat at the origin; while

(ii) if $t > \frac{3}{4}r$, then

$$\begin{aligned} \left(\varphi(t)r^2 + \varphi(t)^2 t^4 \right)^{1-\varepsilon+\delta'} &\approx \varphi(t)^{1-\varepsilon+\delta'} r^{2-2\varepsilon+2\delta'} + \varphi(t)^{2-2\varepsilon+2\delta'} t^{4-4\varepsilon+4\delta'} \\ &\gtrsim \begin{cases} \varphi(t)r & \text{if } r \geq \varphi(t)^{\frac{\varepsilon-\delta'}{1-2\varepsilon+2\delta'}} \\ \varphi(t)r & \text{if } r < \varphi(t)^{\frac{\varepsilon-\delta'}{1-2\varepsilon+2\delta'}} \end{cases} \end{aligned}$$

since if $r < \varphi(t)^{\frac{\varepsilon-\delta'}{1-2\varepsilon+2\delta'}}$, then

$$\varphi(t)r < \varphi(t)^{1+\frac{\varepsilon-\delta'}{1-2\varepsilon+2\delta'}} = \varphi(t)^{\frac{1-\varepsilon+\delta'}{1-2\varepsilon+2\delta'}} \lesssim \varphi(t)^{2-2\varepsilon+2\delta'} t^{4-4\varepsilon+4\delta'}$$

since

$$\frac{1-\varepsilon+\delta'}{1-2\varepsilon+2\delta'} > 2-2\varepsilon+2\delta', \quad \text{i.e.}$$

for some $\delta' > 0$, i.e. $\frac{1-\varepsilon}{1-2\varepsilon} > 2-2\varepsilon$, i.e.

$$1-\varepsilon > (1-2\varepsilon)(2-2\varepsilon), \quad \text{i.e. } \frac{1}{4} < \varepsilon < 1.$$

The case $r \geq \varphi(t)^{\frac{\varepsilon-\delta'}{1-2\varepsilon+2\delta'}}$ is straightforward.

Now we turn to the case $|\mu| = 2$ where using that both φ and ψ are nearly monotone on $(-1, 1)$, and from Theorem 8, we have

$$\begin{aligned} |D^\mu a_{1,1}(x, y, z, t)| &\lesssim \varphi(t)^{1-\eta} |W|^2 + \varphi(t)^{1-\eta} |W| + \varphi(t) + |\psi'(t)| + |\psi''(t)| \\ &\quad + |\varphi''(r)| h\left(\frac{t}{r}\right) + \varphi'(r) \left| h'\left(\frac{t}{r}\right) \right| \frac{1}{r} \left(1 + \frac{t}{r}\right) \\ &\quad + \varphi(r) \left| h''\left(\frac{t}{r}\right) \right| \frac{1}{r^2} \left(1 + \frac{t}{r}\right)^2 \\ &\lesssim \left(\varphi(t)r^2 + \psi(t) + \varphi(r)h\left(\frac{t}{r}\right) \right)^{1-2\varepsilon+\delta'} \approx a_{1,1}(x, y, z, t)^{[1-|\mu|\varepsilon]_+ + \delta'} \end{aligned}$$

provided we choose η so that $1 - \eta > 1 - 2\varepsilon + \delta'$, and provided

$$(4.11) \quad \varphi(t)^{1-\eta} r \lesssim \left(\varphi(t)r^2 + \psi(t) + \varphi(r)h\left(\frac{t}{r}\right) \right)^{1-2\varepsilon+\delta'},$$

which of course holds for $\varepsilon > \frac{1}{4}$. Similar calculations hold for $|\mu| = 3, 4$. ■

We have thus demonstrated sharpness of Theorem 22 when $\varepsilon = \frac{1}{4}$ in the following sense. We do not know if similar sharpness holds for $\frac{1}{4} < \varepsilon < 1$.

Theorem 42. *Let $0 < \beta < 1$. The diagonally elliptic smooth flat matrix function $\mathbf{F}_{\varphi, \psi}$ constructed above satisfies the diagonal estimates (2.6) for all $\varepsilon > \frac{1}{4}$ and $\delta > 0$, and the off diagonal estimates (2.7) for all $\varepsilon < \frac{1}{4}$ and $\delta > 0$, yet is not SOS_{1, ω_β} , hence not SOS_{1, ω_1} . Of course, if for a diagonally elliptic smooth flat matrix function $\mathbf{F}$, both (2.6) and (2.7) hold for $\varepsilon = \frac{1}{4}$ and some $\delta > 0$, then Theorem 22 shows that $\mathbf{F}$ is SOS_{2, ω_δ} , hence SOS_{1, ω_1} .*

4.3.2. *A Grushin type subordinate matrix function, not a finite sum of squares plus a subordinate quasiconformal block.* It is not hard to modify the above example to obtain a Grushin type subordinate matrix function, with diagonal entries that are finite sums of squares of $C^{2,\delta}$ functions, and that cannot be decomposed as a finite sum of squares of vector fields plus a quasiconformal block². Consider first the 7×7 matrix function in block form,

$$\mathbf{M}(x, y, z, t, u, v, w, s) \equiv \begin{bmatrix} \mathbb{I}_4 & \mathbf{0}_{4 \times 3} \\ \mathbf{0}_{3 \times 4} & \mathbf{F}_{\varphi, \psi}(x, y, z, t) \end{bmatrix},$$

where $\mathbb{I}_4$ is the 4×4 identity matrix, and $\mathbf{0}_{m \times n}$ is the $m \times n$ zero matrix. Then if $f_{\varphi, \psi}(x, y, z, t) \equiv \text{trace } \mathbf{F}_{\varphi, \psi}(x, y, z, t)$, we have using $\mathbf{L}(x, y, z) \approx |(x, y, z)|^2 \mathbb{I}_3$ that

$$\mathbf{M}(x, y, z, t, u, v, w) \approx \begin{bmatrix} \mathbb{I}_4 & \mathbf{0}_{4 \times 3} \\ \mathbf{0}_{3 \times 4} & f_{\varphi, \psi}(x, y, z, t) \mathbb{I}_3 \end{bmatrix},$$

where $f_{\varphi, \psi}(x, y, z, t)$ is ω_s -monotone for all $0 < s < s_0$ by [KoSa1, Theorem 37], and hence is a finite sum of squares of $C^{2,\delta}$ functions by Theorem 4, yet $\mathbf{M}(x, y, z, t, u, v, w)$ is not a finite sum of squares of $C^{1,1}$ vector functions. This example shows in a striking way that additional conditions must be assumed on the off-diagonal entries of the matrix function $\mathbf{M}(x, y, z, t, u, v, w)$ in order for $\mathbf{M}$ to be a finite sum of squares of $C^{2,\delta}$ vector functions.

However, by Theorem 22, the matrix function $\mathbf{M}$ can be decomposed as a sum of squares plus the subordinate quasiconformal block $\mathbf{F}_{\varphi, \psi}(x, y, z, t)$, and we must work just a bit harder to prevent this. We consider instead the example

$$\mathbf{N}(x, y, z, t, u, v, w, s) \equiv \begin{bmatrix} \mathbf{M}(x, y, z, t, u, v, w) & \mathbf{0}_{7 \times 1} \\ \mathbf{0}_{1 \times 7} & \mathbf{G}(x, y, z, t) \end{bmatrix} \sim \begin{bmatrix} \mathbb{I}_4 & \mathbf{0}_{4 \times 3} & \mathbf{0}_{4 \times 1} \\ \mathbf{0}_{3 \times 4} & f_{\varphi, \psi}(x, y, z, t) \mathbb{I}_3 & \mathbf{0}_{3 \times 1} \\ \mathbf{0}_{1 \times 4} & \mathbf{0}_{1 \times 3} & g_{\varphi, \psi}(x, y, z, t) \mathbb{I}_3 \end{bmatrix},$$

where

$$\begin{aligned} \mathbf{F}(x, y, z, t) &= \varphi(t) \mathbf{L}(W) + \left\{ (\varphi(t) t^2)^4 + \varphi(r) h\left(\frac{t}{r}\right) \right\} \mathbb{I}_3, \\ \mathbf{G}(x, y, z, t) &= \rho(t) \mathbf{L}(W) + \left\{ (\rho(t) t^2)^4 + \rho(r) h\left(\frac{t}{r}\right) \right\} \mathbb{I}_3, \end{aligned}$$

are both examples of a 3×3 matrix function that cannot be written as a finite sum of squares of C^2 vector fields, and where $\varphi(t)$ and $\rho(t)$ incomparable.

Note that the hypotheses of Theorem 22 fail here since the final block is $\mathbf{G}(x, y, z, t)$, and Theorem 22 then requires $\mathbf{M}(x, y, z, t, u, v, w)$ to be a sum of squares of $C^{2,\delta}$ vector functions, which it is not since $\mathbf{F}$ is embedded in $\mathbf{M}$. The same observation holds even if we permute rows and columns of $\mathbf{N}$ and declare a final block of the permuted matrix to be the Grushin block. Indeed, the Grushin block will be comparable to $\lambda(x, y, z, t) \mathbb{I}_p$ where $\lambda(x, y, z, t) \in \{1, f_{\varphi, \psi}(x, y, z, t), g_{\varphi, \psi}(x, y, z, t)\}$, and then the remaining block will have either $\mathbf{F}$ or $\mathbf{G}$ embedd in it, hence cannot be a sum of squares.

In another direction, suppose that $f(x, y, z, w, t)$ and $g(x, y, z, w, t)$ are two elliptical flat smooth functions that cannot be written as a finite sum of squares of $C^{2,\delta}$ functions, such as can be found in [KoSa1]. Then a diagonally elliptical 7×7 matrix function $\mathbf{P}(x, y, z, w, t, u, v)$, whose diagonal elements $\{p_{1,1}, p_{2,2}, p_{3,3}, p_{4,4}, p_{5,5}, p_{6,6}, p_{7,7}\}$ are comparable to $\{1, 1, 1, 1, 1, f, g\}$, cannot be decomposed as a finite sum of squares of $C^{2,\delta}$ vector fields plus a subordinate quasiconformal block.

REFERENCES

- [Bon] J.-M. BONY, *Sommes de Carrés de fonctions dérivables*, Bull. Soc. math. France **133** (4), 2005, p. 619–639 .
- [BoBrCoPe] J.-M. BONY, F. BROGLIA, F. COLOMBINI AND L. PERNAZZA, *Nonnegative functions as squares or sums of squares*, Journal of Functional Analysis **232** (2006) p. 137 – 147.
- [Cho] M.-D. CHOI, *Positive semidefinite biquadratic forms*, Lin. Alg. and its Appl. **12** (1975), 95–100.
- [Chr] M. CHRIST, *Hypoellipticity in the infinitely degenerate regime*, Complex Analysis and Geometry, Ohio State Univ. Math. Res. Instl Publ. **9**, Walter de Gruyter, New York (2001), 59–84.
- [FePh] C. FEFFERMAN AND D. H. PHONG, *Subelliptic eigenvalue problems*, Conf. in Honor of A. Zygmund, Wadsworth Math. Series 1981.

²The hypoellipticity theorem in [KoSa3] doesn't apply to the operator $\mathbf{L} = \nabla^{\text{tr}} \mathbf{A}(x) \nabla$ with a matrix function $A(x)$ of this form.

- [Gua] P. GUAN, *C^2 a priori estimates for degenerate Monge-Ampère equations*, Duke Math. J. **86** (1997), 323-346.
- [HiNi] C. J. HILLAR AND J. NIE, *An elementary and constructive solution to Hilbert's 17th problem for matrices*, Proc. A.M.S. **136** (2008), 73-76.
- [Hor] L. HÖRMANDER, *Hypoelliptic second order differential equations*, Acta. Math. **119** (1967), 141-171.
- [KoSa1] L. KOROBENKO AND E. SAWYER, *Sums of squares of scalar $C^{2,\delta}$ functions*, arXiv:2107.12840.
- [KoSa3] L. KOROBENKO AND E. SAWYER, *Hypoellipticity via sums of squares in the infinitely degenerate regime*, arXiv:2107.12451.
- [MaSaUrVu] M. J. MARTÍN, E. T. SAWYER, I. URIARTE-TUERO AND D. VUKOTIĆ, *The Krzyż conjecture revisited*, Advances in Math. **273** (2015), 716-745.
- [RiSaWh] C. RIOS, E. T. SAWYER AND R. L. WHEEDEN, *Regularity of subelliptic Monge-Ampère equations*, Adv. Math. **217** (2008), no. 3, 967-1026.
- [RoSt] L. P. ROTHSCHILD AND E. M. STEIN, *Hypoelliptic differential operators and nilpotent groups*, Acta. Math. **137** (1976), 419-440.
- [Tat] D. TATARU, *On the Fefferman-Phong inequality and related problems*, preprint.

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