

SUMS OF SQUARES III: HYPOELLIPTICITY IN THE INFINITELY DEGENERATE REGIME

LYUDMILA KOROBENKO AND ERIC SAWYER

ABSTRACT. This is the third paper in a series of three dealing with sums of squares and hypoellipticity in the infinitely degenerate regime. We establish a $C^{2,\delta}$ generalization of M. Christ's smooth sum of squares theorem, and then use a bootstrap argument with the sum of squares decomposition for matrix functions, obtained in our second paper of this series, to prove a hypoellipticity theorem that generalizes some cases of the results of Christ, Hoshiro, Koike, Kusuoka and Stroock and Morimoto for sums of squares, and of Fedĭi and Kohn for degeneracies not necessarily a sum of squares.

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1. INTRODUCTION

The regularity theory of second order *subelliptic* linear equations with smooth coefficients is well established, see e.g. Hörmander [Ho] and Fefferman and Phong [FePh]. In [Ho], Hörmander obtained hypoellipticity of sums of squares of smooth vector fields plus a lower order term, whose Lie algebra spans at every point. In [FePh], Fefferman and Phong considered general nonnegative semidefinite smooth self-adjoint linear operators, and characterized subellipticity in terms of a containment condition involving Euclidean balls and "subunit" balls related to the geometry of the nonnegative semidefinite form associated to the operator. Of course subelliptic operators L with smooth coefficients are hypoelliptic, namely every distribution solution u of $Lu = \phi$ is smooth when ϕ is smooth.

However, the question of hypoellipticity in general remains largely a mystery. A necessary condition is that the principal symbol $\sigma(x, \xi)$ of L cannot vanish at any point x , but this is far from sufficient. A possible form for a characterization involving the effective symbol $\tilde{\sigma}(x, \xi)$ (when it exists) is given by Christ in [Chr2], motivated by his main hypoellipticity theorem for sums of squares in the infinitely degenerate regime in [Chr,

2020 *Mathematics Subject Classification.* 35B65, 35J70, 35H10.

Key words and phrases. infinite degeneracy, elliptic operators, hypoellipticity, sums of squares, vector fields.

The second author is partially supported by NSERC grant number 12409 and the McKay Research Chair grant at McMaster University.

see Main Theorem 2.3]. We will generalize this latter theorem of Christ to hold for $C^{2,\delta}$ symbols, which will play a major role in Theorem 9 below on hypoellipticity in the infinitely degenerate regime.

Thus a basic obstacle to understanding hypoellipticity in general arises when ellipticity degenerates to infinite order in some directions, and we briefly review what is known in this infinite regime here. The theory has only had its surface scratched so far, as evidenced by the results of Fedii [Fe], Kusuoka and Strook [KuStr], Kohn [Koh], Koike [Koi], Korobenko and Rios [KoRi], Morimoto [Mor], Akhunov, Korobenko and Rios [AkKoRi], and the aforementioned paper of Christ [Chr], to name just a few. In the *rough* infinitely differentiable regime, Rios, Sawyer and Wheeden had earlier obtained results analogous to those in [KoRi], where L is (rough) hypoelliptic if every *weak* solution u of $Lu = \phi$ is continuous when ϕ is bounded.

In [Fe], Fedii proved that the two-dimensional operator $\frac{\partial}{\partial x^2} + f(x)^2 \frac{\partial}{\partial y^2}$ is hypoelliptic merely under the assumption that f is smooth and positive away from $x = 0$. In [KuStr], Kusuoka and Strook showed using probabilistic methods that under the same conditions on $f(x)$, the three-dimensional analogue $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + f(x)^2 \frac{\partial^2}{\partial z^2}$ of Fedii's operator is hypoelliptic *if and only if*

$$\lim_{x \rightarrow 0} x \ln f(x) = 0.$$

Morimoto [Mor] and Koike [Koi] introduced the use of nonprobabilistic methods, and further refinements of this approach were obtained in Christ [Chr], using a general theorem on hypoellipticity of sums of squares of smooth vector fields in the infinite regime, i.e. where the Lie algebra does *not* span at all points. In particular, for the operator $L_3 = \frac{\partial^2}{\partial x^2} + a^2(x) \frac{\partial^2}{\partial y^2} + b^2(x) \frac{\partial^2}{\partial z^2}$ in $\mathbb{R}^3$, Christ proved that if $a, b \in C^\infty$ are even, elliptic, nondecreasing on $[0, \infty)$, and $a(x) \geq b(x)$ for all x , and if in addition $\limsup_{x \rightarrow 0} |x \ln a(x)| \neq 0$, and the coefficient b satisfies

$$\lim_{x \rightarrow 0} b(x)x |\ln a(x)| = 0,$$

then L_3 is hypoelliptic. Moreover, he showed that if some partial derivative of b is nonzero at $x = 0$, then L_3 is hypoelliptic *if and only if* the above condition holds.

On the other hand, the novelty in Kohn [Koh], which was generalized in [KoRi], was the absence of any assumption regarding sums of squares of vector fields. This is relevant since it is an open problem whether or not there are smooth nonnegative functions λ on the real line vanishing only at the origin, and to infinite order there, such that they **cannot** be written as a finite sum $\lambda = \sum_{n=1}^N f_n^2$ of squares of smooth functions f_n . The existence of such examples are attributed to Paul Cohen in both [Bru] and [BoCoRo], but apparently no example has ever appeared in the literature, and the existence of such an example is an open problem, see [Pie, Remark 5.1]¹. This extends moreover to matrices since if a matrix is a sum of squares (equivalently a sum of positive rank one matrices), then each of its diagonal elements is as well. On the other hand, Kohn makes the additional assumption that $\lambda(x)$ vanishes only at the origin in $\mathbb{R}^m$, something not necessarily assumed in the other aforementioned works. More importantly, Kohn's theorem applies only to operators of Grushin type $L(x, D) + \lambda(x) L(y, D)$, where the degeneracy $\lambda(x)$ factors out of the operator $\lambda(x) L(y, D)$, a restriction that this paper will in part work to remove.

Missing then is a treatment of more general smooth operators $L = \nabla A(x) \nabla +$ lower order terms, whose matrix $A(x)$ is *comparable* to an operator in diagonal form of the types considered above - see Definition 1 below. Our purpose in this paper is to address this more general case in the following setting of real-valued differential operators. Suppose $1 \leq m < p \leq n$. Let $L = \nabla A(x) \nabla$ where $A(x) \sim D_\lambda(\tilde{x})$ with $\tilde{x} = (x_1, \dots, x_m)$, $x = (x_1, \dots, x_n)$ and where $D_\lambda(\tilde{x})$ has C^2 nonnegative diagonal entries $\lambda_1(\tilde{x}), \dots, \lambda_n(\tilde{x})$ depending only on $\tilde{x}$ and positive away from the origin in $\mathbb{R}^m$:

$$A(x) \sim D_\lambda(\tilde{x}) = \begin{bmatrix} \mathbb{I}_m & \mathbf{0}_{m \times (p-m-1)} & \mathbf{0}_{m \times (n-p+1)} \\ \mathbf{0}_{(p-m-1) \times m} & D_{\{\lambda_{m+1}(\tilde{x}), \dots, \lambda_{p-1}(\tilde{x})\}} & \mathbf{0}_{(p-m-1) \times (n-p+1)} \\ \mathbf{0}_{(n-p+1) \times m} & \mathbf{0}_{(n-p+1) \times (p-m-1)} & \lambda_p(\tilde{x}) \mathbb{I}_{n-p+1} \end{bmatrix}.$$

We will refer to a diagonal matrix having this form for any $m < p \leq n$ as a *Grushin matrix function of type m* . Note that the comparability $A(x) \sim D_\lambda(\tilde{x})$ implies that $a_{k,k}(x) \approx \lambda_k(\tilde{x})$ for all the diagonal entries, so that $\lambda_k(\tilde{x}) \approx a_{k,k}(\tilde{x}, 0)$ may be assumed smooth without loss of generality. Moreover $A(x) \sim A_{\text{diag}}(\tilde{x}, 0)$ (see [KoSa2, after Definition 10]).

¹See also <https://mathoverflow.net/a/106072>

All of our theorems will apply to operators L having a Grushin matrix function $A(x)$ of type m that is also *elliptical* in the sense that $A(x)$ is positive definite for $x \neq 0$. Moreover, we will require in addition that the intermediate diagonal entries $\{a_{k,k}(\tilde{x})\}_{k=m+1}^{p-1}$ (there won't be any such entries in the case $p = m + 1$) are smooth and *strongly* $C^{4,2\delta}$ (see [KoSa1]) for some $\delta > 0$ (we show in [KoSa2] that such functions can be written as a sum of squares of $C^{2,\delta}$ functions, and moreover give a sharp ω -monotonicity criterion for strongly $C^{4,2\delta}$), and that the off diagonal entries of $A(x)$ satisfy certain strongly subordinate inequalities. We emphasize that no additional assumptions are made on the last $n - p + 1$ entries of $D(\tilde{x})$, which are all equal to $\lambda_p(\tilde{x})$.

Our approach is broadly divided into four separate steps, the first and second of which are the subject of the first two papers in this series:

- (1) First, a proof that a $C^{3,1}$ function can be written as a finite sum of squares of $C^{1,1}$ functions first appeared in Guan [Gua], who attributed the result to Fefferman. In [KoSa1] we adapted treatments of this result from Tataru [Tat] and Bony [Bon] to establish conditions under which a $C^{4,2\delta}$ nonnegative function can be written as a finite sum of squares of $C^{2,\delta}$ functions for some $\delta > 0$. The methods of Tataru and Bony were in turn modelled on a localized splitting of a nonnegative symbol a , due to Fefferman and Phong [FePh], who used it to establish a strong form of Gårding's inequality, and is the main idea behind the result of Fefferman appearing in [Gua]. That splitting used the implicit function theorem to write a nonnegative symbol a as a sum of squares plus a symbol depending on fewer variables, so that induction could be applied. This same scheme was used in [KoSa1] to obtain a sum of squares of $C^{2,\delta}$ functions, but taking care to arrange assumptions so that the implicit function theorem applied.
- (2) Second, in [KoSa2], we showed that under analogous conditions on the diagonal entries of a matrix-valued function M , and strong subordinate-type inequalities on the off diagonal entries, M can then be written as a finite sum of squares of $C^{2,\delta}$ vector fields for some $\delta > 0$.
- (3) Third, we here extend a theorem of M. Christ on hypoellipticity of sums of smooth squares of vector fields to the setting of $C^{2,\delta}$ vector fields, with the appropriate notion of gain in a range of Sobolev spaces.
- (4) Fourth, we here adapt arguments of M. Christ together with the above steps to obtain hypoellipticity of linear operators L of the form

$$(1.1) \quad L = \nabla^{\text{tr}} A(x) \nabla + D(x),$$

where the matrix A and scalar D are smooth functions of $x \in \mathbb{R}^n$, and with $\tilde{x} = (x_1, \dots, x_m)$, we have

$$(1.2) \quad A(x) \sim \begin{bmatrix} \mathbb{I}_m & 0 \\ 0 & D_{\lambda}(\tilde{x}) \end{bmatrix},$$

where $\mathbb{I}_m$ is the $m \times m$ identity matrix, and $D_{\lambda}(\tilde{x})$ is the $(n - m) \times (n - m)$ diagonal matrix with the components of $\lambda(\tilde{x}) = (\lambda_{m+1}(\tilde{x}), \dots, \lambda_n(\tilde{x}))$ along the diagonal. The component functions $\lambda_{\ell}(\tilde{x})$ satisfy certain natural conditions described explicitly below.

We will end this section by stating our main results on hypoellipticity. Then in the next section, we use a result on calculus of rough symbols from the 1980's [Saw] to derive a rough version of M. Christ's hypoellipticity theorem for sums of smooth vector fields in the infinitely degenerate regime, where symbol splitting is inadequate. Finally in the last sections, we use a bootstrap argument that exploits the $C^{2,\delta}$ regularity of the vector fields, to bring all of these results to bear on proving hypoellipticity for linear partial differential operators L of the form (1.1).

But first we recall the main results from the second paper in this series [KoSa2] on sums of squares of matrix functions that we will use here.

Definition 1. Let A and B be real symmetric positive semidefinite $n \times n$ matrices. We define $A \preceq B$ if $B - A$ is positive semidefinite. Let $\beta < \alpha$ be positive constants. A real symmetric positive semidefinite $n \times n$ matrix A is said to be (β, α) -comparable to a symmetric $n \times n$ matrix B , written $A \sim_{\beta, \alpha} B$, if $\beta B \preceq A \preceq \alpha B$, i.e.

$$(1.3) \quad \beta \xi^{\text{tr}} B \xi \leq \xi^{\text{tr}} A \xi \leq \alpha \xi^{\text{tr}} B \xi, \quad \text{for all } \xi \in \mathbb{R}^n.$$

We say A is comparable to B , written $A \sim B$, if $A \sim_{\beta, \alpha} B$ for some $0 < \beta < \alpha < \infty$.

Note that if A is *comparable* to B , then both A and B are positive semidefinite. Indeed, both $0 \leq (\alpha - \beta) \xi^{\text{tr}} B \xi$ and $0 \leq \left(\frac{1}{\beta} - \frac{1}{\alpha}\right) \xi^{\text{tr}} A \xi$ hold for all $\xi \in \mathbb{R}^n$.

Definition 2. A matrix function $\mathbf{A}(x)$ is subordinate if $\left| \frac{\partial \mathbf{A}}{\partial x_k}(x) \cdot \xi \right|^2 \leq C \xi^{\text{tr}} A(x) \xi$ for all $\xi \in \mathbb{R}^n$, equivalently $\frac{\partial \mathbf{A}}{\partial x_k}(x)^{\text{tr}} \frac{\partial \mathbf{A}}{\partial x_k}(x) \preceq C \mathbf{A}(x)$.

Finally recall the following seminorm from [Bon],

$$(1.4) \quad [h]_{\alpha, \delta}(x) \equiv \limsup_{y, z \rightarrow x} \frac{|D^\alpha h(y) - D^\alpha h(z)|}{|y - z|^\delta}.$$

Here is the sum of squares decomposition with a quasiformal block of order $(n - p + 1) \times (n - p + 1)$, where $1 < p \leq n$. We say that a symmetric matrix function $\mathbf{Q}_p(x)$ is quasiconformal if the eigenvalues $\lambda_i(x)$ of $\mathbf{Q}_p(x)$ are nonnegative and comparable.

Theorem 3. Let

$$1 < p \leq n, \quad \frac{1}{4} \leq \varepsilon < 1, \quad 0 < \delta < \delta'' < 1, \quad M \geq 1,$$

and

$$\delta' = \frac{2\delta(1 + \delta)}{2 + \delta}.$$

Suppose that $\mathbf{A}(x)$ is a $C^{4, 2\delta}$ symmetric $n \times n$ matrix function of a variable $x \in \mathbb{R}^M$, which is comparable to a diagonal matrix function $\mathbf{D}(x)$, hence comparable to its associated diagonal matrix function $\mathbf{A}_{\text{diag}}(x)$.

- (1) Moreover, assume $a_{p,p}(x) \approx a_{p+1,p+1}(x) \approx \dots \approx a_{n,n}(x)$ and that the diagonal entries $a_{1,1}(x), \dots, a_{p-1,p-1}(x)$ satisfy the following differential estimates up to fourth order,

$$(1.5) \quad \begin{aligned} |D^\mu a_{k,k}(x)| &\lesssim a_{k,k}(x)^{[1-|\mu|\varepsilon]_+ + \delta'}, \quad 1 \leq |\mu| \leq 4 \text{ and } 1 \leq k \leq p-1, \\ [a_{k,k}]_{\mu, 2\delta}(x) &\lesssim 1, \quad |\mu| = 4 \text{ and } 1 \leq k \leq p-1. \end{aligned}$$

- (2) Furthermore, assume the off diagonal entries $a_{k,j}(x)$ satisfy the following differential estimates up to fourth order,

$$(1.6) \quad \begin{aligned} |D^\mu a_{k,j}| &\lesssim \left(\min_{1 \leq s \leq j} a_{s,s} \right)^{\left[\frac{1}{2} + (2-|\mu|\varepsilon)_+ + \delta''\right]}, \quad 0 \leq |\mu| \leq 4 \text{ and } 1 \leq k < j \leq p-1, \\ [a_{k,j}]_{\mu, 2\delta} &\lesssim 1, \quad |\mu| = 4 \text{ and } 1 \leq k < j \leq p-1, \\ |D^\mu a_{k,j}| &\lesssim \left(\min_{1 \leq s \leq k} a_{s,s} \right)^{\left[\frac{1}{2} + (2-|\mu|\varepsilon)_+ + \delta''\right]}, \quad 0 \leq |\mu| \leq 4 \text{ and } 1 \leq k \leq p-1 < j \leq n, \\ [a_{k,j}]_{\mu, 2\delta} &\lesssim 1, \quad |\mu| = 4 \text{ and } 1 \leq k \leq p-1 < j \leq n. \end{aligned}$$

- (3) Then there is a positive integer $I \in \mathbb{N}$ such that the matrix function $\mathbf{A}$ can be written as a finite sum of squares of $C^{2,\delta}$ vectors $X_{k,j}$, plus a matrix function $\mathbf{A}_p$,

$$\mathbf{A}(x) = \sum_{k=1}^{p-1} \sum_{i=1}^I X_{k,j}(x) X_{k,j}(x)^{\text{tr}} + \mathbf{A}_p(x), \quad x \in \mathbb{R}^M,$$

where the vectors $X_{k,i}(x)$, $1 \leq k \leq p-1$, $1 \leq i \leq I$ are $C^{2,\delta}(\mathbb{R}^M)$, $\mathbf{A}_p(x) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_p(x) \end{bmatrix}$, and $\mathbf{Q}_p(x) \in C^{4, 2\delta}(\mathbb{R}^M)$ is quasiconformal. Moreover, $Z_k \equiv \sum_{i=1}^I X_{k,i} X_{k,i}^{\text{tr}} \in C^{4, 2\delta}(\mathbb{R}^M)$ and

$$(1.7) \quad \begin{aligned} ca_{k,k} \mathbf{e}_k \otimes \mathbf{e}_k &\prec Z_k Z_k^{\text{tr}} + \sum_{m=k+1}^n a_{m,m} \mathbf{e}_m \otimes \mathbf{e}_m \prec C \sum_{m=k}^n a_{m,m} \mathbf{e}_m \otimes \mathbf{e}_m, \quad 1 \leq k \leq p-1, \\ \mathbf{Q}_p(x) &\sim a_{p,p}(x) \mathbb{I}_{n-p+1}. \end{aligned}$$

Finally, if in addition $\mathbf{A}(x)$ is subordinate, then $\mathbf{Q}_p(x)$ is also subordinate².

²A more general assumption is that of *semisubordinaticity*, namely $\frac{\partial \mathbf{A}}{\partial x_k} = \mathbf{S}_k^1 + (\mathbf{S}_k^2)^{\text{tr}}$ where $[\mathbf{S}_k^j]^{\text{tr}} \mathbf{S}_k^j \preceq C \mathbf{A}$ for $j = 1, 2$ and $k = 1, 2, \dots, n$, whose importance arises from the fact that semisubordinaticity of $\mathbf{Q}_p$ can be used in place of subordinaticity

Remark 4. If in addition $a_{k,k}(x) \approx 1$ for $1 \leq k \leq m < p$, then the conditions (1.5) and (1.6) in (1) and (2) are vacuous for $1 \leq k \leq m$, and moreover the proof shows that the vectors $X_{k,i}$ are actually in $C^{4,2\delta}(\mathbb{R}^M)$ for $1 \leq k \leq m, 1 \leq i \leq I$.

These remarks yield the following corollary in which conditions (1.5) and (1.6) in (1) and (2) play no role.

Corollary 5. Suppose $\mathbf{A}(x)$ is a $C^{4,\delta}(\mathbb{R}^M)$ symmetric $n \times n$ matrix function that is comparable to a diagonal matrix function. In addition suppose that $a_{k,k}(x) \approx 1$ for $1 \leq k \leq p-1$ and $a_{k,k}(x) \approx a_{p,p}(x)$ for $p \leq k \leq n$. Then

$$\mathbf{A}(x) = \sum_{k=1}^{p-1} X_k(x) X_k(x)^{\text{tr}} + \mathbf{Q}_p(x), \quad x \in \mathbb{R}^M,$$

where $X_k, \mathbf{Q}_p \in C^{4,\delta}(\mathbb{R}^M)$ and (1.7) holds for $1 \leq k \leq p-1$.

Remark 6. If the diagonal entry $a_{k,k}(x)$ is smooth and ω_s -montone on $\mathbb{R}^n$ for some $s > 1 - \varepsilon$, then the diagonal differential estimates (1.5) above hold for $a_{k,k}(x)$ since $|D^\mu a_{k,k}(x)| \leq C_{s,s'} a_{k,k}(x)^{s'}$ for any $s' < s$ ([KoSa2, Theorem 18]).

Remark 7. If in Theorem 3, we drop the hypothesis (1.5) that the diagonal entries satisfy the differential estimates, and even slightly weaken the off diagonal hypotheses (1.6), then using the Fefferman-Phong theorem for sums of squares of scalar functions, the proof of Theorem 3 shows that the operator $L = \nabla^{\text{tr}} \mathbf{A} \nabla$ can be written as $L = \sum_{j=1}^N X_j^{\text{tr}} X_j$ where the vector fields X_j are $C^{1,1}$ for $j = 1, 2, \dots, N$. However, unlike the situation for scalar functions, the example in Theorem 38 of [KoSa2] shows that we cannot dispense entirely with the off diagonal hypotheses (1.6) in (2). Moreover, the space $C^{1,1}$ seems not to be sufficient for gaining a positive degree δ of smoothness for solutions to a second order operator, and so this result will neither be used nor proved here.

In this paper we will apply the sums of squares representations for matrix functions obtained in [KoSa2] to a rough generalization of a theorem of M. Christ, that then leads to our main hypoellipticity theorem via a bootstrap argument.

2. STATEMENT OF MAIN HYPOELLIPTICITY THEOREMS

We begin with the following general hypoellipticity theorem in the infinitely degenerate regime as in Step (4) of the introduction. We emphasize that we make no assumptions regarding the order of vanishing of the matrix function $A(x)$ at the origin. Since we only consider degeneracies at the origin, it is useful to make the following definition.

Definition 8. We say that a $q \times q$ matrix function $f : \mathbb{R}^n \rightarrow \mathbb{R}^{q^2}$ on $\mathbb{R}^n$ is elliptical if $f(x)$ is positive definite for $x \neq 0$. A scalar function f corresponds to the case $q = 1$.

Theorem 9. Suppose $1 \leq m < p \leq n$. Let L be a second order real self-adjoint divergence form partial differential operator in $\mathbb{R}^n$ given by

$$(2.1) \quad L = \nabla^{\text{tr}} A(x) \nabla + D(x),$$

where the matrix A and scalar D are smooth real functions of $x \in \mathbb{R}^n$, and $A(x)$ is subordinate, i.e. $\frac{\partial A}{\partial x_k} \nabla$ is subunit with respect to $\nabla^{\text{tr}} A(x) \nabla$.

(1) Suppose further that with $\tilde{x} = (x_1, \dots, x_m)$ we have the following Grushin assumption,

$$(2.2) \quad A(x) \sim \begin{bmatrix} \mathbb{I}_m & 0 \\ 0 & D_\lambda(\tilde{x}) \end{bmatrix},$$

of $\mathbf{Q}_p$ in the proof of Theorem 9 below. However, the semisubordinate condition is much harder to pass through the 1-SD in [KoSa2] than is the subordinate condition, and it is ultimately as difficult to deal with as the sum of squares decomposition itself.

where $\mathbb{I}_m$ is the $m \times m$ identity matrix, and $D_{\lambda}(\tilde{x})$ is the $(n-m) \times (n-m)$ diagonal matrix with the components of $\lambda(\tilde{x}) = (\lambda_{m+1}(\tilde{x}), \dots, \lambda_n(\tilde{x}))$ along the diagonal, i.e.

$$(2.3) \quad D_{\lambda}(\xi) = \begin{bmatrix} \lambda_{m+1}(\tilde{x}) & 0 & \cdots & 0 \\ 0 & \lambda_{m+2}(\tilde{x}) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n(\tilde{x}) \end{bmatrix}.$$

- (a) Moreover, we suppose that the component functions λ_{ℓ} are elliptical in $\mathbb{R}^m$, and $\lambda_p(\tilde{x}) \approx \lambda_{p+1}(\tilde{x}) \approx \cdots \approx \lambda_n(\tilde{x})$.
- (b) We also suppose that there are positive numbers $0 < \delta < \delta'' < \frac{1}{2}$, $\frac{1}{4} \leq \varepsilon < 1$, such that for $\delta' = \frac{2\delta(1+\delta)}{2+\delta}$ and for $k < j \leq n$ and $1 \leq k \leq p-1$, the entries $a_{k,j}(x)$ of $A(x)$ satisfy the differential size inequalities³ in (1.5) and (1.6) for all $x \in \mathbb{R}^n$.
- (2) Then L is hypoelliptic if

$$(2.4) \quad \lim_{\tilde{x} \rightarrow 0} \mu(|\tilde{x}|, \sqrt{\max\{\lambda_{m+1}, \dots, \lambda_p\}(\tilde{x})}) \ln \min\{\lambda_{m+1}, \dots, \lambda_p\}(\tilde{x}) = 0,$$

where

$$\mu(t, g) \equiv \max\{g(z)(t - |z|) : 0 \leq |z| \leq t\}.$$

Moreover, condition (2.4) is necessary for hypoellipticity if in addition $A(x)$ is a diagonal matrix with monotone entries.

Remark 10. Note that when $m = 1$, it suffices to assume only smoothness of the diagonal entries $\lambda_{\ell}(\tilde{x})$ in place of (1.5), in view of Bony's sum of squares theorem [Bon, Théorème 1].

Here is a variation, without any special hypotheses on the diagonal entries, that will be used to prove Theorem 9 in conjunction with the sum of squares decomposition in Theorem 3. However, the proof of this next result will require a generalization of M. Christ's sum of squares theorem to include $C^{2,\delta}$ vector fields.

Theorem 11. Let L be a real second order divergence form partial differential operator in $\mathbb{R}^n$ satisfying (2.1). Let $1 \leq m < p \leq n+1$, and write

$$x = (x_1, \dots, x_m, x_{m+1}, \dots, x_{p-1}, x_p, \dots, x_n) = (\tilde{x}, \tilde{x}, \hat{x}) \in \mathbb{R}^m \times \mathbb{R}^{p-m-1} \times \mathbb{R}^{n-p+1},$$

where the middle factor $\mathbb{R}^{p-m-1}$ vanishes if $p = m+1$, and the final factor vanishes if $p = n+1$.

- (1) Suppose that there exist $C^{2,\delta}$ vector fields $X_j(x) \in \text{Op}(C^{2,\delta}S_{1,0}^1)$ for $1 \leq j \leq N$, and an $(n-p+1) \times (n-p+1)$ matrix function $\mathbf{Q}_p(x) \in C^{4,2\delta}$ that is elliptical, quasiconformal and subordinate, such that

$$L = \left(\sum_{j=1}^N X_j^{\text{tr}} X_j + \hat{\nabla}^{\text{tr}} \mathbf{Q}_p(x) \hat{\nabla} \right) + \sum_{j=1}^N A_j X_j + \sum_{j=1}^N X_j^{\text{tr}} \tilde{A}_j + A_0,$$

where $\hat{\nabla} = (\partial_{x_p}, \dots, \partial_{x_n})$ and $A_j, \tilde{A}_j \in \text{Op}(C^{1,\delta}S_{1,0}^0)$, $A_0 \in \mathcal{O}_{(-\delta/2, \delta/2)}^{-\delta/2+\varepsilon}$ for all $\varepsilon > 0$.

- (2) Suppose further that there are elliptical scalar functions $\lambda_{m+1}(\tilde{x}), \dots, \lambda_p(\tilde{x}) \in C^2(\mathbb{R}^n)$ with $0 \leq \lambda_j \leq 1$ for all j , such that $\mathbf{Q}_p(x) \sim \lambda_p(\tilde{x}) \mathbb{I}_{n-p+1}$ and such that the following inequalities hold for all Lipschitz functions v :

$$(2.5) \quad \begin{aligned} \sum_{k=1}^m |\partial_{x_k} v|^2 + \sum_{k=m+1}^{p-1} \lambda_k(\tilde{x}) |\partial_{x_k} v|^2 &\lesssim \sum_{j=1}^N |X_j v|^2 + \lambda_p(\tilde{x}) \sum_{k=p}^n |\partial_{x_k} v|^2, \\ \sum_{j=1}^N |X_j v|^2 &\lesssim \sum_{k=1}^m |\partial_{x_k} v|^2 + \sum_{k=m+1}^{p-1} \lambda_k(\tilde{x}) |\partial_{x_k} v|^2 + \lambda_p(\tilde{x}) \sum_{k=p}^n |\partial_{x_k} v|^2 \end{aligned}$$

³The diagonal inequalities become more demanding the smaller ε is, while the off diagonal inequalities become less demanding.

(3) Finally set

$$\Lambda_{\text{sum}}(\tilde{x}) \equiv \sum_{k=m+1}^p \lambda_k(\tilde{x}) \text{ and } \Lambda_{\text{product}}(\tilde{x}) \equiv \prod_{k=m+1}^p \lambda_k(\tilde{x}),$$

and define the Koike functional $\mu(t, g)$ for any function $g(\tilde{x})$ by

$$(2.6) \quad \mu(t, g) \equiv \max\{g(\tilde{x})(t - |\tilde{x}|) : 0 \leq |\tilde{x}| \leq t\}.$$

(4) Then the operator L is hypoelliptic if

$$(2.7) \quad \lim_{x \rightarrow 0} \mu(|\tilde{x}|, \sqrt{\Lambda_{\text{sum}}}) \ln \Lambda_{\text{product}}(\tilde{x}) = 0.$$

This is sharp in the sense that (2.7) holds if L is both hypoelliptic and diagonal with monotone entries.

Here is our rough version, in the setting of sums of squares of real vector fields, of M. Christ's hypoellipticity theorem as needed in Step (3) of the introduction. Note in particular that the vector fields X_j appearing below are only assumed to be $C^{2,\delta}$, while the sum of their squares $\sum_j X_j^{\text{tr}} X_j$ is assumed to be smooth.

Theorem 12. Suppose $1 \leq p \leq n$ and $N \geq 1$. Let $R \subset T^*V$, the cotangent bundle of an open set $V \subset \mathbb{R}^n$, be any ray, and assume that the operator L has the form

$$(2.8) \quad L = \sum_{j=1}^N X_j^{\text{tr}} X_j + \sum_{j=1}^N A_j X_j + \sum_{j=1}^N X_j^{\text{tr}} \tilde{A}_j + R_1 + A_0 + \widehat{\nabla}^{\text{tr}} \cdot \mathbf{Q}_p(x) \widehat{\nabla},$$

where the vector fields X_j , $j = 1, 2, \dots, N$ are $C^{2,\delta}(\mathbb{R}^n)$ differential operators, and $\mathbf{Q}_p(x)$ is a $C^{4,2\delta}(\mathbb{R}^m)$ $(n-p+1) \times (n-p+1)$ matrix that is subordinate and quasiconformal, and $\widehat{\nabla} = (\partial_{x_p}, \dots, \partial_{x_n})$.

(1) Assume further that $\mathbf{Q}_p = \mathbf{Q}_p(x) \approx a(x) \mathbb{I}_{n-p+1}$ with $a \in C^{4,2\delta}(\mathbb{R}^n)$ elliptical, $L \in \text{Op}(S_{1,0}^2)$, $X_j \in \text{Op}(C^{2,\delta} S_{1,0}^1)$ and $A_j, \tilde{A}_j \in \text{Op}(C^{1,\delta} S_{1,0}^0)$, $A_0 \in \mathcal{O}_{(-\delta/2, \delta/2)}^{-\delta/2+\varepsilon}$ for all $\varepsilon > 0$, in some conic neighbourhood V of R .

(a) In addition, assume $R_1 = \sum_{k=1}^n S_k \Theta_k \circ \widehat{\nabla}$, where each $S_k \in C^{1,\delta}(\mathbb{R}^{m \times m})$ is subunit with respect to $\mathbf{Q}_p$, and $\Theta_k = (\Theta_{kp}, \dots, \Theta_{kn})$ is a multiplier of order zero.

(b) Suppose there exists $w \in C^\infty$ satisfying $w(\xi) \rightarrow \infty$ as $|\xi| \rightarrow \infty$ such that

$$(2.9) \quad \int_{\mathbb{R}^d} w^2(\xi) |\dot{u}(\xi)|^2 d\xi \leq C \sum_j \|X_j u\|^2 + C \|\sqrt{a} \widehat{\nabla} u\|^2 + C \|u\|^2 \quad \forall u \in C_0^1(V),$$

(c) Finally, suppose that for each small conic neighborhood Γ of R there exist scalar valued symbols $\psi, p \in S_{1,0}^0$ such that ψ is everywhere nonnegative, ψ does not depend on ξ in Γ , $\psi \equiv 0$ in some smaller conic neighborhood of R , $\psi \geq 1$ on $T^*V \setminus \Gamma$, $p \equiv 0$ in a conic neighborhood of the closure of Γ , and such that for each $\delta > 0$ there exists $C_\delta < \infty$ such that for any relatively compact open subset $U \Subset V$ and for all $u \in C_0^2(U)$ and each index i ,

$$(2.10) \quad \begin{aligned} \|\text{Op}[\log\langle \xi \rangle \{\psi, \sigma(X_i)\}] u\|^2 &\leq \delta \sum_j \|X_j u\|^2 + \delta \|\sqrt{a} \widehat{\nabla} u\|^2 + C_\delta \|u\|^2 + C_\delta \|\text{Op}(p)u\|_{H^1}^2, \\ \|\sqrt{\mathbf{Q}_p} \text{Op}[\log\langle \xi \rangle \{\psi, \widehat{\xi}\}] u\|^2 &\leq \delta \sum_j \|X_j u\|^2 + \delta \|\sqrt{a} \widehat{\nabla} u\|^2 + C_\delta \|u\|^2 + C_\delta \|\text{Op}(p)u\|_{H^1}^2, \end{aligned}$$

where $\widehat{\xi} = (\xi_p, \dots, \xi_n)$.

(2) Then there exists $\gamma > 0$ such that for any $u \in L_{loc}^2$ we have $Lu \in H^\gamma(R) \implies u \in H^\gamma(R)$.

Remark 13. The term R_1 arises from the conjugation of $\widehat{\nabla} \cdot \mathbf{Q}_p(x) \widehat{\nabla}$ by $\Lambda_s = (1 + |\xi|^2)^{s/2}$, needed in the bootstrap procedure. Indeed, we have denoting $q_{ij} = (\mathbf{Q}_p)_{ij}$

$$\Lambda_s \widehat{\nabla} \cdot \mathbf{Q}_p(x) \widehat{\nabla} \Lambda_{-s} - \widehat{\nabla} \cdot \mathbf{Q}_p(x) \widehat{\nabla} = \sum_{i,j=p}^n [\Lambda_s, q_{ij}] \Lambda_{-s} \partial_{x_i} \partial_{x_j}.$$

Using rough pseudodifferential calculus we have

$$\sigma([\Lambda_s, q_{ij}]\Lambda_{-s}\partial_{x_i}) = -i \sum_{|\alpha|=1} D^\alpha q_{ij} \frac{\xi^\alpha \xi_i}{\langle \xi \rangle^2} = -i \sum_{k=1}^n \partial_{x_k} q_{ij} \frac{\xi_k \xi_i}{\langle \xi \rangle^2} \mod \mathcal{O}_{(-\delta, \delta)}^{-\varepsilon}.$$

Denoting

$$S_k = \partial_{x_k} \mathbf{Q}_p, \quad (\theta_k(\xi))_i = -i \frac{\xi_k \xi_i}{\langle \xi \rangle^2},$$

we have that $R_1 = \sum_{k=1}^n S_k \Theta_k \circ \widehat{\nabla}$ has the desired properties since $\mathbf{Q}_p$ is subordinate, and

$$\Lambda_s \widehat{\nabla} \cdot \mathbf{Q}_p(x) \widehat{\nabla} \Lambda_{-s} = \widehat{\nabla} \cdot \mathbf{Q}_p(x) \widehat{\nabla} + R_1 \mod \mathcal{O}_{(-\delta, \delta)}^{-\varepsilon}.$$

We end this section on statements of the main hypoellipticity theorems, by outlining the four steps taken in order to get to the point where we can apply Theorems 3, 11 and 12 to obtain our hypoellipticity Theorem 9.

2.1. Summary of the steps. Consider the operator $L = \nabla A(\hat{x}) \nabla + D(x)$ with smooth coefficients.

- (1) We first apply Theorem 3 to write $\nabla A(\hat{x}) \nabla = \mathbf{X}^{\text{tr}} \mathbf{X}$ plus a quasiconformal subordinate term $\widehat{\nabla} \cdot \mathbf{Q}_p(x) \widehat{\nabla}$, where the vector fields $\mathbf{X}$ belong to $\mathcal{C}^{2, \delta} S_{1,0}^1$ for some $\delta > 0$, and $\mathbf{Q}_p \in C^{4, \delta}$.
- (2) We then use the smooth pseudodifferential calculus to write

$$\Lambda_s L \Lambda_{-s} = L + \widehat{\nabla} \cdot \mathbf{Q}_p(x) \widehat{\nabla} + V \mathbf{X} + \mathbf{X}^{\text{tr}} U + A_0(x, \xi) + R_1$$

where the pseudodifferential operators $V \mathbf{X}, U \mathbf{X} \in \text{Op} \mathcal{C}^{1, \delta} S_{1,0}^1$, and $R_1 \in \text{Op} \mathcal{C}^{1, \delta} S_{1,0}^1$ is subunit with respect to the quasiconformal term, and where $A_0 \in \mathcal{C}^{0, \delta} S_{1,0}^0$.

- (3) We next show that the operator $L = \nabla A(\hat{x}) \nabla + D(x)$ is hypoelliptic if and only if for every integer $s \in \mathbb{Z}$, there is $\gamma = \gamma([s]) > 0$ depending only on the integer part $[s]$ of s , such that

$$u \in H^0 \text{ and } \Lambda_s L \Lambda_{-s} u \in H^\gamma \text{ implies } u \in H^\gamma, \text{ for } 0 \leq \gamma \leq 1.$$

- (4) Finally, we apply Theorem 12 and Theorem 11 to obtain hypoellipticity of L .

Remark 14. Note that if we apply symbol splitting as in [Tay] to the vector fields $\mathbf{X}$ to obtain $\mathbf{X} = \mathbf{X}^{\text{h}} + \mathbf{X}^{\text{b}}$ where $\mathbf{X}^{\text{h}} \in \text{Op} S_{1, \eta}^1$ and $\mathbf{X}^{\text{b}} \in \text{Op} \mathcal{C}^{2, \delta} S_{1, \eta}^{1-\eta(2+\delta)}$, then the subunit property of the vector field $\mathbf{X}$ is not inherited by the smooth vector field $\mathbf{X}^{\text{h}}$. Indeed, the definition of $\mathbf{X}^{\text{h}}$ shows that it is obtained by applying a mollification of size $2^{-j\eta}$ to a Littlewood-Paley projection onto frequencies of size 2^j , and such mollifications are not comparable when applied to infinitely degenerate fields, even suitably away from the degeneracies.

3. A ROUGH VARIANT OF M. CHRIST'S THEOREM

We now prove our extension of M. Christ's hypoellipticity theorem, namely Theorem 12, to the case of a sum of squares of *rough* vector fields, whose sum of squares is nevertheless smooth. We will assume the rough symbols are in the classes $\mathcal{C}^{2, \delta} S_{1,0}^\alpha$, but we could just as well formulate and prove a variant for the symbol classes $\mathcal{C}^{2, \delta} S_{\rho, \eta}^\alpha$, which we leave for the interested reader, as we will not use such a variant in our applications. The proof of this rough theorem is accomplished by adapting the sum of squares argument of Christ [Chr] in the smooth case. For this we begin with some preliminaries.

3.1. Preliminaries. Here we recall definitions and properties of symbols, Gårding's inequality, parametrices, rough symbols, and wave front sets.

3.1.1. Symbols. We begin by recalling in $\mathbb{R}^n$, the definition of symbols $S_{\rho, \eta}^m$ from Stein [Ste, Chapter VI], the definition of symbols $S_{\rho, \eta}^{m, k}$ and $S_{\rho, \eta}^{m+}$ from Christ [Chr], and then some results on rough versions of the symbol classes $S_{\rho, \eta}^m$ from [Saw] and [Tay]. See also Treves [Tre] for symbols defined in open sets $\Omega \subset \mathbb{R}^n$.

Definition 15. Let $a(x, \xi)$ be a smooth function on $\mathbb{R}^n \times \mathbb{R}^n$, $0 \leq \eta < \rho \leq 1$, and $-\infty < m < \infty$.

- (1) Define $a \in S_{\rho, \eta}^m$, referred to as a symbol of type (ρ, η) and order m , if

$$(3.1) \quad \left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi) \right| \leq C_{\alpha, \beta} \langle \xi \rangle^{m - \rho|\beta| + \eta|\alpha|} \quad x \in \mathbb{R}^n, \xi \in \mathbb{R}^n, (\alpha, \beta) \in \mathbb{Z}_+^n \times \mathbb{Z}_+^n.$$

(2) Define $a \in S_{\rho,\eta}^{m,k}$ if

$$\left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi) \right| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\beta|+\eta|\alpha|} (\log \langle \xi \rangle)^{k+|\alpha|+|\beta|}.$$

(3) Define

$$S_{\rho,\eta}^{m+} \equiv \bigcap_{\varepsilon>0} S_{\rho-\varepsilon,\eta+\varepsilon}^{m,\varepsilon}, \quad m \in \mathbb{R}.$$

For a symbol $a \in S_{\rho,\eta}^m$, the associated pseudodifferential operator $A : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$, also denoted by $A = \text{Op } a$, is defined on the space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^n)$ on $\mathbb{R}^n$ by

$$(3.2) \quad Au(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} a(x, \xi) \widehat{u}(\xi) d\xi, \quad x \in \mathbb{R}^n.$$

It follows with some work (see e.g. [Ste]) that $\text{Op } a : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ is continuous, and moreover, if a_k converges pointwise to a on $\mathbb{R}^n$, and (3.1) holds for $a = a_k$ uniformly in k , then $a \in S_{\rho,\eta}^m$ as well. By duality $\text{Op } a : \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ is a continuous map from the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ to itself, and the asymptotic formulas for adjoints and compositions holds without restriction, e.g. if $a \in S_{\rho,\eta}^{m_1}$ and $b \in S_{\rho,\eta}^{m_2}$, then $\text{Op } a \circ \text{Op } b = \text{Op } (a \circ b)$ where for all $M \in \mathbb{N}$,

$$a \circ b = \sum_{\ell=0}^M \frac{1}{i^\ell \ell!} \widehat{\nabla}_\xi^\ell a \cdot \nabla_x^\ell b + E_M;$$

$$\text{with } E_M \in S_{\rho,\eta}^{m_1+m_2-M-1}.$$

It follows immediately from the definitions that the asymptotic formulas for adjoints and compositions extend to the symbol classes $S_{\rho,\eta}^{m+}$. For example, by uniqueness of the expansions, we have

$$E_M \in S_{\rho-\varepsilon,\eta+\varepsilon}^{m_1+\varepsilon+m_2+\varepsilon-M-1} \subset S_{\rho-2\varepsilon,\eta+2\varepsilon}^{m_1+m_2-M-1,2\varepsilon}$$

for each $\varepsilon > 0$, and so

$$E_M \in \bigcap_{\varepsilon>0} S_{\rho-2\varepsilon,\eta+2\varepsilon}^{m_1+m_2-M-1,2\varepsilon} = S_{\rho,\eta}^{m_1+m_2-M-1+}.$$

Now $S_{\rho,\eta}^{m+} \subset S_{\rho,\eta}^{m,k}$, and it turns out that for our purposes, we apply the pseudodifferential calculus to the symbol classes $S_{\rho,\eta}^{m+}$, as well as to the classes $S_{\rho,\eta}^{m,k}$ that arise naturally from the hypotheses of the theorems. We will not necessarily make explicit mention of this distinction in the sequel however.

3.1.2. Parametrices. Let $a(x, \xi) \in S_{1,\eta}^m$ be elliptic of order m , i.e. there are strictly positive continuous functions $\rho(x)$ and $c(x)$ in Ω such that the symbol $a(x, \xi)$ satisfies

$$c(x) |\xi|^m \leq |a(x, \xi)|, \quad \xi \in \mathbb{R}^n \text{ with } |\xi| \geq \rho(x), x \in \Omega.$$

Proposition 16. *Let $a(x, \xi) \in S_{1,\eta}^m(\Omega)$. If $a(x, \xi)$ is elliptic of order m , then there is $b(x, \xi) \in S_{1,\eta}^{-m}$ such that $a \circ b = 1$. Conversely, if there is $b(x, \xi) \in S_{1,\eta}^{-m}$ such that $a \circ b = 1$, then $a(x, \xi)$ is elliptic of order m .*

Proof. Determine recursively symbols b_j from the relations

$$(3.3) \quad \begin{aligned} b_0(x, \xi) a(x, \xi) &= 1, \\ b_j(x, \xi) a(x, \xi) &= - \sum_{1 \leq |\alpha| \leq j} \frac{1}{\alpha!} \partial_\xi^\alpha a(x, \xi) D_x^\alpha b_{j-|\alpha|}(x, \xi), \quad j \geq 1, \end{aligned}$$

which make sense only for $|\xi| \geq \rho(x)$. The first three such symbols are given by

$$\begin{aligned} b_0(x, \xi) &= \frac{1}{a(x, \xi)}, \\ b_1(x, \xi) &= -b_0(x, \xi) \sum_{i=1}^n \frac{\partial}{\partial \xi_i} a(x, \xi) \frac{1}{i} \frac{\partial}{\partial x_i} b_0(x, \xi) = -\frac{1}{i} b_0(x, \xi) \nabla_\xi a(x, \xi) \cdot \nabla_x b_0(x, \xi), \\ b_2(x, \xi) &= -b_0(x, \xi) \sum_{i=1}^n \frac{\partial}{\partial \xi_i} a(x, \xi) \frac{1}{i} \frac{\partial}{\partial x_i} b_1(x, \xi) - b_0(x, \xi) \sum_{|\alpha|=2} \frac{1}{\alpha!} \partial_\xi^\alpha a(x, \xi) D_x^\alpha b_0(x, \xi) \\ &= -\frac{1}{i} b_0(x, \xi) \nabla_\xi a(x, \xi) \cdot \nabla_x b_1(x, \xi) - b_0(x, \xi) \frac{1}{2!} \nabla_\xi^2 a(x, \xi) \cdot \nabla_x^2 b_0(x, \xi). \end{aligned}$$

To deal with the requirement that $|\xi| \geq \rho(x)$, we select a monotone increasing sequence of continuous functions $\rho_{j+1}(x) > \rho_j(x) > \rho(x)$ and a sequence of smooth cutoff functions $\chi_j(x, \xi) \in C^\infty(\Omega \times \mathbb{R}^n)$ satisfying

$$\chi_j(x, \xi) = \begin{cases} 0 & \text{if } |\xi| \leq \rho_j(x) \\ 1 & \text{if } |\xi| \leq 2\rho_j(x) \end{cases}.$$

One can easily prove by induction on j that $\chi_j b_j \in S^{-m-j}(\Omega)$, and moreover that for carefully chosen such χ_j the series $\sum_{j=1}^\infty \chi_j b_j$ converges in $S^{-m}(\Omega)$ to a symbol b satisfying $a \circ b = 1$. Indeed, if $\{K_j\}_{j=1}^\infty$ is a standard exhausting sequence of compact sets for Ω , and if the constants $C_{\alpha, \beta}^{(j)}(K_i)$ satisfy

$$|\partial_\xi^\alpha \partial_x^\beta (\chi_j b_j)| \leq C_{\alpha, \beta}^{(j)}(K_i) |\xi|^{-m-j-|\alpha|}, \quad \text{for } x \in K_i, \xi \in \mathbb{R}^n \setminus \{0\},$$

then we need only require in addition that $\rho_j(x) \geq 2 \sup_{i \leq j, |\alpha|+|\beta| \leq j} C_{\alpha, \beta}^{(j)}(K_i)^{\frac{1}{j}}$.

The converse is an easy exercise using only the consequence

$$a(x, \xi) b(x, \xi) - 1 \in S^{-1}(\Omega),$$

which implies that for every compact set $K \subset \Omega$, there is a constant C_K such that

$$|a(x, \xi) b(x, \xi) - 1| \leq C_K \frac{1}{1 + |\xi|}.$$

■

Corollary 17. *Let A belong to $S_{1,0}^m(\Omega)$. Then A is elliptic of order m if and only if there is $B \in S_{1,0}^{-m}(\Omega)$ with*

$$AB = BA = I \quad \text{mod } S^{-\infty}(\Omega),$$

where $S^{-\infty}(\Omega) = \bigcap_{m \in \mathbb{R}} S_{1,0}^{-m}(\Omega)$.

3.1.3. Rough symbols. The following definitions are taken from [Tay] and [Saw].

Definition 18. *A symbol $\sigma : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ belongs to the rough symbol class $\mathcal{C}^M S_{\rho, \delta}^m$ (where $M \in \mathbb{Z}_+$ and $0 \leq \rho, \delta \leq 1$) if for all multiindices α, β with $|\alpha| \leq M$, there are constants $C_{\alpha, \beta}$ such that*

$$\left| D_x^\alpha D_\xi^\beta \sigma(x, \xi) \right| \leq C_{\alpha, \beta} (1 + |\xi|)^{m + \delta|\alpha| - \rho|\beta|}, \quad x \in \mathbb{R}^n, \xi \in \mathbb{R}^n.$$

If $0 < \mu < 1$, then $\sigma \in \mathcal{C}^{M+\mu} S_{\rho, \delta}^m$ if in addition we have

$$\left| D_\xi^\beta \sigma(x + h, \xi) - \sum_{\ell=0}^M \frac{(h \cdot \nabla_x)^\ell}{\ell!} D_\xi^\beta \sigma(x, \xi) \right| \leq C_{M, \beta} |h|^{M+\mu} (1 + |\xi|)^{m + \delta(M+\mu) - \rho|\beta|}, \quad x \in \mathbb{R}^n, \xi \in \mathbb{R}^n.$$

Definition 19. *A symbol $\sigma : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ belongs to the operator class $\mathcal{O}_I^m$ if its associated operator*

$$(\text{Op } \sigma) u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma(x, \xi) \widehat{u}(\xi) d\xi, \quad x \in \mathbb{R}^n,$$

admits a bounded extension from $H_{p, \text{comp}}^{s+m}$ to $H_{p, \text{loc}}^s$ (resp. $\Lambda_{p, \text{comp}}^{s+m}$ to $\Lambda_{p, \text{loc}}^s$) for $s \in I$ (resp. $s \in I \cap [0, \infty)$) and all $1 < p < \infty$.

The symbol σ belongs to the operator class $\overline{\mathcal{O}}_I^m$ if in addition $\text{Op } \sigma$ is bounded from $\Lambda_{p, \text{comp}}^{t+m}$ to $\Lambda_{p, \text{loc}}^t$ where t is the right endpoint of the interval I .

Here the subscript comp means compactly supported distributions in the space, while the subscript loc means distributions locally in the space.

The following result of Bourdaud is well known, see also [Tay, Section 2.1] and [Saw, Theorem 3].

Theorem 20 ([Bou, Bou]). *For all real m , and all $\nu > 0$ and $0 \leq \delta < 1$ we have*

$$\mathcal{C}^\nu S_{1, \delta}^m \subset \overline{\mathcal{O}}_{(-(1-\delta)\nu, \nu)}^m.$$

3.1.4. *Rough pseudodifferential calculus.* While symbol smoothing is a very effective and relatively simple tool for use in elliptic and finite type situations, it fails to sufficiently preserve the subunit property of vector fields in the infinitely degenerate regime. For this reason we will instead use the pseudodifferential calculus from [Saw], to which we now turn.

If $\sigma \in \mathcal{C}^\nu S_{1,\delta_1}^{m_1}$ and $\tau \in \mathcal{C}^{M+\mu+\nu} S_{1,\delta_2}^{m_2}$ have compact support in $\mathbb{R}^n \times \mathbb{R}^n$, then the composition $\text{Op } \sigma \circ \text{Op } \tau$ of the operators $\text{Op } \sigma$ and $\text{Op } \tau$ equals the operator $\text{Op } (\sigma \circ \tau)$ where

$$(\sigma \circ \tau)(x, \eta) \equiv \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y) \cdot (\xi-\eta)} \sigma(x, \xi) \tau(y, \eta) dy d\xi,$$

and the double integral on the right hand side is absolutely convergent under the compact support assumption, thus justifying the claim. Given such symbols without the assumption of compact support, we may then consider instead the symbols σ_ε and τ_ε where $a_\varepsilon(x, \xi) \equiv \psi(\varepsilon x, \varepsilon \xi) a(x, \xi)$. Provided $\psi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is 1 on the unit ball, the symbols a_ε are uniformly in the same symbol class as a , and hence the above formula persists in the limit when the operators are restricted to acting on the space $\mathcal{S}$ of rapidly decreasing functions. Of course it may happen that the resulting symbol $\sigma \circ \tau$ fails to belong to any reasonable rough symbol class $\mathcal{C}^{M+\mu} S_{\rho,\delta}^m$ - see [Saw, Subsection 5.3]. Nevertheless, we have the following useful symbol expansion of $\sigma \circ \tau$ valid up to an error operator in an appropriate class $\overline{\mathcal{O}}_I^m$.

Theorem 21. ([Saw, Theorem 4]) *Suppose $\sigma \in \mathcal{C}^\nu S_{1,\delta_1}^{m_1}$ and $\tau \in \mathcal{C}^{M+\mu+\nu} S_{1,\delta_2}^{m_2}$ where M is a nonnegative integer, $0 < \mu, \delta_1, \delta_2 < 1$, $\nu > 0$ and $M + \mu \geq m_1 \geq 0$. Let $\delta \equiv \max\{\delta_1, \delta_2\}$. Then*

$$\begin{aligned} \sigma \circ \tau &= \sum_{\ell=0}^M \frac{1}{i^\ell \ell!} \nabla_\xi^\ell \sigma \cdot \nabla_x^\ell \tau + E; \\ E &\in \mathcal{O}_{(-1-\delta)\nu, \nu}^{m_1+m_2+(M+\mu)(\delta_2-1)+\varepsilon}, \quad \text{for every } \varepsilon > 0. \end{aligned}$$

There is an analogous expansion for the symbol of the adjoint operator $(\text{Op } \sigma)^{\text{tr}}$.

3.1.5. *Smooth distributions and wave front sets.* The following definitions are taken from Treves [Tre].

Definition 22. *A distribution u in an open set $\Omega \subset \mathbb{R}^n$ is said to be C^∞ in some neighbourhood of a point $(x_0, \xi^0) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$ if there is a function $g \in C_c^\infty(\mathbb{R}^n)$ equal to 1 in a neighbourhood of x_0 , and an open cone $\Gamma^0 \subset \mathbb{R}^n$ containing ξ^0 such that for every $M > 0$ there is a positive constant C_M satisfying*

$$|\widehat{gu}(\xi)| \leq C_M (1 + |\xi|)^{-M}, \quad \xi \in \Gamma^0.$$

Definition 23. *A distribution u in an open set $\Omega \subset \mathbb{R}^n$ is said to be C^∞ in a conic open subset $\Gamma \subset \Omega \times (\mathbb{R}^n \setminus \{0\})$ if it is C^∞ in some neighbourhood of every point of Γ . The wave front set $WF(u)$ of u is the complement in $\Omega \times (\mathbb{R}^n \setminus \{0\})$ of the union of all conic open sets in which u is C^∞ :*

$$WF(u) \equiv \Omega \times (\mathbb{R}^n \setminus \{0\}) \setminus \bigcup \{ \Gamma \text{ conic open } \subset \Omega \times (\mathbb{R}^n \setminus \{0\}) : u \text{ is } C^\infty \text{ in } \Gamma \}.$$

For $\gamma \in \mathbb{R}$, the H^γ wave front set of u is defined analogously, where H^γ is the Sobolev space of order γ .

3.2. Proof of Theorem 12, the limited smoothness variant of Christ's theorem. Now we can begin our proof of the limited smoothness variant Theorem 12, in the setting of real vector fields, of M. Christ's theorem. Let $u \in \mathcal{D}'(V)$ and $0 < \gamma < \delta$ be given. Suppose that the H^γ wave front set of Lu is disjoint from some open conic neighbourhood Γ_0 of a point $(x_0, \xi_0) \in T^*V$. Without loss of generality we may assume that $u \in \mathcal{E}'(V)$. Fix an integer $K \in \mathbb{Z}$ (possibly quite large) such that $u \in H^{-K}$. We will show that $(x_0, \xi_0) \notin WF_{H^\gamma}(u)$ by first constructing a pseudodifferential operator Λ , that is elliptic of order γ in a conic neighbourhood of (x_0, ξ_0) , and then showing that $\Lambda u \in H^0(\mathbb{R}^d)$.

To do this, let ψ be as in part (1) (c) of Theorem 12. Recall the definitions of the symbol classes $S_{\rho,\eta}^m$, $S_{\rho,\eta}^{m,k}$ and $S_{\rho,\eta}^{m+}$:

$$\begin{aligned} a &\in S_{\rho,\eta}^m \text{ if } \left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi) \right| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\beta|+\eta|\alpha|}, \\ a &\in S_{\rho,\eta}^{m,k} \text{ if } \left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi) \right| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\beta|+\eta|\alpha|} (\log \langle \xi \rangle)^{k+|\alpha|+|\beta|}, \\ S_{\rho,\eta}^{m+} &\equiv \bigcap_{\varepsilon>0} S_{\rho-\varepsilon,\eta+\varepsilon}^{m,\varepsilon}. \end{aligned}$$

Then following Christ we define a symbol of nonconstant order, depending on parameters γ and N_0 by

$$(3.4) \quad \lambda(x, \xi) = \begin{cases} |\xi|^\gamma e^{-N_0(\log|\xi|)\psi(x, \xi)} & \text{if } |\xi| \geq e \\ C^\infty \text{ and nonvanishing} & \text{if } |\xi| < e \end{cases}.$$

The nonnegativity of ψ implies that $\lambda \in S_{1,0}^{\gamma+}$. Moreover, $\lambda \in S_{1,0}^{\gamma,0}$. With γ fixed, there exists $\theta > 0$ such that for each N_0 , we have $\lambda \in S_{1,0}^{-\theta N_0+}$ on the closure of the complement of Γ_1 . Now choose N_0 so that $-\theta N_0 < -K$. Then with

$$\Lambda = \text{Op}(\lambda),$$

we have $\Lambda u \in H^{-K+\theta N_0} \subset H^0$ microlocally on the complement of Γ_1 .

Define cutoff functions $\eta_1, \eta_2 \in C_c(\mathbb{R}^d)$ such that $\eta_2 \equiv 1$ in a neighbourhood of the support of u , $\eta_1 \equiv 1$ in a neighbourhood of the support of η_2 , and $\text{Supp } \eta_1 \subset V$.

Recall that if $a \in S_{\rho, \eta}^m$ and $b \in S_{\rho, \eta}^n$, and $\rho > \eta$, then $\text{Op}(a) \circ \text{Op}(b)$ has a symbol $a \odot b$ with an asymptotic expansion

$$(3.5) \quad a \odot b(x, \xi) \sim \sum_{\alpha} c_{\alpha} \partial_{\xi}^{\alpha} a(x, \xi) \partial_x^{\alpha} b(x, \xi), \quad c_{\alpha} = \frac{(-i)^{\alpha}}{\alpha!}.$$

The notation $\sim$ means that for every N , the operator

$$\text{Op}(a) \circ \text{Op}(b) - \text{Op}\left(\sum_{|\alpha| < N} c_{\alpha} \partial_{\xi}^{\alpha} a(x, \xi) \partial_x^{\alpha} b(x, \xi)\right)$$

is smoothing of order $m + n - N(\rho - \eta)$ in the scale of Sobolev spaces. The next lemma is taken verbatim from [Chr], as it involves only symbols of type $(1, 0)$.

Lemma 24 (Lemma 4.1 in [Chr]). *There exists an operator $\Lambda^{-1} \in S_{1,0}^{m+}$ for some $m = m(\gamma)$ depending on γ , such that $\Lambda \circ \Lambda^{-1} -$ is smoothing of infinite order. Moreover, such an operator may be constructed with a symbol of the form*

$$(1 + f) \lambda^{-1}, \quad f \in S_{1,0}^{-1,2}.$$

Proof. Write $f \sim \sum_{k=1}^{\infty} f_k$. Solve the equation

$$\lambda \odot [(1 + f) \lambda^{-1}] \sim 1$$

using the asymptotic expansion (3.5) and the usual iterative procedure as given in (3.3). One obtains $f_1 \in S^{-1,2}$, and by induction, each $f_k \in S_{1,0}^{-k+}$. Choose Λ to be an operator whose full symbol has expansion $\sum_{k=1}^{\infty} f_k$, so that the error is smoothing of all orders in the scale of Sobolev spaces. ■

To prove an analogue of Lemma 4.2 in [Chr] we will need an auxiliary lemma.

Lemma 25. *Let $P \in \text{Op}(C^{\nu} S_{1,0}^{m,l})$ where $m, l \in \mathbb{N}$, and let Λ be the operator in (3.4), where where we recall that ψ is everywhere nonnegative, vanishes identically in a small conic neighbourhood of (x_0, ξ_0) , and is strictly positive on the complement of Γ_1 . Then*

$$\Lambda P \Lambda^{-1} = P + R_1 + R_2 + E,$$

where $R_1 \in \text{Op}(C^{\nu-1} S_{1,0}^{m-1,l+1})$, $R_2 \in \text{Op}(C^{\nu-2} S_{1,0}^{m-2,l+2})$, and $E \in \mathcal{O}_{(-\nu, \nu)}^{m-M-\varepsilon}$ for every $m \leq M < \nu$ and some $0 < \varepsilon < 1$. Moreover, the operator R_1 has the form

$$R_1 = \text{Op}(\{\log \lambda, \sigma(P)\}).$$

Proof. Using Theorem 21 we see that the symbol of $\Lambda P - P \Lambda$ divided by λ equals

$$\begin{aligned} & \frac{1}{\lambda} \{\lambda \odot \sigma(P) - \sigma(P) \odot \lambda\} = \left\{ \sum_{|\alpha|=1} + \sum_{2 \leq |\alpha| \leq M} \right\} c_{\alpha} \left[\frac{\partial_{\xi}^{\alpha} \lambda}{\lambda} \partial_x^{\alpha} \sigma(P) - \partial_{\xi}^{\alpha} \sigma(P) \frac{\partial_x^{\alpha} \lambda}{\lambda} \right] + E \\ &= \sum_{|\alpha|=1} c_{\alpha} [\partial_{\xi}^{\alpha} \log \lambda \partial_x^{\alpha} \sigma(P) - \partial_{\xi}^{\alpha} \sigma(P) \partial_x^{\alpha} \log \lambda] + \text{symbol in } C^{\nu-M} S_{1,0}^{m-2,l+2} + E \\ &= \{\log \lambda, \sigma(P)\} + \text{symbol in } C^{\nu-M} S_{1,0}^{m-2,l+2} + E, \end{aligned}$$

where $E \in \mathcal{O}_{(-\nu, \nu)}^{m-M-\varepsilon}$ for every $M < \nu$ and some $0 < \varepsilon < 1$; and $\{\log \lambda, \sigma(P)\}$ is the Poisson bracket of $\log \lambda$ and $\sigma(P)$, and is a symbol in $\mathcal{C}^{\nu-1} S_{1,0}^{m-1, l+1}$. ■

Define

$$(3.6) \quad L_1 = \sum_j X_j^{\text{tr}} X_j + \sum_j A_j X_j + \sum_j X_j^{\text{tr}} \tilde{A}_j + A_0,$$

so that $L = L_1 + R_1 + \widehat{\nabla} \cdot \mathbf{Q}_p \widehat{\nabla}$. This next lemma is our first analogue of Lemma 4.5 in [Chr].

Lemma 26 (Lemma 4.5 in [Chr]). *Let Λ be the operator with symbol λ in (3.4). Suppose that L_1 takes the form (3.6). Define*

$$b_j \equiv \text{Op} \{ \log \lambda, \sigma(X_j^{\text{tr}}) \} \quad \text{and} \quad \tilde{b}_j \equiv \text{Op} \{ \log \lambda, \sigma(X_j) \}.$$

Then there exists a pseudodifferential operator G of the form

$$(3.7) \quad G = \sum_j B_j \circ X_j + \sum_j X_j^{\text{tr}} \circ \tilde{B}_j + B_0,$$

such that

$$(L_1 + G) \eta_1 \Lambda \eta_2 = \eta_1 \Lambda L \eta_2 + R,$$

where

$$(3.8) \quad \begin{aligned} B_j &= b_j + c_j \quad \text{and} \quad \tilde{B}_j = \tilde{b}_j + \tilde{c}_j \quad \text{for every } j \geq 1, \\ B_0 &= \sum_j \left(b_j \circ \tilde{b}_j + A_j \tilde{b}_j + \tilde{A}_j b_j \right) \mod \text{Op} \left(C^{0, \delta} S_{1,0}^{-1, 2} \right), \end{aligned}$$

where each $c_j, \tilde{c}_j \in \text{Op} \left(C^{0, \delta} S^{-1, 1} \right)$, and where $A_j, \tilde{A}_j \in C^{1, \delta} S_{1,0}^0$ are the coefficients of the differential operator L in (3.6), and $R \in \mathcal{O}_{(-\delta, \delta)}^{-\varepsilon}$.

Proof. In constructing the symbol of G we will work formally, ignoring the cutoff functions η_1 and η_2 . This is permissible by pseudolocality since $\eta_1 \eta_2 = \eta_2$. The desired equation $(L_1 + G) \Lambda = \Lambda L + R$ is then equivalent to

$$\begin{aligned} G &= \Lambda L_1 \Lambda^{-1} - L_1 + R \Lambda^{-1} \\ &= \sum_j [\Lambda X_j^{\text{tr}} X_j \Lambda^{-1} - X_j^{\text{tr}} X_j] + R \Lambda^{-1} \\ &\quad + \sum_j \Lambda \left(A_j X_j + X_j^{\text{tr}} \tilde{A}_j + A_0 \right) \Lambda^{-1} - \sum_j \left(A_j X_j + X_j^{\text{tr}} \tilde{A}_j + A_0 \right) \\ &\equiv G_{\text{top}} + \Lambda G_{\text{lower}} \Lambda^{-1} - G_{\text{lower}}; \\ \text{where } G_{\text{top}} &= \sum_j [\Lambda X_j^{\text{tr}} X_j \Lambda^{-1} - X_j^{\text{tr}} X_j] + R \Lambda^{-1} \\ &= \sum_j [(\Lambda X_j^{\text{tr}} \Lambda^{-1}) (\Lambda X_j \Lambda^{-1}) - X_j^{\text{tr}} X_j] + R \Lambda^{-1}; \\ \text{and } G_{\text{lower}} &= \sum_j \left(A_j X_j + X_j^{\text{tr}} \tilde{A}_j + A_0 \right). \end{aligned}$$

We first consider G_{top} . Using Lemma 25 with $P = X_j$, $m = 1, l = 0$ we have

$$\begin{aligned} \Lambda X_j \Lambda^{-1} &= X_j + \text{Op} \{ \log \lambda, \sigma(X_j) \} + \text{symbol in } C^{0, \delta} S_{1,0}^{-1, 2} \mod \mathcal{O}_{(-\delta, \delta)}^{-1-\varepsilon} \\ &= X_j + b_j + c_j \mod \mathcal{O}_{(-\delta, \delta)}^{-1-\varepsilon}, \end{aligned}$$

where $b_j = \text{Op} \{ \log \lambda, \sigma(X_j) \} \in C^{1, \delta} S_{1,0}^{0, 1}$ and c_j has a symbol in $C^{0, \delta} S_{1,0}^{-1, 2}$. Since both $\{\log \lambda, \sigma(X_j)\}$ and $\{\log \lambda, \sigma(X_j^{\text{tr}})\}$ belong to $C^{1, \delta} S_{1,0}^{0, 1}$, inserting these equations into the identity derived for G_{top} in the preceding paragraph shows that

$$G_{\text{top}} = \sum_j B_j \circ X_j + \sum_j X_j^{\text{tr}} \circ \tilde{B}_j + B_0$$

where the operators $B_j, \widetilde{B}_j \in \text{Op} \left(C^{1,\delta} S_{1,0}^{0,1} \right)$ and $B_0 \in \text{Op} \left(C^{0,\delta} S_{1,0}^{0,2} \right)$ satisfy (3.8).

Now consider G_{lower} . We can write

$$\Lambda G_{\text{lower}} \Lambda^{-1} = \sum_j \Lambda \left(A_j X_j + X_j^{\text{tr}} \widetilde{A}_j + A_0 \right) \Lambda^{-1} = \sum_j \left(\Lambda A_j \Lambda^{-1} \Lambda X_j \Lambda^{-1} + \Lambda X_j^{\text{tr}} \Lambda^{-1} \Lambda \widetilde{A}_j \Lambda^{-1} + \Lambda A_0 \Lambda^{-1} \right).$$

Applying Lemma 25 to A_j and X_j we have

$$\begin{aligned} \Lambda A_j \Lambda^{-1} &= A_j + \text{symbol in } \text{Op} \left(C^{0,\delta} S_{1,0}^{-1,1} \right) \mod \mathcal{O}_{(-\delta,\delta)}^{-1-\varepsilon}, \\ \Lambda X_j \Lambda^{-1} &= X_j + \text{Op}(\{\log \lambda, \sigma(X_j)\}) + \text{symbol in } C^{0,\delta} S_{1,0}^{-1,2} \mod \mathcal{O}_{(-\delta,\delta)}^{-1-\varepsilon}. \end{aligned}$$

Using Theorem 21 this gives

$$\Lambda A_j \Lambda^{-1} \Lambda X_j \Lambda^{-1} = A_j X_j + c_j X_j + \text{symbol in } C^{0,\delta} S_{1,0}^{0,1} \mod \mathcal{O}_{(-\delta,\delta)}^{-\varepsilon},$$

where $c_j \in C^{0,\delta} S_{1,0}^{-1,1}$, and the symbol in $C^{0,\delta} S_{1,0}^{0,1}$ has the form $A_j \tilde{b}_j + \text{symbol in } C^{0,\delta} S_{1,0}^{0,1}$ with $\tilde{b}_j = \{\log \lambda, \sigma(X_j)\}$. Analyzing the other terms in $\Lambda G_{\text{lower}} \Lambda^{-1}$ in the same way we obtain

$$\begin{aligned} \Lambda G_{\text{lower}} \Lambda^{-1} &= B_j X_j + X_j^{\text{tr}} \tilde{B}_j + \tilde{B}_0 \mod \mathcal{O}_{(-\delta,\delta)}^{-\varepsilon}, \\ \text{where } B_j, \tilde{B}_j &\text{ as in 3.8,} \end{aligned}$$

and $\tilde{B}_0 \in \text{Op} \left(C^{0,\delta} S_{1,0}^{0,1} \right)$ and has the structure as in (3.8). Combining with the estimate for G_{top} we obtain the result. ■

Lemma 27 (Lemma 4.6 in [Chr]). *Suppose that L, ψ, p satisfy the hypotheses of Theorem 12. Then for any $N \geq 0$, and for any fixed relatively compact subset $U \subset V$, any $\delta > 0$ and any $f \in C^{\gamma+3}$ supported in U , the operator G constructed in Lemma 26 satisfies*

$$(3.9) \quad |\langle Gf, f \rangle| \leq \delta \sum_j \|X_j f\|^2 + \delta \|\sqrt{a} \widehat{\nabla} f\|^2 + C_\delta \|f\|^2 + C_\delta \|\text{Op}(p) f\|_{H^1}^2.$$

Proof. We first note

$$\sigma(b_j) = \{\log \lambda, \sigma(X_j^{\text{tr}})\} = -N_0 \log |\xi| \{\psi, \sigma(X_j^{\text{tr}})\} + \text{symbol in } C^{1,\delta} S_{1,0}^0,$$

and similarly for $\tilde{b}_j$. Using this together with (3.8) and hypothesis (2.10) with $\delta = \delta_0$ we therefore obtain

$$\begin{aligned} |\langle B_j \circ X_j f, f \rangle| &= |\langle (b_j + c_j) \circ X_j f, f \rangle| \\ &\leq \varepsilon \|X_j f\|^2 + C_\varepsilon \|\tilde{b}_j f\|^2 + C_\varepsilon \|f\|^2 \\ &\leq \varepsilon \|X_j f\|^2 + C_\varepsilon \|\log |\xi| \{\psi, \sigma(X_j^{\text{tr}})\} f\|^2 + C_\varepsilon \|f\|^2 \\ &\leq \varepsilon \|X_j f\|^2 + C_\varepsilon \left(\delta_0 \sum_j \|X_j u\|^2 + \delta_0 \|\sqrt{a} \widehat{\nabla} f\|^2 + C_{\delta_0} \|f\|^2 + C_{\delta_0} \|\text{Op}(p) f\|_{H^1}^2 \right) + C_\varepsilon \|f\|^2. \end{aligned}$$

Choosing $\delta_0 = \varepsilon / C_\varepsilon$ this gives

$$|\langle B_j \circ X_j f, f \rangle| \leq \varepsilon \sum_j \|X_j u\|^2 + \varepsilon \|\sqrt{a} \widehat{\nabla} f\|^2 + C_\varepsilon \|f\|^2 + C_\varepsilon \|\text{Op}(p) f\|_{H^1}^2.$$

The rest of the terms in (3.7) are handled in the same way, giving (3.9). ■

To handle the Grushin type term $\widehat{\nabla} \cdot \mathbf{Q}_p(x) \widehat{\nabla}$ in (2.8) we will need the following two lemmas

Lemma 28. *There holds*

$$\left(\widehat{\nabla} \cdot \mathbf{Q}_p \widehat{\nabla} \eta_1 + \mathbf{E} \right) \Lambda \eta_2 = \eta_1 \Lambda \widehat{\nabla} \cdot \mathbf{Q}_p \widehat{\nabla} \eta_2 + \mathbf{R},$$

where $\mathbf{R} \in \mathcal{O}_{(-\delta, \delta)}^{-\varepsilon}$, and with $\widehat{\xi} = (\xi_p, \dots, \xi_n)$, the matrix operator $\mathbf{E}$ takes the form

$$(3.10) \quad \mathbf{E} = H \circ \mathbf{Q}_p \widehat{\nabla} + \widehat{\nabla} \left(\sum_{|\alpha|=1} D^\alpha \mathbf{Q}_p \right) \circ H_0 + H \circ \left(\sum_{|\alpha|=1} D^\alpha \mathbf{Q}_p \right) \circ H_0 \\ + H_3 \circ \mathbf{Q}_p \widehat{\nabla} + H \circ \mathbf{Q}_p H + \tilde{H}_0 \quad \mathcal{O}_{(-\delta, \delta)}^{-\varepsilon};$$

where $H = \text{Op} \left\{ \log \lambda, \widehat{\xi} \right\} \in \text{Op} \left(S_{1,0}^{0,1} \right)$, $H_0 \in \text{Op} \left(S_{1,0}^0 \right)$, $\tilde{H}_0 \in \text{Op} \left(C^{0,\delta} S_{1,0}^0 \right)$, $H_3 \in \text{Op} \left(S_{1,0}^{-1,1} \right)$.

Proof. In constructing the symbol of $\mathbf{E}$ we will work formally, ignoring the cutoff functions η_1 and η_2 . This is permissible by pseudolocality since $\eta_1 \eta_2 = \eta_2$. Let $\mathbf{L}_2 \equiv \widehat{\nabla} \cdot \mathbf{Q}_p \widehat{\nabla}$, the desired equation $(\mathbf{L}_2 + \mathbf{E}) \Lambda = \Lambda \mathbf{L}_2 + \mathbf{R}$ is then equivalent to

$$(3.11) \quad \mathbf{E} = \Lambda \mathbf{L}_2 \Lambda^{-1} - \mathbf{L}_2 + \mathbf{R} \Lambda^{-1} \\ = \Lambda \widehat{\nabla} \cdot \mathbf{Q}_p \widehat{\nabla} \Lambda^{-1} - \widehat{\nabla} \cdot \mathbf{Q}_p \widehat{\nabla} + \mathbf{R} \Lambda^{-1} \\ = \left(\Lambda \widehat{\nabla} \Lambda^{-1} \right) \cdot \left(\Lambda \mathbf{Q}_p \widehat{\nabla} \Lambda^{-1} \right) - \widehat{\nabla} \cdot \mathbf{Q}_p \widehat{\nabla} + \mathbf{R} \Lambda^{-1}.$$

Next using Lemma 25 we have

$$\Lambda \widehat{\nabla} \Lambda^{-1} = \widehat{\nabla} + \{ \log \lambda, \widehat{\xi} \} + \text{symbol in } S_{1,0}^{-1,1} \equiv \widehat{\nabla} + H + H_3,$$

where $H = \{ \log \lambda, \widehat{\xi} \} \in \text{Op} \left(S_{1,0}^{0,1} \right)$, and $H_3 \in \text{Op} \left(S_{1,0}^{-1,1} \right)$. To estimate $\Lambda \mathbf{Q}_p \widehat{\nabla} \Lambda^{-1}$ we will need a refinement of Lemma 25, namely, the estimate obtained in the proof

$$\frac{1}{\lambda} \{ \lambda \odot \sigma(P) - \sigma(P) \odot \lambda \} = \left\{ \sum_{|\alpha|=1} + \sum_{|\alpha|=2} \right\} c_\alpha \left[\frac{\partial_\xi^\alpha \lambda}{\lambda} \partial_x^\alpha \sigma(P) - \partial_\xi^\alpha \sigma(P) \frac{\partial_x^\alpha \lambda}{\lambda} \right] + S \\ = \{ \log \lambda, \sigma(P) \} + \sum_{|\alpha|=2} c_\alpha \left[\partial_\xi^\alpha \log \lambda \partial_x^\alpha \sigma(P) - \partial_\xi^\alpha \sigma(P) \partial_x^\alpha \log \lambda \right] + S,$$

where $S \in \mathcal{O}_{(-\nu, \nu)}^{-1-\varepsilon}$ for some $0 < \varepsilon < 1$ and $0 < \nu < \delta$. Now $\sigma(P) = \sigma(\mathbf{Q}_p \widehat{\nabla}) = \mathbf{Q}_p \widehat{\xi}$, so

$$\sum_{|\alpha|=2} c_\alpha \left[\partial_\xi^\alpha \log \lambda \partial_x^\alpha \sigma(P) - \partial_\xi^\alpha \sigma(P) \partial_x^\alpha \log \lambda \right] = \sum_{|\alpha|=2} c_\alpha \partial_\xi^\alpha \log \lambda \partial_x^\alpha \mathbf{Q}_p \widehat{\xi} = \text{symbol in } \mathcal{C}^{0,\delta} S_{1,0}^{-1,0},$$

where the last equality holds since ψ does not depend on ξ in Γ , and therefore no logarithmic terms arise from differentiation of $\log \lambda$ with respect to ξ . Altogether we thus have

$$\Lambda \mathbf{Q}_p \widehat{\nabla} \Lambda^{-1} = \mathbf{Q}_p \widehat{\nabla} + \text{Op} \left(\{ \log \lambda, \mathbf{Q}_p \widehat{\xi} \} \right) + \text{symbol in } \mathcal{C}^{0,\delta} S_{1,0}^{-1,0} \quad \text{mod } \mathcal{O}_{(-\delta, \delta)}^{-1-\varepsilon} \\ = \mathbf{Q}_p \widehat{\nabla} + \sum_{|\alpha|=1} (D^\alpha \mathbf{Q}_p) \widehat{\xi} D_\xi^\alpha \log \lambda + \mathbf{Q}_p \cdot \{ \log \lambda, \widehat{\xi} \} + \text{symbol in } \mathcal{C}^{0,\delta} S_{1,0}^{-1,0} \quad \text{mod } \mathcal{O}_{(-\delta, \delta)}^{-1-\varepsilon} \\ = \mathbf{Q}_p \widehat{\nabla} + \sum_{|\alpha|=1} (D^\alpha \mathbf{Q}_p) \cdot \text{symbol in } S_{1,0}^0 + \mathbf{Q}_p \cdot H + \text{symbol in } \mathcal{C}^{0,\delta} S_{1,0}^{-1,0} \quad \text{mod } \mathcal{O}_{(-\delta, \delta)}^{-1-\varepsilon}$$

where we note that $\widehat{\xi} D_\xi^\alpha \log \lambda \in S_{1,0}^0$ for each α with $|\alpha| = 1$ since ψ does not depend on ξ , and therefore no logarithmic terms arise from differentiation of $\log \lambda$ with respect to ξ . This gives

$$\left(\Lambda \widehat{\nabla} \Lambda^{-1} \right) \cdot \left(\Lambda \mathbf{Q}_p \widehat{\nabla} \Lambda^{-1} \right) = \widehat{\nabla} \cdot \mathbf{Q}_p \widehat{\nabla} + H \circ \mathbf{Q}_p \widehat{\nabla} + H_3 \circ \mathbf{Q}_p \widehat{\nabla} + \widehat{\nabla} \left(\sum_{|\alpha|=1} D^\alpha \mathbf{Q}_p \right) \circ H_0 \\ + H \circ \left(\sum_{|\alpha|=1} D^\alpha \mathbf{Q}_p \right) \circ H_0 + H \circ \mathbf{Q}_p H + \tilde{H}_0 \quad \text{mod } \mathcal{O}_{(-\delta, \delta)}^{-\varepsilon}.$$

■

Lemma 29. *Let $\mathbf{E}$ be a pseudodifferential operator of the form (3.10). Then for any fixed relatively compact subset $U \subset V$, any $\delta > 0$ and any $f \in C_c^\infty$ supported in U , we have*

$$(3.12) \quad |\langle \mathbf{E}f, f \rangle| \leq \delta \sum_j \|X_j f\|^2 + \delta \|\sqrt{a} \widehat{\nabla} f\|^2 + C_\delta \|f\|^2 + C_\delta \|\text{Op}(p) f\|_{H^1}^2.$$

Proof. Here is where we will need to use that the matrix $\mathbf{Q}_p$ is subordinate - in the case $p = n$, then $\mathbf{Q}_n$ is simply a scalar and the subordinate inequality is that of Malgrange. We will use (3.10) and the notation $\mathbf{Q}'_p = \sum_{|\alpha|=1} D^\alpha \mathbf{Q}_p$. We have

$$\begin{aligned} \langle \mathbf{E}f, f \rangle &= - \left\langle \sqrt{\mathbf{Q}_p} H^{\text{tr}} f, \sqrt{\mathbf{Q}_p} \widehat{\nabla} f \right\rangle - \left\langle H_0 f, \mathbf{Q}'_p \widehat{\nabla} f \right\rangle + \left\langle H_0 f, \mathbf{Q}'_p H^{\text{tr}} f \right\rangle \\ &\quad + \left\langle H_3^{\text{tr}} f, \mathbf{Q}_p \nabla' f \right\rangle + \left\langle \sqrt{\mathbf{Q}_p} H f, \sqrt{\mathbf{Q}_p} H^{\text{tr}} f \right\rangle + \langle H_0 f, f \rangle. \end{aligned}$$

Now we use the crucial fact that $\mathbf{Q}_p$ is subordinate, i.e. $|\mathbf{Q}'_p| \leq C \mathbf{Q}_p$, and together with Cauchy-Schwartz this gives

$$|\langle \mathbf{E}f, f \rangle| \leq \delta \left\| \sqrt{\mathbf{Q}_p} \widehat{\nabla} f \right\|^2 + C_\delta \left\| \sqrt{\mathbf{Q}_p} H^{\text{tr}} f \right\|^2 + C_\delta \left\| \sqrt{\mathbf{Q}_p} H f \right\|^2 + C_\delta \|f\|^2.$$

Finally, using the definition of λ we obtain

$$\sigma(H) = \{\log \lambda, \widehat{\xi}\} = -N_0 \log |\xi| \{\psi, \widehat{\xi}\},$$

which together with the fact that $\mathbf{Q}_p \approx a \mathbb{I}_{n-p+1}$ shows

$$|\langle \mathbf{E}f, f \rangle| \leq \delta \left\| \sqrt{a} \widehat{\nabla} f \right\|^2 + C_\delta \left\| \sqrt{a} \text{Op} \left(\log \langle \xi \rangle \{\psi, \widehat{\xi}\} \right) f \right\|^2 + C_\delta \|f\|^2.$$

Combining with estimate (2.10) as in the proof of Lemma 27 we conclude (3.12). ■

Finally, we obtain an estimate on the subunit term R_1 .

Lemma 30. *Let $R_1 = \sum_{k=1}^n S_k \Theta_k \circ \widehat{\nabla}$, where each $S_k \in C^{1,\delta}(\mathbb{R}^{m \times m})$ is subunit with respect to $\mathbf{Q}_p$, and $\Theta_k = (\Theta_{kp}, \dots, \Theta_{kn})$ is a multiplier of order zero. Then*

$$(3.13) \quad (R_1 \eta_1 + J) \Lambda \eta_2 = \eta_1 \Lambda R_1 \eta_2 + R,$$

where $J \in \text{Op}(C^{0,\delta} S_{1,0}^{0,1})$, $R \in \mathcal{O}_{(-\delta, \delta)}^{-1-\varepsilon}$, and

$$(3.14) \quad |\langle Jf, f \rangle| \leq \delta \sum_j \|X_j f\|^2 + \delta \|\sqrt{a} \widehat{\nabla} f\|^2 + C_\delta \|f\|^2 + C_\delta \|\text{Op}(p) f\|_{H^1}^2,$$

any $\delta > 0$ and any $f \in C_c^\infty$.

Proof. Proceeding as in the proof of Lemma 28 we have

$$\begin{aligned} \Lambda S_k \Theta_k \circ \widehat{\nabla} \Lambda^{-1} &= S_k \Theta_k \circ \widehat{\nabla} + \sum_{|\alpha|=1} (D^\alpha S_k) \widehat{\xi} \theta_k(\xi) D_\xi^\alpha \log \lambda + S_k \{\log \lambda, \widehat{\xi} \theta_k(\xi)\} + \text{symbol in } C^{0,\delta} S_{1,0}^{-1,0} \quad \text{mod } \mathcal{O}_{(-\delta, \delta)}^{-1-\varepsilon} \\ &= S_k \Theta_k \circ \widehat{\nabla} + \text{symbol in } C^{0,\delta} S_{1,0}^0 + S_k H_k + \text{symbol in } C^{0,\delta} S_{1,0}^{-1,0} \quad \text{mod } \mathcal{O}_{(-\delta, \delta)}^{-1-\varepsilon} \\ &\equiv S_k \Theta_k \circ \widehat{\nabla} + J_k \quad \text{mod } \mathcal{O}_{(-\delta, \delta)}^{-1-\varepsilon}. \end{aligned}$$

where $H_k \in \text{Op}(S_{1,0}^{0,1})$. Defining $J \equiv \sum_{k=1}^n J_k$ and using the fact that S_k is subunit together with $\mathbf{Q}_p \approx a \mathbb{I}_{n-p+1}$ and (2.10) we obtain (3.14). ■

We are now ready to prove a generalization of Lemma 4.4 in [Chr], which is the main estimate we need.

Lemma 31 (Lemma 4.4 in [Chr]). *Let L take the form (2.8) and satisfy (2.9) and (2.10). Let $0 < \gamma < \delta$ be fixed. If N_0 is chosen sufficiently large in the definition of Λ , then for any fixed relatively compact $U \Subset V$ and any $u \in C^{2,\delta}(U)$,*

$$(3.15) \quad \|\eta_1 \Lambda u\|_{L^2(\mathbb{R}^n)} \leq C \|\eta_1 \Lambda L u\|_{L^2(\mathbb{R}^n)} + C \|u\|_{H^0(\mathbb{R}^n)}.$$

Proof. Recall that

$$L = \sum_j X_j^{\text{tr}} X_j + \sum_j A_j X_j + \sum_j X_j^{\text{tr}} \tilde{A}_j + A_0 + R_1 + \widehat{\nabla} \cdot \mathbf{Q}_p \widehat{\nabla} \equiv L_1 + L_2 + R_1,$$

where we used the notation $L_2 = \widehat{\nabla} \cdot \mathbf{Q}_p \widehat{\nabla}$. If we set

$$v \equiv \eta_1 \Lambda u \in C^2(\mathbb{R}^n),$$

we have

$$\begin{aligned} \langle (L_1 + G)v, v \rangle &= \langle L_1 v, v \rangle + \langle Gv, v \rangle \\ &= \sum_j \|X_j v\|_{L^2}^2 + \sum_j \langle A_j \circ X_j v, v \rangle + \sum_j \langle X_j^{\text{tr}} \circ \tilde{A}_j v, v \rangle + \langle A_0 v, v \rangle + \langle Gv, v \rangle \\ &= \sum_j \|X_j v\|_{L^2}^2 + \sum_j \langle X_j v, A_j^{\text{tr}} v \rangle + \sum_j \langle \tilde{A}_j v, X_j v \rangle + \langle A_0 v, v \rangle + \langle Gv, v \rangle \\ &= \sum_j \|X_j v\|_{L^2}^2 + O\left(\sqrt{\sum_j \|X_j v\|_{L^2}^2} \|v\|_{L^2} + \|v\|_{L^2}^2\right) + \langle Gv, v \rangle, \end{aligned}$$

since the operators A_j and $\tilde{A}_j$ have order 0. Similarly

$$\begin{aligned} \langle (L_2 + E)v, v \rangle &= \langle \nabla' \cdot \mathbf{Q}_p \widehat{\nabla} v, v \rangle + \langle Ev, v \rangle \\ &= \int |\sqrt{\mathbf{Q}_p} \widehat{\nabla} v|^2 + \langle Ev, v \rangle, \\ \langle (R_1 + J_0)v, v \rangle &= \left\langle \sum_{i=1}^n S_i \Theta_i \widehat{\nabla} v, v \right\rangle + \langle Jv, v \rangle \\ &\leq \delta \int a |\widehat{\nabla} v|^2 + C_\delta \|v\|_{L^2}^2 + \langle J_0 v, v \rangle. \end{aligned}$$

We also have from Lemmas 27, 28, and 30 that

$$\begin{aligned} (L_1 + G)v &= (L_1 + G) \eta_1 \Lambda \eta_2 u = \eta_1 \Lambda L_1 \eta_2 u + Ru = \eta_1 \Lambda L_1 u + Ru, \\ (L_2 + E)v &= (L_2 + E) \eta_1 \Lambda \eta_2 u = \eta_1 \Lambda L_2 \eta_2 u + Ru = \eta_1 \Lambda L_2 u + Ru, \\ (R_1 + J_0)v &= (R_1 + J) \eta_1 \Lambda \eta_2 u = \eta_1 \Lambda R_1 \eta_2 u + Ru = \eta_1 \Lambda R_1 u + Ru \end{aligned}$$

since $\eta_2 u = u$, and hence adding together

$$|\langle (L + G + E + J)v, v \rangle| \leq |\langle \eta_1 \Lambda L \eta_2 u, v \rangle| + |\langle Ru, v \rangle| \leq \frac{1}{2} \|\eta_1 \Lambda L u\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{2} \|Ru\|_{L^2(\mathbb{R}^n)}^2 + \|v\|_{L^2(\mathbb{R}^n)}^2$$

Thus from (3.9), (3.12), (3.14), and the above we conclude that

$$\begin{aligned} \sum_j \|X_j v\|_{L^2}^2 + \|\sqrt{\mathbf{Q}_p} \widehat{\nabla} v\|^2 &= \langle (L_1 + G)v, v \rangle - \langle Gv, v \rangle + C \left(\sqrt{\sum_j \|X_j v\|_{L^2}^2} \|v\|_{L^2} + \|v\|_{L^2}^2 \right) \\ &\quad + \langle (L_2 + E)v, v \rangle - \langle Ev, v \rangle \\ &\quad + \langle (R_1 + J)v, v \rangle - \langle J_0 v, v \rangle - \left\langle \sum_{i=1}^n S_i \Theta_i \widehat{\nabla} v, v \right\rangle \\ &\leq \frac{1}{2} \|\eta_1 \Lambda L u\|_{L^2}^2 + \frac{1}{2} \|Ru\|_{L^2}^2 + C_\delta \|v\|_{L^2}^2 \\ &\quad + \delta \sum_j \|X_j v\|_{L^2}^2 + 4\delta \|\sqrt{a} \widehat{\nabla} v\|_{L^2}^2 + C_\delta \|\text{Op}(p)v\|_{H^1}^2 \\ &\quad + C \left(\sqrt{\sum_j \|X_j v\|_{L^2}^2} \|v\|_{L^2} + \|v\|_{L^2}^2 \right). \end{aligned}$$

Combining this with the inequality

$$\sqrt{\sum_j \|X_j v\|_{L^2}^2} \|v\|_{L^2} \leq \delta \sum_j \|X_j v\|_{L^2}^2 + C_\delta \|v\|_{L^2}^2$$

and the condition $\mathbf{Q}_p \approx a\mathbb{I}_{n-p+1}$ we obtain, choosing δ smaller if necessary,

$$\begin{aligned} \sum_j \|X_j v\|_{L^2}^2 + \|\sqrt{a}\widehat{\nabla}v\|^2 &\leq \frac{1}{2} \|\eta_1 \Lambda Lu\|_{L^2}^2 + \frac{1}{2} \|Ru\|_{L^2}^2 + C_\delta \|v\|_{L^2}^2 + C_\delta \|\text{Op}(p)v\|_{H^1}^2 \\ &\quad + \delta \sum_j \|X_j v\|_{L^2}^2 + \delta \|\sqrt{a}\widehat{\nabla}v\|_{L^2}^2. \end{aligned}$$

Absorbing the terms $\delta \sum_j \|X_j v\|_{L^2}^2$ and $\delta \|\sqrt{a}\widehat{\nabla}v\|_{L^2}^2$ into the left hand side, and then using that the order of the error term R is $-\varepsilon$, we obtain

$$(3.16) \quad \sum_j \|X_j v\|_{L^2}^2 + \|\sqrt{a}\widehat{\nabla}v\|_{L^2}^2 \leq \|\eta_1 \Lambda Lu\|_{L^2(\mathbb{R}^n)}^2 + C \|v\|_{L^2}^2 + C \|u\|_{H^{-\varepsilon}}^2,$$

where the term involving the H^1 norm of $\text{Op}(p)\Lambda u$ may be absorbed into $\|u\|_{H^{-\varepsilon}}^2$ since Λ may be made to be regularizing of arbitrary high order in a conic neighborhood of the symbol p , by choosing N_0 to be sufficiently large.

Next we write

$$\begin{aligned} \|v\|_{L^2}^2 &= \int_{\{\xi \in \mathbb{R}^n; |\xi| \leq N\}} |\widehat{v}(\xi)|^2 d\xi + \int_{\{\xi \in \mathbb{R}^n; |\xi| > N\}} |\widehat{v}(\xi)|^2 d\xi \\ &\leq N^{2\gamma} \int_{\{\xi \in \mathbb{R}^n; |\xi| \leq N\}} \langle \xi \rangle^{-2\gamma} |\widehat{v}(\xi)|^2 d\xi + \frac{1}{w^2(N)} \int_{\{\xi \in \mathbb{R}^n; |\xi| > N\}} w^2(\langle \xi \rangle) |\widehat{v}(\xi)|^2 d\xi \\ &\leq N^{2\gamma} \|u\|_{H^0}^2 + \frac{1}{w^2(N)} \|w(\langle \xi \rangle) \widehat{v}(\xi)\|_{L^2}^2 \\ &\leq N^{2\gamma} \|u\|_{H^0}^2 + \frac{C}{w^2(N)} \left(\sum_j \|X_j v\|_{L^2}^2 + \|\sqrt{a}\widehat{\nabla}v\|_{L^2}^2 + \|v\|_{L^2}^2 \right) \end{aligned}$$

where for the last inequality we used (2.9). Let $\delta = C/w^2(N)$ and note that δ can be made arbitrarily small by choosing N sufficiently large, we combine the above equality with (3.16) to obtain

$$\begin{aligned} \|v\|_{L^2}^2 &\leq C_\delta \|u\|_{H^0}^2 + \delta \left(\sum_j \|X_j v\|_{L^2}^2 + \|\sqrt{a}\widehat{\nabla}v\|_{L^2}^2 + \|v\|_{L^2}^2 \right) \\ &\leq C_\delta \|u\|_{H^0}^2 + \delta \left(\|\eta_1 \Lambda Lu\|_{L^2(\mathbb{R}^n)}^2 + C \|v\|_{L^2}^2 + C \|u\|_{H^{-\varepsilon}}^2 \right) \end{aligned}$$

Choosing δ sufficiently small to absorb the norm $\|v\|_{L^2}^2$ to the left hand side we conclude

$$\|\eta_1 \Lambda u\|_{L^2(\mathbb{R}^n)}^2 = \|v\|_{L^2(\mathbb{R}^n)}^2 \leq C_\gamma \|\eta_1 \Lambda Lu\|_{L^2(\mathbb{R}^n)}^2 + C_\gamma \|u\|_{H^0(\mathbb{R}^n)}^2,$$

for a constant C_γ depending on γ . ■

3.2.1. Removal of the smoothness assumption. It remains to remove the smoothness assumption $u \in C^{2,\delta}(U)$ in Lemma 31, and to convert the above *a priori* estimate (3.15) to the desired conclusion $\Lambda u \in H^0$ of Theorem 12. For this we fix a strictly positive smooth function $r \in C^\infty(\mathbb{R}^n)$ such that

$$r(\xi) \equiv \begin{cases} |\xi|^{-1} & \text{for } |\xi| \geq 2 \\ 1 & \text{for } |\xi| \leq 1 \end{cases},$$

and we fix a large exponent q . For $\varepsilon > 0$ small define a mollified symbol

$$\begin{aligned} \lambda_\varepsilon(x, \xi) &= r_\varepsilon(\xi) \cdot \lambda(x, \xi) = r(\varepsilon\xi)^q \cdot \lambda(x, \xi); \\ \text{where } r_\varepsilon(\xi) &\equiv r(\varepsilon\xi)^q. \end{aligned}$$

where $\lambda(x, \xi) = |\xi|^\gamma e^{-N_0(\log|\xi|)\phi(x, \xi)}$ for $|\xi| \geq e$ as in (3.4). Let $\Lambda_\varepsilon = \text{Op } \lambda_\varepsilon$. The symbols $r_\varepsilon(\xi)$ satisfy

$$(3.17) \quad \frac{|\partial_\xi^\alpha r_\varepsilon|}{r_\varepsilon} \leq C_{\alpha, q} |\xi|^{-|\alpha|}, \quad \text{uniformly in } \varepsilon > 0 \text{ and } \xi \in \mathbb{R}^n.$$

If q is chosen sufficiently large relative to the order of the distribution u , then $\Lambda_\varepsilon u \in C^2$ for all $\varepsilon > 0$, and since Λ_ε is elliptic of order γ in a conic neighbourhood of (x_0, ξ_0) , it suffices to show that the L^2 norm of $\eta_1 \Lambda_\varepsilon u$ remains uniformly bounded as $\varepsilon \searrow 0$. However, Lemma 31 fails to apply since we do not know that the distribution u is a function in $C^{2, \delta}(U)$, and we now work to circumvent this difficulty.

The parameter N_0 in (3.4) can be chosen sufficiently large that $\eta_1 \Lambda Lu \in L^2$ because ϕ is strictly positive in a conic neighbourhood of the H^γ wave front set of u , and hence Λ is regularizing there of order at least $\gamma - \sigma N_0$ for some constant $\sigma > 0$. The L^2 norm of $\eta_1 \Lambda_\varepsilon Lu$ is bounded uniformly in $\varepsilon > 0$ and tends to the L^2 norm of $\eta_1 \Lambda Lu$.

As in the proof of Lemma 31, we have for each $\varepsilon > 0$, an operator G_ε and an identity

$$\begin{aligned} (L_1 + G_\varepsilon) \eta_1 \Lambda_\varepsilon u &= \eta_1 \Lambda_\varepsilon L_1 u + R_\varepsilon u, \\ (L_2 + E_\varepsilon) \eta_1 \Lambda_\varepsilon u &= \eta_1 \Lambda_\varepsilon L_2 u + R_\varepsilon u, \\ (R_1 + J_\varepsilon) \eta_1 \Lambda_\varepsilon u &= \eta_1 \Lambda_\varepsilon R_1 u + R_\varepsilon u, \end{aligned}$$

with both sides of the equation in C^2 for each $\varepsilon > 0$. Moreover, the differential inequalities (3.17) ensure that the proof of Lemma 31 carries through for each $\varepsilon > 0$ with Λ replaced by Λ_ε , so that G_ε takes the form (3.7), i.e.

$$\begin{aligned} G_\varepsilon &= \sum_j B_{j, \varepsilon} \circ X_j + \sum_j X_j^{\text{tr}} \circ \widetilde{B_{j, \varepsilon}} + B_{0, \varepsilon}, \\ B_{0, \varepsilon} &\in \text{Op} \left(\mathcal{C}^{0, \delta} S_{1, \eta}^{0, 2} \right) \text{ and } B_{j, \varepsilon}, \widetilde{B_{j, \varepsilon}} \in \text{Op} \left(\mathcal{C}^{1, \delta} S_{1, \eta}^{0, 1} \right), \end{aligned}$$

where the pseudodifferential operator coefficients $B_{0, \varepsilon}$, $B_{j, \varepsilon}$ and $\widetilde{B_{j, \varepsilon}}$ lie uniformly in the indicated operator classes. A similar argument holds for E_ε and J_ε . All functions have sufficient differentiability for the proof of Lemma 31 to apply, and this proof, together with the above identity, yield

$$\|\eta_1 \Lambda_\varepsilon u\|_{L^2(R)} \leq C \|\eta_1 \Lambda_\varepsilon Lu\|_{L^2(R)} + C \|u\|_{H^0(R)},$$

uniformly in $\varepsilon > 0$. We conclude as desired that the L^2 norm of $\eta_1 \Lambda_\varepsilon u$ remains bounded as $\varepsilon \searrow 0$.

Thus we have proved that for any distribution $u \in \mathcal{D}'(V)$, and any $0 < \gamma < \delta$, there is a symbol Λ as in (3.4) that is elliptic of order γ on the conical set Γ , and satisfies

$$\|\eta_1 \Lambda u\|_{L^2(R)} \leq C \|\eta_1 \Lambda Lu\|_{L^2(R)} + C \|u\|_{H^0(R)}.$$

The proof of Theorem 12 is now complete.

Combined with the bootstrapping argument above, this shows that $u \in H_{\text{loc}}^s(R)$ for all $s \in \mathbb{R}$. Indeed, $\eta_2 u \in H^{-M}(R)$ for some M sufficiently large, and thus we can begin the bootstrapping argument at $s = -M$.

4. PROOF OF THEOREM 11

We now prove Theorem 11. The first step is to use a bootstrapping argument to reduce matters to the level of $L^2(\mathbb{R}^n)$. Consider the general second order divergence form operator

$$Lu(x) \equiv \nabla^{\text{tr}} A(x) \nabla u(x) + D(x) u(x),$$

where A and D are real and smooth, and where $A(x)$ satisfies appropriate form comparability conditions. In order to conclude hypoellipticity of L it is enough to show that there is $\gamma > 0$ such that for every $s \in \mathbb{R}$, we have the bootstrapping inequality

$$u \in H_{\text{loc}}^s(\mathbb{R}^n) \text{ and } Lu \in H_{\text{loc}}^{s+\gamma}(\mathbb{R}^n) \implies u \in H_{\text{loc}}^{s+\gamma}(\mathbb{R}^n) \quad \text{for all } s \in \mathbb{R}.$$

Now with $\widehat{\Lambda}_s(\xi) \equiv (1 + |\xi|^2)^{\frac{s}{2}}$, and $\gamma > 0$ fixed, it suffices to show

$$u \in H_{\text{loc}}^0(\mathbb{R}^n) \text{ and } \Lambda_s L \Lambda_{-s} u \in H_{\text{loc}}^\gamma(\mathbb{R}^n) \implies u \in H_{\text{loc}}^\gamma(\mathbb{R}^n) \quad \text{for all } s \in \mathbb{R}.$$

For $s \geq 0$ we use

$$\Lambda_s L \Lambda_{-s} = (\Lambda_s L - L \Lambda_s) \Lambda_{-s} + L = [\Lambda_s, L] \Lambda_{-s} + L,$$

and for $s \leq 0$ we use

$$\Lambda_s L \Lambda_{-s} = -\Lambda_s (\Lambda_{-s} L - L \Lambda_{-s}) + L = -\Lambda_s [\Lambda_{-s}, L] + L,$$

to conclude that it suffices to prove

$$(4.1) \quad \begin{aligned} u &\in H_{\text{loc}}^0(\mathbb{R}^n) \text{ and } [\Lambda_s, L] \Lambda_{-s} u \in H_{\text{loc}}^\gamma(\mathbb{R}^n) \implies u \in H_{\text{loc}}^\gamma(\mathbb{R}^n) & \text{for all } s \geq 0, \\ u &\in H_{\text{loc}}^0(\mathbb{R}^n) \text{ and } [\Lambda_{-s}, L] \Lambda_s u \in H_{\text{loc}}^\gamma(\mathbb{R}^n) \implies u \in H_{\text{loc}}^\gamma(\mathbb{R}^n) & \text{for all } s \leq 0. \end{aligned}$$

The second step is to use the sum of squares assumption in part (1) of Theorem 11 to show that it is sufficient to establish the conditions of Theorem 12. So define

$$(4.2) \quad \tilde{G} \equiv [\Lambda_s, L] \Lambda_{-s} = \Lambda_s L \Lambda_{-s} - L,$$

and suppose for the moment that the operator L has the simple form

$$(4.3) \quad L = \sum_j X_j^{\text{tr}} X_j,$$

where $L \in S_{1,0}^2$ is smooth and $X_j \in C^{2,\delta}$. We first establish the properties of $\tilde{G}$ we need using the rough version of asymptotic expansion from [Saw] given in Theorem 21 above, which we repeat here for the reader's convenience.

Suppose $\sigma \in \mathcal{C}^\nu S_{1,\delta_1}^{m_1}$ and $\tau \in \mathcal{C}^{M+\mu+\nu} S_{1,\delta_2}^{m_2}$ where M is a nonnegative integer, $0 < \mu, \delta_1, \delta_2 < 1$, $\nu > 0$ and $M + \mu \geq m_1 \geq 0$. Let $\delta \equiv \max\{\delta_1, \delta_2\}$. Then

$$\begin{aligned} \sigma \circ \tau &= \sum_{\ell=0}^M \frac{1}{i^\ell \ell!} \nabla_\xi^\ell \sigma \cdot \nabla_x^\ell \tau + E; \\ E &\in \mathcal{O}_{(-(1-\delta)\nu, \nu)}^{m_1+m_2+(M+\mu)(\delta_2-1)+\varepsilon}, \quad \text{for every } \varepsilon > 0. \end{aligned}$$

Lemma 32. *Let L and $\tilde{G}$ be as in (4.3) and (4.2). Then*

$$(4.4) \quad \begin{aligned} \tilde{G} &= \sum_j B_j \circ X_j + \sum_j X_j^{\text{tr}} \circ \widetilde{B_j} + B_0, \\ B_0 &\in \mathcal{O}_{(-\delta/2, \delta/2)}^{-\delta/2+\varepsilon} \text{ for every } \varepsilon > 0, \text{ and } B_j, \widetilde{B_j} \in \text{Op}(C^{1,\delta} S_{1,0}^0). \end{aligned}$$

Proof. First we note that

$$[\Lambda_s, L] = \sum_j [\Lambda_s, X_j^{\text{tr}}] X_j + X_j^{\text{tr}} [\Lambda_s, X_j],$$

and so we investigate operators $[\Lambda_s, X_j^{\text{tr}}]$ and $[\Lambda_s, X_j]$. The analysis is similar, so we only give details for $[\Lambda_s, X_j]$. Using Theorem (21) with $m_1 = s$, $m_2 = 1$, $M = 1$, $\mu = 1 + \delta/2$, $\nu = \delta/2$ and $\delta_1 = \delta_2 = 0$ we have

$$\sigma([\Lambda_s, X_j]) = C \nabla_\xi \left(1 + |\xi|^2\right)^{\frac{s}{2}} \cdot \nabla_x \sigma(X_j) + E,$$

where $E \in \mathcal{O}_{(-\delta/2, \delta/2)}^{1+s-(2+\delta/2)+\varepsilon}$. Composing with Λ_{-s} and using $\text{Op}\left(\nabla_\xi \left(1 + |\xi|^2\right)^{\frac{s}{2}}\right) = R^{-1} \circ \Lambda_s$, where $R^{-1} \in S_{1,0}^{-1}$, we obtain

$$X_j^{\text{tr}} [\Lambda_s, X_j] \Lambda_{-s} = X_j^{\text{tr}} \circ \widetilde{B_j} + R$$

with $\widetilde{B_j} \in C^{1,\delta} S_{1,0}^0$ and $R \in \mathcal{O}_{(-\delta/2, \delta/2)}^{-\delta/2+\varepsilon}$. ■

Now we start with an operator $L \in S_{1,0}^2$ of the more general form

$$(4.5) \quad L = \sum_j X_j^{\text{tr}} X_j + A_0 + \widehat{\nabla}^{\text{tr}} \cdot \mathbf{Q}_p(x) \widehat{\nabla},$$

where $X_j \in C^{2,\delta}$ and $A_0 \in S_{1,0}^1$. Using Lemma 32 for any operator L in the form (4.5) and Remark 13 we can show that the operator $\Lambda_s L \Lambda_{-s}$ has the form

$$\Lambda_s L \Lambda_{-s} = \sum_j X_j^{\text{tr}} X_j + \sum_j B_j X_j + \sum_j X_j^{\text{tr}} \widetilde{B_j} + B_0 + R_1 + \widehat{\nabla}^{\text{tr}} \mathbf{Q}_p(x) \widehat{\nabla},$$

where X_j , B_j , $\tilde{B}_j$, and B_0 are as in Lemma 32 and R_1 is as in Theorem 12. Thus to show hypoellipticity of the operator (4.5), it is sufficient to show that it satisfies the hypotheses of Theorem 12, which completes the second step of the proof.

We prepare for the final step of the proof with an auxiliary Lemma (see [Chr, Lemma 5.1]), and its corollary to be used later for showing condition (2.9).

Lemma 33. *Let $\varphi \in C_0^2(\mathbb{R}^n)$, $f \in C^\infty(\mathbb{R}^n)$ simply positive, and $s > 0$. Then for any $l \in \{1, \dots, n\}$ there exists a constant C_l independent of s such that*

$$(4.6) \quad \|\varphi\|^2 \leq C_l \left(\frac{1}{\tau^2 [\min_{|x| \geq s} f(x)]^2} + s^2 \right) \left(\|\partial_{x_l} \varphi\|^2 + \int \tau^2 f(x)^2 \varphi(x)^2 dx \right),$$

where the minimum is taken over all $x \in \text{supp } \varphi$ s.t. $|x| \geq s$.

Proof. Fix $s > 0$, for any $x \in \mathbb{R}^n$ we have

$$\begin{aligned} \varphi(x) &= \varphi \left(x + s \frac{x_l}{|x_l|} \right) - \int_1^{1+s/|x_l|} \frac{\partial \varphi}{\partial t}(x_1, \dots, x_{l-1}, tx_l, x_{l+1}, \dots, x_n) dt \\ \varphi^2(x) &\lesssim \varphi^2 \left(x + s \frac{x_l}{|x_l|} \right) + \left(\int_1^{1+s/|x_l|} \nabla \varphi(x_1, \dots, x_{l-1}, tx_l, x_{l+1}, \dots, x_n) \cdot (0, \dots, x_l, 0, \dots, 0) dt \right)^2 \\ \int_{|x_l| \leq s} \varphi^2(x) dx &\lesssim \int_{|x_l| \leq s} \varphi^2 \left(x + s \frac{x_l}{|x_l|} \right) dx \\ &\quad + \int_{|x_l| \leq s} \left(\int_1^{1+s/|x_l|} |\partial_l \varphi(x_1, \dots, x_{l-1}, tx_l, x_{l+1}, \dots, x_n) t^{3/4} x_l|^2 dt \int_1^{1+s/|x_l|} t^{-3/2} dt \right) dx \\ \int_{|x_l| \leq s} \varphi^2(x) dx &\lesssim \int_{s \leq |x_l| \leq 2s} \varphi^2(x) dx + \int_{|x_l| \leq s} \int_1^{1+s/|x_l|} |\partial_l \varphi(x_1, \dots, x_{l-1}, tx_l, x_{l+1}, \dots, x_n) t^{3/4} x_l|^2 dt dx. \end{aligned}$$

Switching the order of integration in the last term on the right and making a change of variables $y = (x_1, \dots, x_{l-1}, tx_l, x_{l+1}, \dots, x_n)$ we obtain

$$\int_{|x_l| \leq s} \int_1^{1+s/|x_l|} |\partial_l \varphi(x_1, \dots, x_{l-1}, tx_l, x_{l+1}, \dots, x_n) t^{3/4} x_l|^2 dt dx \leq \int_1^\infty \int_{|y_l| \leq 2s} |\partial_l \varphi(y) y_l|^2 t^{-1/2} \frac{dy}{t} dt \lesssim s^2 \int |\partial_l \varphi(y)|^2 dy,$$

which combining with the above gives

$$\int_{|x_l| \leq s} \varphi^2(x) dx \lesssim \int_{s \leq |x_l| \leq 2s} \varphi^2(x) dx + s^2 \int |\partial_l \varphi(x)|^2 dx.$$

Finally,

$$\int_{|x_l| \geq s} \tau^2 f(x)^2 \varphi(x)^2 dx \geq \tau^2 [\min_{|x_l| \geq s} f(x)]^2 \int_{|x_l| \geq s} \varphi^2(x) dx,$$

and thus altogether

$$\int \varphi^2(x) dx \lesssim \frac{1}{\tau^2 [\min_{|x_l| \geq s} f(x)]^2} \int_{|x_l| \geq s} \tau^2 f(x)^2 \varphi(x)^2 dx + s^2 \int |\partial_l \varphi(x)|^2 dx$$

which implies (4.6). ■

Lemma 34. *Let φ and f as in Lemma 33. There exists a strictly positive continuous function w satisfying $w(\tau) \rightarrow \infty$ as $\tau \rightarrow \infty$ such that for every $l \in \{1, \dots, n\}$ and some constant $C_l > 0$*

$$(4.7) \quad \int w(\tau)^2 \varphi(x)^2 dx \leq C_l \int (|\partial_l \varphi(x)|^2 + \tau^2 f(x)^2 \varphi(x)^2) dx.$$

Proof. For all $s \geq 0$ define

$$f_0(s) \equiv \min_{x \in \text{supp } \varphi: |x| \geq s} f(x),$$

and note that $f_0(0) = 0$, $f_0(s) > 0$ for $s \neq 0$, and f_0 is nondecreasing on $[0, \infty)$. Let $r = r(\tau) > 0$ be the unique point satisfying

$$(4.8) \quad \frac{1}{r} = \tau f_0(r).$$

Define the function w by

$$w(\tau) = \inf_{0 < s < \infty} \left(\frac{1}{s} + \tau f_0(s) \right),$$

since $1/s$ is nonincreasing and $f_0(s)$ nondecreasing in s we have $w(\tau) \approx 1/r$ where r is given by (4.8). Therefore, $w(\tau) \rightarrow \infty$ as $\tau \rightarrow \infty$ and using (4.6) with $s = r$ we obtain

$$\begin{aligned} \int w(\tau)^2 \varphi(x)^2 dx &\leq C_l \frac{1}{r^2} \left(\frac{1}{\tau^2 f_0(r)^2} + r^2 \right) \int (|\partial_l \varphi(x)|^2 + \tau^2 f(x)^2 \varphi(x)^2) dx \\ &\leq C_l \int (|\partial_l \varphi(x)|^2 + \tau^2 f(x)^2 \varphi(x)^2) dx. \end{aligned}$$

■

4.1. Sufficiency. We can now proceed to complete the sufficiency part of Theorem 11. We note that without loss of generality we may assume that the diagonal entries $\lambda_k(\tilde{x})$ are smooth. Indeed, from $A(x) \sim D_\lambda(x)$ we obtain $A(x) \sim A_{\text{diag}}(x)$ and hence

$$(4.9) \quad \lambda_k(\tilde{x}) \approx a_{k,k}(x) \approx a_{k,k}(\tilde{x}, 0, 0),$$

where the functions $a_{k,k}(\tilde{x}, 0, 0)$ are smooth for $1 \leq k \leq n$ by assumption.

Proof of sufficiency in Theorem 11. Let $(\xi_1, \dots, \xi_m, \eta_{m+1}, \dots, \eta_n)$ denote the dual variables, and denote $\xi = (\xi_1, \dots, \xi_m)$, $\eta = (\eta_{m+1}, \dots, \eta_n)$, $\tilde{x} = (x_1, \dots, x_m)$. Define

$$R = \{(x, \xi, \eta) : x = 0, \xi = 0, \eta_{m+1}, \dots, \eta_n > 0\}.$$

The principal symbol of L vanishes on the manifold $\tilde{x} = \xi = 0$, so it suffices to prove that $Lu \in H^s(\mathfrak{N}(R)) \implies u \in H^s(\mathfrak{N}(R))$ for some conical neighbourhood $\mathfrak{N}(R)$ of the ray R . We start with verifying condition (2.9). Let $\mathcal{F}(u)(\tilde{x}, \eta)$ be the partial Fourier transform of u in $n - m$ variables η , then from Lemma 34 with $x = \tilde{x}$ and $\varphi(\tilde{x}) = \mathcal{F}(u)(\tilde{x}, \eta)$, we have for $k = m + 1, \dots, p - 1$

$$\int w(\eta_k)^2 \mathcal{F}(u)(\tilde{x}, \eta)^2 d\tilde{x} \leq C \int (|\nabla_{\tilde{x}} \mathcal{F}(u)(\tilde{x}, \eta)|^2 + \eta_k^2 \lambda_k(\tilde{x}) \mathcal{F}(u)(\tilde{x}, \eta)^2) d\tilde{x},$$

and for $k = p, \dots, n$

$$\int w(\eta_k)^2 \mathcal{F}(u)(\tilde{x}, \eta)^2 d\tilde{x} \leq C \int (|\nabla_{\tilde{x}} \mathcal{F}(u)(\tilde{x}, \eta)|^2 + \eta_k^2 \lambda_p(\tilde{x}) \mathcal{F}(u)(\tilde{x}, \eta)^2) d\tilde{x},$$

where $w(s) \rightarrow \infty$ as $s \rightarrow \infty$. Adding the inequalities together gives

$$\begin{aligned} &\int w(|\eta|)^2 \mathcal{F}(u)(\tilde{x}, \eta)^2 d\tilde{x} \\ &\leq C \int \left(|\nabla_{\tilde{x}} \mathcal{F}(u)(\tilde{x}, \eta)|^2 + \left[\sum_{k=m+1}^{p-1} \eta_k^2 \lambda_k(\tilde{x}) + \sum_{k=p}^n \eta_k^2 \lambda_p(\tilde{x}) \right] \mathcal{F}(u)(\tilde{x}, \eta)^2 \right) d\tilde{x}, \end{aligned}$$

where $w(|\eta|) \rightarrow \infty$ as $|\eta| \rightarrow \infty$. Combining with the first line in (2.5) we obtain

$$\int_{\mathbb{R}^n} w(|(\eta)|)^2 \mathcal{F}(u)(\tilde{x}, \eta)^2 d\tilde{x} d\eta \leq C \sum_j \|X_j u\|^2 + C \left\| \sqrt{\lambda_p} \widehat{\nabla} u \right\|^2,$$

which gives upon using the first condition in (2.5) again

$$\begin{aligned} \int \min\{(|(\xi, \eta)|, w(|(\xi, \eta)|))\}^2 \hat{u}(\xi, \eta)^2 d\xi d\eta &\lesssim \int_{|\xi| \leq |\eta|} w(|\eta|)^2 |\hat{u}(\xi, \eta)|^2 d\xi d\eta + \int_{|\xi| \geq |\eta|} |\xi|^2 |\hat{u}(\xi, \eta)|^2 d\xi d\eta \\ &\leq C \sum_j \|X_j u\|^2 + C \left\| \sqrt{\lambda_p} \widehat{\nabla} u \right\|^2. \end{aligned}$$

We proceed to verify (2.10) with $p \equiv 0$ and ψ constructed below. Since the principal symbol of the operator vanishes on $\mathbb{R}^{n-m} \times \mathbb{R}^{n-m}$, namely when $\tilde{x} = \xi = 0$, we need to localize matters to a strip $\left| \left(\tilde{x}, \frac{\xi}{|\eta|} \right) \right| < \rho$

where ψ enjoys favorable commutation relations with the symbol $\sigma(X_j)$ of the vector field X_j . So let $p \equiv 0$ and let $\rho > 0$. Let $\psi \in C^\infty(T^{\text{tr}}V)$ be homogeneous of degree 0 with respect to (ξ, η) and satisfy

$$\begin{cases} \psi = 1, & \text{if } \left| \left(x, \frac{\xi}{|\eta|} \right) \right| \geq 3\rho \\ \psi = 0, & \text{if } \left| \left(x, \frac{\xi}{|\eta|} \right) \right| \leq \rho \\ \psi = \psi(x_{m+1}, \dots, x_n), & \text{if } \left| \left(\tilde{x}, \frac{\xi}{|\eta|} \right) \right| \leq 2\rho \end{cases}.$$

Thus ψ is 1 outside a large ball of radius 3ρ , vanishes inside a small ball of radius ρ , and makes the transition from 0 to 1 in the strip while depending only on the variables $\tilde{x}$ in the strip $\left| \left(\tilde{x}, \frac{\xi}{|\eta|} \right) \right| \leq 2\rho$. In the strip $\left| \left(\tilde{x}, \frac{\xi}{|\eta|} \right) \right| < \rho$, ψ is a function of variables $x_{m+1}, \dots, x_n$ only, and the main step of Christ's application of his theorem occurs now: for each $j = 1, \dots, k$ there exist $a_\ell^j(\tilde{x})$, $\ell = m+1, \dots, n$ such that

$$\{\psi, \sigma(X_j)\} = i \sum_{\ell=m+1}^n a_\ell^j(\tilde{x}) \partial_{x_\ell} \psi,$$

$$|a_\ell^j(\tilde{x})| \lesssim \sqrt{\lambda_\ell(\tilde{x})}, \quad \ell = m+1, \dots, p-1, \quad |a_\ell^j(\tilde{x})| \lesssim \sqrt{\lambda_p(\tilde{x})}, \quad \ell = p, \dots, n$$

using conditions (2.5), and

$$\{\psi, \eta\} = i \hat{\nabla} \psi.$$

Using the condition $|\xi| \leq \rho |\eta|$ this gives for each $j = 1, \dots, N$

$$\begin{aligned} \|\text{Op}[\log\langle(\xi, \eta)\rangle\{\psi, \sigma(X_j)\}]u\|^2 &\lesssim \sum_{\ell=m+1}^{p-1} \left\| \text{Op}\left[\sqrt{\lambda_\ell(\tilde{x})} \log\langle\eta\rangle\right]u \right\|^2 + \left\| \text{Op}\left[\sqrt{\lambda_p(\tilde{x})} \log\langle\eta\rangle\right]u \right\|^2 \\ &= \int \Lambda_{\text{sum}}(\tilde{x}) \log\langle\eta\rangle^2 \mathcal{F}(u)(\tilde{x}, \eta)^2 d\tilde{x} d\eta, \end{aligned}$$

and

$$\begin{aligned} \left\| \sqrt{\mathbf{Q}_p} \text{Op}[\log\langle(\xi, \eta)\rangle\{\psi, \eta\}]u \right\|^2 &\lesssim \left\| \sqrt{\lambda_p} \text{Op}[\log\langle\eta\rangle]u \right\|^2 \\ &\lesssim \int \Lambda_{\text{sum}}(\tilde{x}) \log\langle\eta\rangle^2 \mathcal{F}(u)(\tilde{x}, \eta)^2 d\tilde{x} d\eta, \end{aligned}$$

upon using the definition of $\Lambda_{\text{sum}}(\tilde{x})$. To show (2.10) it is therefore sufficient to establish the first inequality in the following display (since the second follows directly from (2.5))

$$\begin{aligned} (4.10) \quad &\int \log\langle\eta\rangle^2 \Lambda_{\text{sum}}(\tilde{x}) \mathcal{F}(u)(\tilde{x}, \eta)^2 d\tilde{x} d\eta \lesssim \delta \int |\nabla_{\tilde{x}} \mathcal{F}(u)(\tilde{x}, \eta, \tau)|^2 d\tilde{x} d\eta d\tau \\ &+ \delta \int \left[\sum_{k=m+1}^{p-1} \eta_k^2 \lambda_k(\tilde{x}) + \sum_{k=p}^n \eta_k^2 \lambda_p(\tilde{x}) \right] \mathcal{F}(u)(\tilde{x}, \eta)^2 d\tilde{x} d\eta + C_\delta \|u\|^2 \\ &\lesssim \delta \sum_{j=1}^N \|X_j u\|^2 + \delta \|\sqrt{\lambda_p} \hat{\nabla} u\|^2 + C_\delta \|u\|^2. \end{aligned}$$

Using the definitions of $\Lambda_{\text{sum}}(\tilde{x})$ and $\Lambda_{\text{product}}(\tilde{x})$ we conclude that it is sufficient to show

$$(4.11) \quad (\log \tau)^2 \|\sqrt{\Lambda_{\text{sum}}} \varphi\|^2 \leq \delta(\tau) \|\nabla_{\tilde{x}} \varphi\|^2 + \delta(\tau) \tau^2 \|\sqrt{\Lambda_{\text{product}}} \varphi\|^2, \quad \text{for all } \varphi \in C_0^1(\mathbb{R}^m),$$

where $\delta(\tau) \rightarrow 0$ as $\tau \rightarrow \infty$. Indeed, (4.11) together with the bound $0 \leq \lambda_j \leq 1$ implies

$$\begin{aligned} &\int \log\langle\eta\rangle^2 \Lambda_{\text{sum}} \varphi(\tilde{x})^2 d\tilde{x} \leq \delta(\langle\eta\rangle) \|\nabla_{\tilde{x}} \varphi\|^2 + \delta(\langle\eta\rangle) \langle\eta\rangle^2 \|\sqrt{\Lambda_{\text{product}}} \varphi\|^2 \\ &\leq \delta(\langle\eta\rangle) \|\nabla_{\tilde{x}} \varphi\|^2 + \delta(\langle\eta\rangle) \left[\sum_{k=m+1}^{p-1} |\eta_k|^2 \|\sqrt{\lambda_k} \varphi\|^2 + \sum_{k=p}^n |\eta_k|^2 \|\sqrt{\lambda_p} \varphi\|^2 \right] + C_\delta \|\varphi\|^2. \end{aligned}$$

This implies (4.10) by splitting the region of integration into $|\eta|$ sufficiently large so that $\delta(\langle\eta\rangle) \leq \delta$, and the region where $|\eta|$ is bounded, and thus the left hand side of (4.10) is bounded by $C||u||^2$.

To establish (4.11), we first recall for convenience the Koike condition

$$(4.12) \quad \lim_{\tilde{x} \rightarrow 0} \mu(|\tilde{x}|, \sqrt{\Lambda_{\text{sum}}}) \ln \Lambda_{\text{product}}(\tilde{x}) = 0.$$

Now let $\phi \in C_0^1(B(0, r))$. Then we then have with $\phi_{\tilde{y}}(\rho) \equiv \phi(\rho\tilde{y})$,

$$(4.13) \quad \begin{aligned} \int_{|\tilde{x}| \leq r} \Lambda_{\text{sum}}(\tilde{x}) \phi(\tilde{x})^2 d\tilde{x} &= \int_{|\tilde{x}| \leq r} \Lambda_{\text{sum}}(\tilde{x}) (r - |\tilde{x}|)^2 \frac{\phi(\tilde{x})^2}{(r - |\tilde{x}|)^2} d\tilde{x} \\ &\leq \mu\left(r, \sqrt{\Lambda_{\text{sum}}}\right)^2 \int_{|\tilde{x}| \leq r} \frac{\phi(\tilde{x})^2}{(r - |\tilde{x}|)^2} d\tilde{x} = \mu\left(r, \sqrt{\Lambda_{\text{sum}}}\right)^2 \int_{\mathbb{S}^{m-1}} \left\{ \int_0^r \left(\frac{1}{r - \rho} \int_\rho^r \phi'_{\tilde{y}}(\rho) \right)^2 \rho^{m-1} d\rho \right\} d\tilde{y} \\ &\leq \mu\left(r, \sqrt{\Lambda_{\text{sum}}}\right)^2 \int_{\mathbb{S}^{m-1}} \left\{ 4 \int_0^r \phi'_{\tilde{y}}(\rho)^2 \rho^{m-1} d\rho \right\} d\tilde{y} \leq 4\mu\left(r, \sqrt{\Lambda_{\text{sum}}}\right)^2 \int |\nabla_{\tilde{x}} \phi(\tilde{x})|^2, \end{aligned}$$

where in the last line we have applied Hardy's inequality.

Fix $\varphi \in C_0^1(\mathbb{R}^m)$ as in (4.11). Let $\chi \in C_0^1(\mathbb{R}^1)$ satisfy $\chi(t) = 1$ for $|t| \leq 1$ and $\chi(t) = 0$ for $|t| \geq 2$, and define the function

$$(4.14) \quad \nu(\tilde{x}) \equiv \chi(\tau \Lambda_{\text{product}}(\tilde{x})),$$

and the set

$$I(\tau) \equiv \{\tilde{x} \in \text{Supp } \varphi : \tau \Lambda_{\text{product}}(\tilde{x}) > 1\}.$$

We can write

$$(4.15) \quad \int \Lambda_{\text{sum}}(\tilde{x}) \varphi(\tilde{x})^2 d\tilde{x} \leq 2 \int \Lambda_{\text{sum}}(\tilde{x}) \nu(\tilde{x})^2 \varphi(\tilde{x})^2 d\tilde{x} + 2 \int \Lambda_{\text{sum}}(\tilde{x}) (1 - \nu(\tilde{x}))^2 \varphi(\tilde{x})^2 d\tilde{x}.$$

To estimate the second integral we notice that it vanishes outside the set $I(\tau)$ and thus

$$(4.16) \quad \begin{aligned} (\log \tau)^2 \int \Lambda_{\text{sum}}(\tilde{x}) (1 - \nu(\tilde{x}))^2 \varphi(\tilde{x})^2 d\tilde{x} &\leq (\log \tau)^2 \int_{I(\tau)} \Lambda_{\text{product}}(\tilde{x}) \varphi(\tilde{x})^2 d\tilde{x} \\ &= \delta(\tau) \tau^2 \int \Lambda_{\text{product}}(\tilde{x}) \varphi(\tilde{x})^2 d\tilde{x}, \end{aligned}$$

where $\delta(\tau) = (\log \tau)^2 \tau^{-1} \rightarrow 0$ as $\tau \rightarrow \infty$.

To estimate the first integral on the right hand side of (4.15) we define

$$r(\tau) \equiv \sup\{|\tilde{y}| : \tilde{y} \in \text{Supp } \varphi : \tau \Lambda_{\text{product}}(\tilde{y}) \leq 2\}.$$

Since $\text{Supp } \varphi$ is compact, the supremum above is attained at some point $\tilde{z} \in \text{Supp } \varphi$, and moreover we have both

$$|\tilde{z}| = r \text{ and } \tau = \frac{2}{\Lambda_{\text{product}}(\tilde{z})}.$$

Thus $\ln \tau \approx \ln \frac{1}{\Lambda_{\text{product}}(\tilde{z})}$ and so

$$\mu\left(r(\tau), \sqrt{\Lambda_{\text{sum}}}\right) \ln r(\tau) \approx \mu\left(|\tilde{z}|, \sqrt{\Lambda_{\text{sum}}}\right) \ln \frac{1}{\Lambda_{\text{product}}(\tilde{z})}.$$

The Koike condition condition (4.12) now implies

$$(4.17) \quad \lim_{\tau \rightarrow \infty} \mu(r(\tau), \sqrt{\Lambda_{\text{sum}}}) \ln r(\tau) = \lim_{\tilde{x} \rightarrow 0} \mu(|\tilde{x}|, \sqrt{\Lambda_{\text{sum}}}) \ln \frac{1}{\Lambda_{\text{product}}(\tilde{x})} = 0,$$

since $r(\tau) \rightarrow 0$ as $\tau \rightarrow \infty$. We now need to combine this result with (4.13) to obtain the desired estimate. Let $\phi(\tilde{x}) = \nu(\tilde{x})\varphi(\tilde{x})$. Then using the definition of $\nu(\tilde{x})$ in (4.14) we obtain

$$\begin{aligned} \int |\nabla_{\tilde{x}} \phi(\tilde{x})|^2 d\tilde{x} &\leq C \int |\nabla_{\tilde{x}} \nu(\tilde{x})|^2 \varphi(\tilde{x})^2 d\tilde{x} + C \int \nu(\tilde{x})^2 |\nabla_{\tilde{x}} \varphi(\tilde{x})|^2 d\tilde{x} \\ &\leq C\tau^2 \int_{I(\tau)} |\nabla_{\tilde{x}} \Lambda_{\text{product}}(\tilde{x})|^2 \varphi(\tilde{x})^2 d\tilde{x} + C \int |\nabla_{\tilde{x}} \varphi(\tilde{x})|^2 d\tilde{x} \\ &\leq C\tau^2 \int_{I(\tau)} \Lambda_{\text{product}}(\tilde{x}) \varphi(\tilde{x})^2 d\tilde{x} + C \int |\nabla_{\tilde{x}} \varphi(\tilde{x})|^2 d\tilde{x}, \end{aligned}$$

where in the last inequality we used the Malgrange inequality, see e.g. [Gla, Lemme I], applied to $\Lambda_{\text{product}}(\tilde{x}) = \prod_{k=m+1}^p \lambda_k(\tilde{x})$, where the functions λ_k are smooth by (4.9). Finally, from the definition of r and (4.14) it follows that

$$\text{Supp } \phi \subset \text{Supp } \nu \subset \left\{ \tilde{y} : \tau < \frac{2}{\Lambda_{\text{product}}(\tilde{y})} \right\} \subset B(0, r(\tau)).$$

since if $|\tilde{x}| > r(\tau)$, then $\tau \Lambda_{\text{product}}(\tilde{y}) > 2$ by the definition of $r(\tau)$.

Combining the above estimate with (4.17) and (4.13) we conclude that

$$\begin{aligned} (\log \tau)^2 \int \Lambda_{\text{sum}}(\tilde{x}) \nu(\tilde{x})^2 \varphi(\tilde{x})^2 d\tilde{x} &= (\log \tau)^2 \int \Lambda_{\text{sum}}(\tilde{x}) \phi(\tilde{x})^2 d\tilde{x} \\ &\leq \delta(\tau) \left(\tau^2 \int \Lambda_{\text{product}}(\tilde{x}) \varphi(\tilde{x})^2 d\tilde{x} + \int |\nabla_{\tilde{x}} \varphi(\tilde{x})|^2 d\tilde{x} \right) \end{aligned}$$

with $\delta(\tau) = C\mu(r, \sqrt{\Lambda_{\text{sum}}})^2 (\log \tau)^2 \rightarrow 0$ as $\tau \rightarrow \infty$. Together with (4.16) this gives (4.11). ■

4.2. Sharpness. We now turn to the sharpness portion of Theorem 11. If the Koike condition (4.12) fails, then

$$\begin{aligned} 0 &< \limsup_{\tilde{x} \rightarrow 0} \mu \left(|\tilde{x}|, \sqrt{\Lambda_{\text{sum}}} \right) \ln \frac{1}{\Lambda_{\text{product}}(\tilde{x})} \\ &= \limsup_{\tilde{x} \rightarrow 0} \mu \left(|\tilde{x}|, \sqrt{\sum_{k=m+1}^p \lambda_k(\tilde{x})} \right) \ln \prod_{k=m+1}^p \frac{1}{\lambda_k(\tilde{x})} \\ &\leq \limsup_{\tilde{x} \rightarrow 0} \mu \left(|\tilde{x}|, \sum_{k=m+1}^p \sqrt{\lambda_k(\tilde{x})} \right) \sum_{j=m+1}^p \ln \frac{1}{\lambda_j(\tilde{x})} \\ &\leq \sum_{k,j=m+1}^p \limsup_{\tilde{x} \rightarrow 0} \mu \left(|\tilde{x}|, \sqrt{\lambda_k(\tilde{x})} \right) \ln \frac{1}{\lambda_j(\tilde{x})} \end{aligned}$$

shows that $p > m+1$ (since $\limsup_{\tilde{x} \rightarrow 0} \mu \left(|\tilde{x}|, \sqrt{\lambda_p(\tilde{x})} \right) \ln \frac{1}{\lambda_p(\tilde{x})} = 0$) and that there is a pair of distinct indices $k, j \in \{m+1, \dots, p\}$ such that

$$\limsup_{\tilde{x} \rightarrow 0} \mu \left(|\tilde{x}|, \sqrt{\lambda_k(\tilde{x})} \right) \ln \frac{1}{\lambda_j(\tilde{x})} > 0.$$

Our sharpness assertion in Theorem 11 now follows immediately from Proposition 36 and Theorem 37 below.

To prove the Proposition and Theorem, we will need the following lemma (see Hoshiro [Hos, (2.7)]), whose short proof we include here for the reader's convenience.

Lemma 35 (T. Hoshiro [Hos]). *Let L be a hypoelliptic operator on $\mathbb{R}^n$. For any multiindex β and any subsets Ω, Ω' of $\mathbb{R}^n$ such that $\Omega' \Subset \Omega$, there exists $N \in \mathbb{N}$ and $C > 0$ such that*

$$(4.18) \quad \|D^\beta u\|_{L^2(\Omega')}^2 \leq C \left(\sum_{|\alpha| \leq N} \|D^\alpha L u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 \right) \quad \forall u \in C^\infty(\overline{\Omega}).$$

Proof. Fix $\Omega' \Subset \Omega$ and consider the set

$$S \equiv \{u \in L^2(\Omega') : D^\alpha Lu \in L^2(\Omega') \text{ for all multiindices } \alpha\}.$$

The family of seminorms $\|u\|_{L^2(\Omega')}, \|D^\alpha Lu\|_{L^2(\Omega')}, |\alpha| \in \mathbb{N}$, makes it a Fréchet space. Since L is hypoelliptic we have $S \subset C^\infty(\Omega')$, and in particular $S \subset C^M(\Omega')$ for any $M > 0$. Now consider the inclusion map

$$T : S \rightarrow C^M(\Omega'),$$

we claim T is closed. Indeed, suppose $\{u_n\} \subset S$ satisfies $u_n \rightarrow u$ in S and $u_n \rightarrow v$ in $C^M(\Omega')$, in particular, $u_n \rightarrow u$ in $L^2(\Omega')$ and $u_n \rightarrow v$ in $L^\infty(\Omega')$. Then for any $n \in \mathbb{N}$

$$\|u - v\|_{L^2(\Omega')} \leq \|u - u_n\|_{L^2(\Omega')} + \|u_n - v\|_{L^2(\Omega')} \leq \|u - u_n\|_{L^2(\Omega')} + \|u_n - v\|_{L^\infty(\Omega')} |\Omega'|^{\frac{1}{2}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This implies $u = v$, i.e. T is closed. By the closed graph theorem T is continuous, and therefore there exists $N \in \mathbb{N}$ and $C > 0$ such that

$$\|u\|_{C^M(\Omega')} \leq C \left(\sum_{|\alpha| \leq N} \|D^\alpha Lu\|_{L^2(\Omega')}^2 + \|u\|_{L^2(\Omega')}^2 \right).$$

Since the choice of M was arbitrary, this implies (4.18). ■

Proposition 36. Fix distinct indices $k, j \in \{m+1, \dots, p\}$ where $p > m+1$. Define

$$\begin{aligned} L_1 &\equiv \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_m^2} + \lambda_k(x_1, \dots, x_m) \frac{\partial^2}{\partial x_k^2} + \lambda_j(x_1, \dots, x_m) \frac{\partial^2}{\partial x_j^2}, \\ L_2 &\equiv \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_m^2} + \sum_{i=m+1}^p \lambda_i(x_1, \dots, x_m) \frac{\partial^2}{\partial x_i^2} + \sum_{i=p+1}^n \lambda_p(x_1, \dots, x_m) \frac{\partial^2}{\partial x_i^2}. \end{aligned}$$

If L_1 is not hypoelliptic in $\mathbb{R}^{m+2}$, then L_2 is not hypoelliptic in $\mathbb{R}^n$.

Proof. Suppose L_1 is not hypoelliptic in $\mathbb{R}^{m+2}$, i.e. there exists a non smooth function $u = u(x_1, \dots, x_m, x_k, x_j)$ such that $L_1 u \in C^\infty(\mathbb{R}^{m+2})$. If we define the function v by

$$v(x_1, \dots, x_n) = u(x_1, \dots, x_m, x_k, x_j),$$

then v is not smooth since u is not smooth. However,

$$L_2 v(x_1, \dots, x_n) = L_1 u(x_1, \dots, x_m, x_k, x_j)$$

and is therefore smooth in $\mathbb{R}^n$. ■

Theorem 37. Suppose that $h, f \in C^\infty(\mathbb{R}^m)$ are strongly monotone, i.e.

$$f(z) \leq f(x) \text{ and } h(z) \leq h(x) \text{ for all } z \in B(0, |x|),$$

and satisfy $h(x), f(x) \geq 0$ and $h(0) = f(0) = 0$ for all $x \in \mathbb{R}^m$. Define

$$(4.19) \quad \mu(t, h) \equiv \max\{h(z)(t - |z|) : 0 \leq |z| \leq t\}.$$

and suppose in addition that

$$(4.20) \quad \liminf_{x \rightarrow 0} \mu(|x|, h) \ln f(x) \neq 0.$$

Then the operator

$$\mathcal{L} \equiv \Delta_x + f^2(x) \frac{\partial^2}{\partial y^2} + h^2(x) \frac{\partial^2}{\partial t^2}$$

fails to be C^∞ -hypoelliptic in $\mathbb{R}^{m+2}$.

Proof. For $a, \eta > 0$ consider the second order operator $L_\eta \equiv -\Delta_x + f^2(x) \eta^2$ and the eigenvalue problem

$$\begin{aligned} L_\eta v(x, \eta) &= \lambda h^2(x) v(x, \eta), & x &\in B(0, a), \\ v(x) &= 0, & x &\in \partial B(0, a). \end{aligned}$$

The least eigenvalue is given by the Rayleigh quotient formula

$$\begin{aligned}
 \lambda_0(a, \eta) &= \inf_{\varphi(\neq 0) \in C_0^\infty(B)} \frac{\langle L_\eta \varphi, \varphi \rangle_{L^2}}{\langle h^2 \varphi, \varphi \rangle_{L^2}} \\
 (4.21) \quad &= \inf_{\varphi(\neq 0) \in C_0^\infty(B)} \frac{\int_B |\nabla \varphi|^2 dx + \int_B f^2(x) \eta^2 \varphi^2 dx}{\int_B h(x)^2 \varphi^2 dx}.
 \end{aligned}$$

Next, from (4.20) it follows that there exists $\varepsilon > 0$ and sequences $\{a_n\}, \{b_n\} \subset \mathbb{R}^m$ s.t. $|a_n| < |b_n| \leq 1$, $b_n \rightarrow 0$, and

$$(4.22) \quad h(a_n) (|b_n| - |a_n|) |\ln f(b_n)| \geq \varepsilon, \quad \forall n \in \mathbb{N}.$$

Let

$$\eta_n = \frac{1}{f(b_n)} \rightarrow \infty \quad \text{as } n \rightarrow \infty,$$

By strong monotonicity of f and h we have

$$\eta_n f(x) \leq 1, \quad h(x) \geq h(a_n) \quad \forall x \in R_n \equiv \{x \in \mathbb{R}^m : |a_n| \leq |x| \leq |b_n|\}.$$

This implies using (4.21)

$$\begin{aligned}
 \lambda_0(|b_n|, \eta_n) &\leq \inf_{\varphi(\neq 0) \in C_0^\infty(R_n)} \frac{\langle L_{\eta_n} \varphi, \varphi \rangle_{L^2}}{\langle h^2 \varphi, \varphi \rangle_{L^2}} \\
 &\leq h(a_n)^{-2} \inf_{\varphi(\neq 0) \in C_0^\infty(R_n)} \{(\|\nabla \varphi\|^2 + \|\varphi\|^2) / \|\varphi\|^2\} \\
 &\leq h(a_n)^{-2} (C(|b_n| - |a_n|)^{-2} + 1) \leq C |\ln f(b_n)|^2 = C (\ln \eta_n)^2,
 \end{aligned}$$

where we used (4.22) and the definition of η_n for the last two inequalities. It also follows from (4.21) and the fact that $|b_n| \leq 1$

$$(4.23) \quad \lambda_0(1, \eta_n) \leq \lambda_0(|b_n|, \eta_n) \leq C_1 (\ln \eta_n)^2.$$

Now let $v_0(x, \eta_n)$ be an eigenfunction on the ball $B = B(0, 1)$ associated with $\lambda_0(1, \eta_n)$ i.e.

$$-\Delta v_0(x, \eta_n) = [\lambda_0(1, \eta_n) h^2(x) - f^2(x) \eta_n^2] v_0(x, \eta_n),$$

and normalized so that

$$(4.24) \quad \|v_0(\cdot, \eta_n)\|_{L^2(B)} = 1.$$

We first claim that

$$(4.25) \quad \|v_0(\cdot, \eta_n)\|_{L^2((1/2)B)} \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Indeed, we have

$$\begin{aligned}
 &\inf_{1/2 < |x| < 1} f^2(x) \eta_n^2 \int_{1/2 < |x| < 1} |v_0(x, \eta_n)|^2 dx \\
 &\leq \int_B f^2(x) \eta_n^2 |v_0(x, \eta_n)|^2 dx \\
 &\leq \int_B |\nabla v_0(x, \eta_n)|^2 dx + \int_B f^2(x) \eta_n^2 |v_0(x, \eta_n)|^2 dx \\
 &= \lambda_0(1, \eta_n) \int_B h^2(x) |v_0(x, \eta_n)|^2 dx \leq C \lambda_0(1, \eta_n).
 \end{aligned}$$

Dividing both sides by $\inf_{1/2 < |x| < 1} f^2(x) \eta_n^2$ and using (4.23) we obtain that

$$\int_{1/2 < |x| < 1} |v_0(x, \eta_n)|^2 dx \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which implies (4.25).

Define a sequence of functions

$$u_n(x, y, t) = e^{iy\eta_n + \sqrt{\lambda_0(1, \eta_n)}t} v_0(x, \eta_n).$$

Then

$$\mathcal{L}u_n = (\Delta v_0(x, \eta_n) - \eta_n^2 f^2(x) v_0(x, \eta_n) + \lambda_0(1, \eta_n) v_0(x, \eta_n)) e^{iy\eta_n + \sqrt{\lambda_0(1, \eta_n)}t} = 0.$$

Now, let $V = B(0, 1) \times [-\pi, \pi] \times [-\delta, \delta]$ and $V' = B(0, 1/2) \times [-\pi/2, \pi/2] \times [-\delta/2, \delta/2]$ for some $\delta > 0$. We have using (4.25)

$$\|\partial_y^k u_n\|_{L^2(V')}^2 = \eta_n^{2k} \|u_n\|_{L^2(V')}^2 \geq \pi \eta_n^{2k} \int_{1/2B} \int_0^{\delta/2} e^{2\sqrt{\lambda_0(1, \eta_n)}t} |v_0(x, \eta_n)|^2 dt dx \geq C \eta_n^{2k},$$

where the constant C is independent of k and n . On the other hand, using (4.23)

$$\|u_n\|_{L^2(V)}^2 \leq C e^{2\sqrt{\lambda_0(1, \eta_n)}\delta} \leq C \eta_n^{2\sqrt{C_1}\delta}.$$

Since $\eta_n \rightarrow \infty$ as $n \rightarrow \infty$, these two inequalities contradict (4.18) for $k > \sqrt{C_1}\delta$, and thus by Lemma 35 the operator $\mathcal{L}$ is not hypoelliptic. ■

5. PROOF OF THEOREM 9

Finally, we prove Theorem 9 by showing that the requirements of Theorem 11 are satisfied. Let L be as in (2.1). We apply Theorem 3 to obtain $\mathbf{A} = \sum_{j=1}^N Y_j Y_j^{\text{tr}} + A_p$, and write the second order term in L as

$$\nabla^{\text{tr}} \mathbf{A} \nabla = \sum_{j=1}^N \nabla^{\text{tr}} Y_j Y_j^{\text{tr}} \nabla = \sum_{j=1}^N X_j^{\text{tr}} X_j + \widehat{\nabla}^{\text{tr}} \mathbf{Q}_p \widehat{\nabla}, \quad \text{where } X_j = Y_j^{\text{tr}} \nabla,$$

and then note that condition (2.7) is satisfied by the assumption (2.4) of Theorem 9. Moreover, condition (2.5) follows from (1.7).

6. OPEN PROBLEMS

6.1. First problem. In Theorem 9 we have shown that the Koike condition is sufficient for the hypoellipticity of an operator L with $n \times n$ matrix $A(x)$ satisfying certain conditions on both its diagonal and nondiagonal entries. However, in the converse direction we only showed that failure of the Koike condition implies failure of hypoellipticity if in addition L is diagonal with strongly monotone entries. In fact the proof shows that we need only assume in addition that $A(x)$ has the block form

$$A(x) = \begin{bmatrix} \begin{bmatrix} a_{1,1}(x) & \cdots & a_{n,1}(x) \\ \vdots & \ddots & \vdots \\ a_{1,n}(x) & \cdots & a_{m,m}(x) \end{bmatrix} & \mathbf{0}_{m \times 1} & \mathbf{0}_{m \times 1} & \cdots & \mathbf{0}_{m \times 1} \\ \mathbf{0}_{1 \times m} & a_{m+1,m+1}(x) & 0 & \cdots & 0 \\ \mathbf{0}_{1 \times m} & 0 & a_{m+2,m+2}(x) & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{1 \times m} & 0 & 0 & \cdots & a_{n,n}(x) \end{bmatrix}.$$

where just $a_{m+1,m+1}(x)$ and $a_{n,n}(x)$ are assumed to be strongly monotone and satisfy (4.20).

Problem 38. *Is the Koike condition actually necessary and sufficient for hypoellipticity under the assumptions of Theorem 9, **without** assuming the above block form for $A(x)$?*

6.2. Second problem. Recall that the main theorem in [KoRi] extends Kohn's theorem in [Koh] to apply with finitely many blocks instead of the two blocks used in [Koh]. These operators are restricted by being of a certain block form, but they are more general in that the elliptic blocks are multiplied by smooth functions that are positive outside the origin, and have more variables than in our theorems, and furthermore that need not be finite sums of squares of regular functions.

Problem 39. *Can Theorem 11 be extended to more general operators that include the operators appearing in [KoRi]?*

6.3. Third problem. What sort of smooth lower order terms of the form $B(x) \nabla$ and $\nabla^{\text{tr}} C(x)$ can we add to the operator L in the main Theorem 9? The natural hypothesis to make on the vector fields $B(x) \nabla$ and $C(x) \nabla$ is that they are subunit with respect to $\nabla^{\text{tr}} A(x) \nabla$. However, if we use Theorem 11 in the proof, we require more, namely that $B(x) \nabla$ and $C(x) \nabla$ are linear combinations, with $C^{2,\delta}$ coefficients, of the $C^{2,\delta}$ vector fields $X_j(x)$ arising in the sum of squares Theorem 3, something which seems difficult to arrange more generally.

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REED COLLEGE, PORTLAND, OREGON, USA, KOROBENKO@REED.EDU

MCMASTER UNIVERSITY, HAMILTON, ONTARIO, CANADA, SAWYER@MCMASTER.CA