CONTINUITY OF INFINITELY DEGENERATE WEAK SOLUTIONS VIA THE TRACE METHOD

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ABSTRACT. In 1971 Fediĭ proved in [Fe] the remarkable theorem that the linear second order partial differential operator

$$L_f u(x,y) \equiv \left\{ \frac{\partial}{\partial x^2} + f(x)^2 \frac{\partial}{\partial y^2} \right\} u(x,y)$$

is *hypoelliptic* provided that $f \in C^\infty(\mathbb{R})$, $f(0)=0$ and f is positive on $(-\infty,0)\cup(0,\infty)$. Variants of this result, with hypoellipticity replaced by continuity of weak solutions, were recently given by the authors, together with Cristian Rios and Ruipeng Shen, in [KoRiSaSh] to infinitely degenerate elliptic divergence form equations

$$\nabla^{\mathrm{tr}} \mathcal{A}(x,u) \nabla u = \phi(x), \qquad x \in \Omega \subset \mathbb{R}^n,$$

where the nonnegative matrix $\mathcal{A}(x,u)$ has *bounded measurable coefficients* with trace roughly 1 and determinant comparable to f^2, and where $F = \ln \frac{1}{f}$ is essentially doubling.

However, in the plane, these variants assumed additional geometric constraints on f, such as $f(r) \geq e^{-r^{-\sigma}}$ for some $0<\sigma<1$, something not required in Fediĭ's theorem. In this paper we in particular remove these additional geometric constraints in the plane for homogeneous equations, and only assume that f is positive away from the origin.

CONTENTS

1. INTRODUCTION

In 1971 Fediĭ proved in [Fe] the remarkable theorem that the linear second order partial differential operator

$$L_f u(x,y) \equiv \left\{ \frac{\partial}{\partial x^2} + f(x)^2 \frac{\partial}{\partial y^2} \right\} u(x,y)$$

is *hypoelliptic*, i.e. every distribution solution $u \in \mathcal{D}'(\mathbb{R}^2)$ to the equation $L_f u = \phi \in C^\infty(\mathbb{R}^2)$ in $\mathbb{R}^2$ is smooth, i.e. $u \in C^\infty(\mathbb{R}^2)$, provided that $f \in C^\infty(\mathbb{R})$, $f(0)=0$ and f is positive on $(-\infty,0)\cup(0,\infty)$. In particular f can vanish to *infinite* order and L_f is infinitely degenerate elliptic. See also [KuStr], [Mor], [Chr] and [Koh] for generalizations to *smooth* equations in higher dimensions, something we do not pursue here.

To handle *quasilinear* operators using linear theory, one is forced to consider matrices with bounded measurable coefficients. In the elliptic case this theory is well-developed and appears for example in Gilbarg and Trudinger [GiTr] and many other sources. The key breakthrough there was the Hölder *apriori* estimate of De Giorgi, and its later generalizations independently by Nash and Moser. The extension of the De Giorgi-Nash-Moser theory to the subelliptic or finite type setting, originated in the work of Lanconelli [Lan] and a series of papers by Franchi and Lanconelli [FrLan1]-[FrLan5],[Fr], and continued by many authors.

The regularity theory of subelliptic linear equations with smooth coefficients is well established, as evidenced by the results of Hörmander [Ho] and Fefferman and Phong [FePh]. In [Ho], Hörmander obtained hypoellipticity of sums of squares of smooth vector fields whose Lie algebra spans at every point. In [FePh], Fefferman and Phong considered general nonnegative semidefinite smooth linear operators, and characterized subellipticity in terms of a containment condition involving Euclidean balls and "subunit" balls related to the geometry of the nonnegative semidefinite form associated to the operator. In the infinitely degenerate regime, these balls become *nondoubling*, and the classical theory breaks down.

Variants of Fediĭ's result were obtained in [KoRiSaSh] to infinitely degenerate elliptic divergence form equations

$$\nabla^{\mathrm{tr}} A(x) \nabla u = \phi(x), \qquad x \in \Omega \subset \mathbb{R}^n,$$

where the nonnegative matrix A has *bounded measurable coefficients* with trace roughly 1 and determinant comparable to f^2. Here and everywhere else in this paper we follow the notation ∇^{tr} to denote the transpose of the gradient vector, or the divergence operator. The concept of hypoellipticity was interpreted there in terms of local boundedness and continuity of weak solutions. However, additional geometric constraints on the degeneracy f were needed for the methods used there, beyond the minimal restriction that $F \equiv \ln \frac{1}{f}$ be a 'structured geometry', i.e. satisfies the five 'log doubling' structure conditions in Definition 1 below (useful for estimating arc length and volume of control balls). While these additional geometric constraints were often necessary in dimension $n \geq 3$, as is the case for the smooth equations in higher dimensions in [KuStr] and [Chr], the case of dimension $n = 2$ was left open in the rough setting. We note that the restrictions imposed in Definition 1 were crucial for the measure estimates for the control balls associated to the geometry F, obtained in [KoRiSaSh].

The main goal of this paper is to extend this type of result to the plane $\mathbb{R}^2$ for all geometries without *any additional geometric constraints*, i.e. all functions f positive outside the origin, even, and strictly increasing on some interval $(0, R)$. More precisely, to certain equations in the plane $\mathbb{R}^2$ with nonegative matrices having bounded measurable coefficients with trace roughly 1 and determinant f^2, and with at least bounded forcing functions ϕ. One immediate application of our main result is weak hypoellipticity of a class of smooth operators. Let $A(x,y) = A(x)$ be smooth and comparable to $\mathrm{diag}\{1, f^2(x)\}$, where f is even and strictly increasing and positive on some interval $(0, R)$. Assume in addition that the matrix A is subordinate as defined in [RSaW], i.e. $|A'(x)\xi|^2 \leq C\xi^{\mathrm{tr}} A(x)\xi$ for all $\xi \in \mathbb{R}^n$ and a.e. $x \in \Omega$. Then every weak solution to $\nabla^{\mathrm{tr}} A \nabla u = 0$ in Ω is smooth. Indeed, Theorem 2.18 in [RSaW] shows that every weak *continuous* solution to such an equation is smooth, and our continuity result allows us to get rid of this additional assumption. We point out that there is no known result in the literature that applies in this situation, as $\nabla^{\mathrm{tr}} A \nabla$ is not assumed to be a sum of squares of smooth vector fields[1], so the results of Christ [Chr] and Morimoto [Mor] do not seem to apply; neither is the operator assumed to have the special form considered by Kohn [Koh]. On the other hand, we must impose the somewhat restrictive additional assumption of subordinaticity in order to be able to apply the result in [RSaW].

To achieve our main goal, we develop a *trace method* that first constructs a region in $\mathbb{R}^2$ on whose boundary a given subsolution u has a suitable trace, and second applies a maximum principle to derive local boundedness and continuity of weak solutions u to some infinitely degenerate equations, resulting in our Trace Method Theorem. We will use both an existing maximum principle for inhomogeneous equations from [KoRiSaSh], that requires a restriction on the geometry F associated with the operator, as well as a new maximum principle for homogeneous equations, valid for all geometries, and all dimensions as well. Table 1 organizes the various conclusions on weak solutions so that they either persist or improve as we move to the right or lower down in the table.

There are three separate notions of admissibilty of inhomogeneous data ϕ appearing in this table: the strongest notion is that used in [KoRiSaSh2] which amounts to assuming the data are very close to L^∞; the weakest notion is that in [KoRiSaSh], denoted $\phi \in X_f$ here; and an intermediate notion, called f-admissible, that is used in the current paper. The boxes are color coded as follows: continuity results in red boxes require additional geometric constraints and the strongest notion of admissibility of inhomogeneous data, and were proved in [KoRiSaSh2][2]; local boundedness results in red boxes require less stringent additional geometric constraints and the weakest notion of admissibility of inhomogeneous data, and were proved in [KoRiSaSh]; results in black boxes are valid for all geometries and homogeneous equations, and are proved here; and the single continuity result in the purple box requires a geometric constraint and the intermediate notion of admissibility of inhomogeneous data, and is also proved here. In all cases the inhomogeneous data include

[1]It is an open question whether or not every smooth nonnegative function on the real line can be written as a finite sum of squares of smooth functions.

[2]The arXiv article [KoRiSaSh2] contains all of the continuity results stated here for inhomogeneous equations, that require the strong constraint $F_{3,\sigma}$ on the geometry, and also a stronger restriction on the inhomogeneous data ϕ. These results were obtained there using an infinitely degenerate Moser scheme that is considerably more complicated than the adaptation of the De Giorgi / Caffarelli / Vasseur scheme used in [KoRiSaSh].

bounded measurable functions ϕ. The above results are described in more detail below, see e.g. Definition 3 for the meaning of f-admissible, and Definition 2 for the meaning of $F_{k,\sigma}$.

Data	$\phi(x,y)$	$\phi(x,\cdot)$ ρ cont unif in x	0
$\mathcal{A}(x,y,u(x,y))$	loc bdd $F_\sigma,\ \sigma<1$ cont $F_{3,\sigma},\ \sigma<1$	loc bdd $F_\sigma,\ \sigma<1$ cont $F_{3,\sigma},\ \sigma<1$	loc bdd all F
$A(x)$	loc bdd $F_\sigma,\ \sigma<1$ cont $F_{3,\sigma},\ \sigma<1$	cont $F_\sigma,\ \sigma<1$	cont all F

TABLE 1. Brief summary of applications

More specifically, we will consider replacing the Fediĭ operator L_f above with a more general second order divergence form special quasilinear operator $\mathcal{L}=\nabla^{\mathrm{tr}}\mathcal{A}(x,y,u)\nabla$ in $\mathbb{R}^2$ with bounded measurable coefficients, and we will consider the special quasilinear equations (special because $\mathcal{A}$ is independent of ∇u) and restricted linear equations (restricted because A is independent of y),

$$\begin{aligned}\textbf{special quasilinear} \quad &: \quad \mathcal{L}u=\nabla^{\mathrm{tr}}\mathcal{A}(x,y,u)\nabla u=\phi,\\ \textbf{restricted linear} \quad &: \quad Lu=\nabla^{\mathrm{tr}}A(x)\nabla u=\phi,\end{aligned}$$

where ϕ is f-admissible as in Definition 3. Roughly speaking, we prove the following five new results for such second order divergence form operators in the plane with 'geometry' f, which is either assumed to be a bounded measurable function which is even, and positive and strictly increasing on some interval $(0,R)$; or satisfy a stronger condition of a 'structured geometry', i.e. $F=\ln\frac{1}{f}$ satisfies Definition 3 below, which essentially says that f is strictly increasing and F is a doubling function with some normalizing conditions.

Notation 1. *We will often make mention of the plane 'geometry' associated with the functions $f(r)$ or $F(r)=\ln\frac{1}{f(r)}$. By this we mean the geometry of metric balls defined in* [KoRiSaSh, Chapter 7] *using the degenerate Riemannian metric $dt^2=dx^2+\frac{1}{f(x)^2}dy^2$, with its associated control distance d_f*[3]*. Given two geometries represented by $F(r)$ and $G(r)$, we say that F is* stronger*, or* less degenerate*, than G, if $F(r)\le G(r)$ for sufficiently small $r>0$. More generally, if the 2×2 matrix $\mathcal{A}(x,y,u(x,y))$ associated with a divergence form operator $\mathcal{L}$ is comparable to a diagonal matrix $D_f=\begin{bmatrix}1&0\\0&f(x)^2\end{bmatrix}$, then we refer to this geometry as being associated with $\mathcal{L}$ or with $\mathcal{A}(x,y,u(x,y))$.*

(1) (**homogeneous maximum principle**) For *any* geometry, a maximum principle holds for weak subsolutions to homogeneous equations $\mathcal{L}u=0$. This has an extension to all dimensions $n\ge 2$.
(2) (**special homogeneous quasilinear equation**) For *any* geometry, weak solutions u to a homogeneous special quasilinear equation $\mathcal{L}u=0$ are locally bounded.
(3) (**special inhomogeneous quasilinear equation**) If the structured geometry F is stronger than F_σ for some $\sigma<1$, i.e. $F(r)\le F_\sigma(r)$ for $r>0$ sufficiently small, then weak solutions u to a special quasilinear equation $\mathcal{L}u=\phi$ are locally bounded provided the forcing function ϕ is f-admissible.
(4) (**restricted linear homogeneous equation**) For *any* geometry, if $\mathcal{A}(x,y,u)=A(x)$ depends only on x, then weak solutions u to the homogeneous restricted linear equation $Lu=0$ are continuous.
(5) (**restricted linear equation with forcing function continuous in** y) If the structured geometry F is stronger than F_σ for some $\sigma<1$, then weak solutions u to a restricted linear equation $Lu=\phi$ are continuous provided the f-admissible forcing function $\phi(x,y)$ is continuous in y with modulus of continuity uniform in x.

Statement (1) is Theorem 2, and the reader should have no difficulty in deriving statements (2), (3), (4) and (5) from the Trace Method Theorem 3 and the maximum principles in Theorems 1 and 2. Indeed, statements (2) and (4) for homogeneous equations use Theorems 2 and 3; while statements (3) and (5) use Theorems 1 and 3.

The two main new results listed above are statements (2) and (4), which require no additional geometric assumptions on the geometry of the operator $\mathcal{L}$ other than f being even, positive away from zero, and strictly

[3]Recall that a vector $\mathbf{v}$ is subunit for an invertible symmetric matrix A, i.e. $(\mathbf{v}\cdot\boldsymbol{\xi})^2\le\boldsymbol{\xi}^{\mathrm{tr}}A\boldsymbol{\xi}$ for all $\boldsymbol{\xi}$, if and only if $\mathbf{v}^{\mathrm{tr}}A^{-1}\mathbf{v}\le 1$, see e.g. [KoRiSaSh, Chapter 7].

increasing on some interval $(0,R)$[4]; thus giving results closer in spirit to Fediĭ's theorem, which required no geometric assumptions other than that f is positive away from 0. After a section on preliminaries, which makes precise the conditions surrounding our equations, the following two sections prove the new homogeneous maximum principle in $\mathbb{R}^n$, and the Trace Method Theorem in $\mathbb{R}^2$ respectively.

1.1. **Preliminaries.** We begin with the second order special quasilinear equation (where only u, and not ∇u, appears nonlinearly),

$$\mathcal{L}u \equiv \nabla^{\mathrm{tr}} \mathcal{A}\left(x,y,u\left(x,y\right)\right) \nabla u = \phi, \qquad (x,y) \in \Omega, \tag{1.1}$$

where Ω is a bounded domain in the plane $\mathbb{R}^2$, and we assume the following quadratic form condition on the 'quasilinear' matrix $\mathcal{A}(x,y,z)$,

$$c\,\xi^T D_f(x)\xi \le \xi^T \mathcal{A}(x,y,z)\xi \le C\,\xi^T D_f(x)\xi\ , \tag{1.2}$$

for a.e. $(x,y) \in \Omega$ and all $z \in \mathbb{R}$, $\xi \in \mathbb{R}^2$, where c, C are positive constants. Equivalently, the 2×2 matrix $\mathcal{A}\left(x,y,z\right)$ has bounded measurable coefficients and is comparable to the following diagonal matrix $D_f\left(x\right)$ depending only on x,

$$D_f\left(x\right) \equiv \left[\begin{array}{cc} 1 & 0 \\ 0 & f\left(x\right)^2 \end{array}\right].$$

Define the f-gradient by

$$\nabla_f = \sqrt{D_f}\left(x\right)\nabla\ , \tag{1.3}$$

and the associated degenerate Sobolev space $W^{1,p}_f\left(\Omega\right)$, $p \ge 1$, to have norm

$$\|v\|_{W^{1,p}_f} \equiv \left(\int_\Omega \left(|v|^p + |\nabla_f v|^p\right)\right)^{\frac{1}{p}}.$$

Note that if $A\left(x\right)$ is comparable to $D_f\left(x\right)$ and $p=2$, then

$$\|v\|_{W^{1,2}_f} = \sqrt{\int_\Omega \left(|v|^2 + \nabla v^{\mathrm{tr}} D_f \nabla v\right)} \approx \sqrt{\int_\Omega \left(|v|^2 + \nabla v^{\mathrm{tr}} A \nabla v\right)},$$

which shows that ∇_f is an appropriate gradient to use in connection with the operator $\mathcal{L}$. We say $u \in W^{1,2}_f\left(\Omega\right)$ is a $W^{1,2}_f\left(\Omega\right)$-weak solution to $\mathcal{L}u = \phi$ if

$$-\int \left(\nabla w\right)^{\mathrm{tr}} \mathcal{A}\left(x,y,u\left(x,y\right)\right)\nabla u = \int \phi w, \qquad \text{for all } w \in W^{1,2}_f\left(\Omega\right)_0\ .$$

If equality is replaced by $\ge$ (or $\le$), u is called a weak sub- (or super-) solution respectively. Throughout this paper we will assume that the degeneracy function $f\left(r\right) = e^{-F(r)}$ is even and strictly increasing on some interval $(0,R)$, and in some cases we will also assume that there is $R > 0$ such that F satisfies the following five structure conditions from [KoRiSaSh] for some constants $C \ge 1$ and $\varepsilon > 0$. Note that in this paper only results for inhomogeneous equations use the full force of these structure assumptions as they rely on the results from [KoRiSaSh] used in the proofs, namely Theorem 1. In particular, the main abstract result, Theorem 3, only requires the function f to be even and strictly increasing on some interval $(0,R)$.

Definition 1 (structure conditions)**.** *A twice continuously differentiable function $F:(0,R) \to \mathbb{R}$ is said to satisfy geometric structure conditions, or to be a structured geometry, if:*

1. $\lim_{t\to 0^+} F\left(t\right) = +\infty$;
2. $F'\left(t\right) < 0$ *and* $F''\left(t\right) > 0$ *for all* $t \in (0,R)$;
3. $\frac{1}{C}\left|F'\left(r\right)\right| \le \left|F'\left(t\right)\right| \le C\left|F'\left(r\right)\right|$ *for* $\frac{1}{2}r < t < 2r < R$;
4. $\frac{1}{-tF'(t)}$ *is increasing in the interval* $(0,R)$ *and satisfies* $\frac{1}{-tF'(t)} \le \frac{1}{\varepsilon}$ *for* $t \in (0,R)$;
5. $\frac{F''(t)}{-F'(t)} \approx \frac{1}{t}$ *for* $t \in (0,R)$.

We let $B(x,r)$ denote a subunit ball centered at x radius r, associated with the geometry F, as defined in [KoRiSaSh, Chapter 7].

[4] These conclusions were obtained in [KoRiSaSh] under stronger geometric assumptions on the operator $\mathcal{L}$, namely $F \le F_\sigma$ for local boundedness, and $F \le F_{3,\sigma}$ for continuity, where $0 < \sigma < 1$.

Definition 2. *Let $\ln^{(k)}$ denote the k-fold composition of the natural logarithm function, and let $r_0 > 0$ be sufficiently small, depending on k, so that $\ln^{(k)}(1/r_0)$ is well defined and positive. For $0 < r < r_0$ define*

$$\begin{aligned} F_\sigma(r) &\equiv \left(\frac{1}{r}\right)^\sigma, \qquad \sigma > 0, \\ F_{k,\sigma}(r) &\equiv \left(\ln\frac{1}{r}\right)\left(\ln^{(k)}\frac{1}{r}\right)^\sigma, \qquad \sigma > 0 \text{ and } k \in \mathbb{N}. \end{aligned}$$

The functions F_σ and $F_{k,\sigma}$ are examples of functions satisfying geometric structure conditions as above. Note that $f_\sigma = e^{-F_\sigma}$ vanishes to infinite order at $r = 0$, and that f_σ vanishes to a faster order than $f_{\sigma'}$ if $\sigma > \sigma'$. A similar remark applies to $f_{k,\sigma} = e^{-F_{k,\sigma}}$. The first part of the next definition originates in [KoRiSaSh, see Definition 4]. Here and throughout the paper we will use the notation $B_{\mathrm{Euc}}((x,y), r)$ to denote the Euclidean ball centered at (x, y), radius r.

Definition 3. *Fix a bounded domain $\Omega \subset \mathbb{R}^2$. Define the space $X_f(\Omega)$ to consist of all functions ϕ on $\mathbb{R}^2$ such that*

$$\|\phi\|_{X_f(\Omega)} \equiv \sup_{v \in \left(W_f^{1,1}\right)_0(\Omega)} \frac{\int_\Omega |v\phi|}{\int_\Omega |\nabla_f v|} < \infty.$$

We say that ϕ is f-admissible *in Ω if both $\phi \in X_f(\Omega)$ and ϕ satisfies the following L^q growth condition in Ω,*

$$\|\phi\|_{L^q_{\mathrm{growth}}(\Omega)} \equiv \sup_{\substack{(x,y)\in\Omega\setminus\{y\text{-}axis\} \\ B_{\mathrm{Euc}}\left((x,y),\frac{|x|}{2}\right)\subset\Omega}} \|\phi\|_{L^q\left(B_{\mathrm{Euc}}\left((x,y),\frac{|x|}{2}\right)\right)} < \infty, \qquad \textit{for some } q > 1.$$

We norm the f-admissible functions with

$$\|\phi\|_{f-\mathrm{adm}(\Omega)} \equiv \|\phi\|_{X_f(\Omega)} + \|\phi\|_{L^q_{\mathrm{growth}}(\Omega)}.$$

The point of including L^q_{growth} in the definition of *f-admissible* is so that we can apply standard elliptic theory as in [GiTr] away from the y-axis with appropriate uniformity. We will apply the definition of f-admissible to forcing functions ϕ only for structured geometries f. In connection with the definition of $\|\phi\|_{X_f(\Omega)}$, note that $\int_\Omega |\nabla_f v| \approx \|v\|_{W_f^{1,1}(\Omega)}$ by the $1-1$ Poincaré inequality analogous to (2.3) in Proposition 2 below. Finally note that bounded functions are f-admissible; $\|\phi\|_{f-\mathrm{adm}(\Omega)} \lesssim \|\phi\|_{L^\infty(\Omega)}$.

Definition 4. *We say a function $u \in W_f^{1,2}(\Omega)$ is* bounded by a constant $\ell \in \mathbb{R}$ *on the boundary $\partial\Omega$ if $(u-\ell)^+ = \max\{u-\ell, 0\} \in \left(W_f^{1,2}\right)_0(\Omega)$. We define $\sup_{x\in\partial\Omega} u(x)$ to be $\inf\left\{\ell \in \mathbb{R} : (u-\ell)^+ \in \left(W_f^{1,2}\right)_0(\Omega)\right\}$.*

Before we start stating our main new results, we recall the geometric maximum principle for weak subsolutions to inhomogeneous equations given in [KoRiSaSh, Theorem 83] under additional restrictions on the geometry F associated with the form A of the operator. We restrict our attention to the case $n = 2$ as that is the only dimension in which we obtain new results for $W_f^{1,2}(\Omega)$-weak solutions. Namely, we assume that $f(x) \neq 0$ if $x \neq 0$, and that F satisfies the five geometric structure conditions in Definition 1. Note that the admissibility requirement for ϕ in the next theorem is the weakest one, $\phi \in X_f(\Omega)$. This is Theorem 83 in [KoRiSaSh]

Theorem 1 (inhomogeneous maximum principle)**.** *Suppose that $F = \ln\frac{1}{f}$ is a structured geometry, i.e. satisfies the structure conditions in Definition 1, and is stronger than F_σ for some $0 < \sigma < 1$. Assume that u is a weak subsolution to $\mathcal{L}u = \phi$ in a domain $\Omega \subset \mathbb{R}^2$, where $\mathcal{L} = \nabla^{\mathrm{tr}}\mathcal{A}\nabla$ and $\mathcal{A} \approx D_f$ has bounded measurable coefficients, and $\phi \in X_f(\Omega)$. Moreover, suppose that u is bounded in the weak sense on the boundary $\partial\Omega$. Then u is globally bounded in Ω and satisfies*

$$\sup_\Omega u \le \sup_{\partial\Omega} u + C\|\phi\|_{X_f(\Omega)}.$$

We will use the above maximum principle when dealing with inhomogeneous equations in the plane. On the other hand, when dealing with homogeneous equations, we will use an improved maximum principle, valid for more general geometries, and which is our first main new theorem.

1.2. **Statement of the two main results.** Since our homogeneous maximum principle holds in higher dimensions as well, we will give the statement and proof for domains $\Omega \subset \mathbb{R}^n$. We refer to [KoRiSaSh] for the straightforward extension of the planar definitions used here to higher dimensions, noting in particular that $D_f(x_1, ..., x_n)$ is the $n \times n$ diagonal matrix with diagonal entries $\left\{1, ...1, f(x_1)^2\right\}$. Our first main theorem is a maximum principle for homogeneous equations in $\mathbb{R}^n$ that holds for all geometries.

Theorem 2 (homogeneous maximum principle)**.** *Suppose that f is a nonnegative bounded measurable function. Assume that u is a weak subsolution to $\mathcal{L}u = 0$ in a domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$, where $\mathcal{L} = \nabla^{\mathrm{tr}} \mathcal{A} \nabla$ and $\mathcal{A} \approx D_f$ has bounded measurable coefficients. Moreover, suppose that u is bounded in the weak sense on the boundary $\partial\Omega$. Then*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u \ .$$

Our second main theorem yields a new method for obtaining local boundedness and continuity of weak solutions in the plane, given that we already have an 'appropriate' maximum principle. In order to combine statements using either the homogeneous or inhomogeneous maximum principles, it is convenient to define precisely what we mean by an 'appropriate' maximum principle in the plane.

Definition 5. *Let $\phi \in L^2_{\mathrm{loc}}(\Omega)$ and $\mathcal{L} = \nabla^{\mathrm{tr}} \mathcal{A} \nabla$ where $\mathcal{A} \approx D_f$ has bounded measurable coefficients and f is even and strictly increasing on some $(0, R)$. An equation $\mathcal{L}v = \phi$ satisfies the* Maximum Principle Property *in Ω, or $\mathcal{MPP}$ for short, if for every open rectangle $R = (a, b) \times (c, d)$ with closure contained in Ω, there is a constant C_R such that*

$$\sup_{R} v \leq \sup_{\partial R} v + C_R \ ,$$

for all weak solutions v of $\mathcal{L}v = \phi$ that are bounded in the weak sense on $\partial\Omega$.

Remark 1. *The constant C_R in the above definition depends on the function ϕ on the right hand side of the equation. In particular, Theorem 1 implies $\mathcal{MPP}$ with $C_R = \|\phi\|_{X_f(R)}$, and Theorem 2 implies $\mathcal{MPP}$ with $C_R = 0$.*

The new method, which we refer to as the trace method, is embodied in the next theorem, for which we need the following somewhat technical definition of a modulus of continuity associated with a geometry f. See the discussion after (3.19) for a more explicit estimate of this modulus. We note here that this theorem does not rely on the structure conditions in Definition 1, but only requires the function f to be strictly increasing on some interval $(0, R)$ (which is needed in order to define the modulus of continuity in (3.19)).

Definition 6. *For any even f strictly increasing on $(0, R)$, let ω_f be the modulus of continuity associated to f as defined in (3.18) and (3.19) below.*

For any modulus of continuity ω, define a difference operator in the second variable by $D^{\omega}_{\mathbf{e}_2,\delta} h(x, y) = \frac{h(x,y+\delta)-h(x,y)}{\omega(\delta)}$. We also introduce the following notation

$$Lip_{\omega}(\Omega)$$

to denote the class of uniformly continuous functions with modulus of continuity ω. For consistency we write

$$Lip_{\gamma}(\Omega) \equiv C^{0,\gamma}(\Omega), \quad \gamma \in (0, 1].$$

Theorem 3 (trace method)**.** *Let ϕ satisfy the L^q growth condition $\|\phi\|_{L^q_{\mathrm{growth}}(\Omega)} < \infty$ for the operator $\mathcal{L}$ in the domain $\Omega \subset \mathbb{R}^2$, for some $q > 1$. Suppose f is nonnegative, even, and strictly increasing on some $(0, R)$, and that the equation $\mathcal{L}v = \nabla^{\mathrm{tr}} \mathcal{A} \nabla v = \phi$ satisfies the $\mathcal{MPP}$, i.e. the Maximum Principle Property, in Ω with $\mathcal{A}$ as above, and suppose that $u \in W^{1,2}_f(\Omega)$ is a weak solution to this equation, i.e. $\mathcal{L}u = \phi$ in Ω. Then*

1. *u is locally bounded;*
2. *if in addition $\phi(x, y)$ has modulus of continuity ρ in the y-variable uniformly in x, and if the equation $\mathcal{L}v = D^{\omega}_{\mathbf{e}_2,\delta}\phi$ with $\omega \equiv \max\{\omega_f, \rho\}$ satisfies the $\mathcal{MPP}$, and if $\mathcal{A}(x, y, u) = A(x)$ is independent of y and u, then $u \in Lip_{\omega}(\Omega')$ for any $\Omega' \subset\subset \Omega$.*

2. Proof of the homogeneous maximum principle in $\mathbb{R}^n$

In this section we prove the new homogeneous maximum principle Theorem 2, valid in all dimensions. We start with the Caccioppoli inequality, generalizing a similar inequality in Caffarelli and Vasseur [CaVa]. A version of this result appears as Proposition 29 in [KoRiSaSh], we repeat it here for the reader's convenience.

Proposition 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and $u \in \left(W_f^{1,2}\right)_0(\Omega)$ be a weak subsolution to $\mathcal{L}u = \phi$ with $\mathcal{L}$ as in (1.1), (1.2), and $\phi \in X_f(\Omega)$. Suppose there is a constant $P > 0$, and a nonnegative function $v \in W_f^{1,2}(\Omega)$ such that*

$$\|\phi\|_{X_f(\Omega)} \leq Pv(x), \quad a.e.\ x \in \{u > 0\} \cap \Omega. \tag{2.1}$$

Then

$$\int_\Omega |\nabla_f u_+|^2\, dx \leq CP^2 \int_\Omega v^2\, dx, \tag{2.2}$$

where C depends only on the constant c in (1.2). A similar result holds with u_- in place of u_+.

Proof. Since u is a weak subsolution to $\mathcal{L}u = \phi$, we have using as test function $w \equiv u_+ \in \left(W_f^{1,2}\right)_0(\Omega)$, that

$$\begin{aligned} \int \nabla(u_+)\mathcal{A}\nabla u_+\, dx &= \int \nabla u_+ \mathcal{A}\nabla u\, dx \leq -\int u_+\phi\, dx \\ &\leq \|\phi\|_{X_f(\Omega)} \int |\nabla_f u_+|\ dx \leq P\int v\,|\nabla_f u_+|\ dx\ , \end{aligned}$$

where for the last inequality we used conditions (1.2) and (2.1). Using Hölder's inequality and (1.2) this gives

$$\int |\nabla_f u_+|^2\ dx \leq \frac{1}{c}\int \nabla(u_+)\mathcal{A}\nabla u_+\, dx \leq CP^2 \int v^2\, dx,$$

which is (2.2). Using $w = u_-$ as a test function we obtain the last statement of the Proposition. ∎

The second ingredient in the proof of the maximum principle is the following 'straight-across' Sobolev inequality.

Proposition 2. *Suppose that f is a nonnegative bounded measurable function. Then the following global $(2,2)$ Sobolev inequality holds with geometry f*

$$\|w\|_{L^2(\Omega)} \leq C(\Omega)\, \|\nabla_f w\|_{L^2(\Omega)}\,, \qquad w \in \left(W_f^{1,2}\right)_0(\Omega)\,. \tag{2.3}$$

Proof. As in the proof of [KoRiSaSh, Proposition 81], it suffices by using a partition of unity to suppose that Ω is bounded. Then choose a ball $B(0, r_0)$ containing Ω and extend w to be 0 outside Ω so that $w \in \left(W_f^{1,2}\right)_0(B(0, r_0))$. Now first suppose that $w \in C_0^\infty(B(0, r_0))$, then we have the following one-dimensional Sobolev inequality

$$\int_{B(0,r_0)} w(x)^2 dx \leq Cr_0^2 \int_{B(0,r_0)} \left(\frac{\partial}{\partial x_1} w(x)\right)^2 dx \leq Cr_0^2 \int_{B(0,r_0)} |\nabla_f w(x)|^2\, dx,$$

where the last inequality follows from Definition 1.3. If $w \in \left(W_f^{1,2}\right)_0(B(0, r_0))$ choose an approximating sequence $\{w_m\}$ in $C_0^\infty(B(0, r_0))$ and take the limit in the above inequality to obtain

$$\int_\Omega w(x)^2 dx = \int_{B(0,r_0)} w(x)^2 dx \leq Cr_0^2 \int_{B(0,r_0)} |\nabla_f w(x)|^2\, dx = C(\Omega)\int_\Omega |\nabla_f w(x)|^2\, dx,$$

which concludes (2.3). ∎

We can now prove the homogeneous maximum principle, Theorem 2.

Proof of Theorem 2. If u is a weak subsolution to $\mathcal{L}u=0$ with a matrix $\mathcal{A}$ satisfying (1.2), then $u-\sup_{\partial\Omega}u$ is a weak subsolution to $\mathcal{L}_1 v=0$, where the corresponding matrix $\mathcal{A}_1(x,y,v)=\mathcal{A}(x,y,v+\sup_{\partial\Omega}u)$ satisfies the same assumptions. Therefore, we can assume $u\leq 0$ on $\partial\Omega$. Since $\phi\equiv 0$, we can use (2.2) with $v\equiv 0$ and (2.3) applied to $w=u_+$ to obtain

$$\int u_+^2 \leq 0,$$

which implies $u\leq 0$ in Ω. Applying this to $u-\sup_{\partial\Omega}u$ gives the result. ∎

3. Proof of the Trace Method Theorem in $\mathbb{R}^2$

We will prove the Trace Method Theorem in eight steps. Conclusion (1) of Theorem 3 will follow from the first three steps, where the first two will establish 'smoothness' properties of functions $u\in W_f^{1,2}(\Omega)$, where it is crucial that Ω is a planar domain, and the third requires that u be a weak solution. Conclusion (2) will then follow from an additional five steps, two of which are refinements of Steps two and three. Since the operator is elliptic away from the y-axis and condition (1.2) is invariant with respect to translations in y, it suffices to consider the case $\Omega=(-1,1)\times(-1,1)$, which we assume in all eight steps below. We also use the notation $\Omega_{a,b}^{c,d}\equiv(a,b)\times(c,d)$ for $-1\leq a<b\leq 1$ and $-1\leq c<d\leq 1$.

3.1. **Local boundedness of weak solutions.** Here we will prove Conclusion (1) of the Trace Method Theorem. We begin with Lebesgue's differentiation theorem and maximal function for Hilbert space valued functions on an interval (c,d).

Lemma 1. *Suppose that $\mathcal{H}$ is a separable Hilbert space and that $F\in L^2_{\mathcal{H}}((c,d))$, i.e. $F:(c,d)\to\mathcal{H}$ and*

$$\|F\|_{L^2_{\mathcal{H}}((c,d))}=\sqrt{\int_c^d |F(y)|^2_{\mathcal{H}}\,dy}<\infty.$$

Then for almost every $y\in(c,d)$ we have both

$$\lim_{I\searrow\{y\}}\frac{1}{|I|}\int_I |F(t)-F(y)|^2_{\mathcal{H}}\,dt=0,$$
$$M_2F(y)=\sup_{I:\ y\in I}\sqrt{\frac{1}{|I|}\int_I |F(t)|^2_{\mathcal{H}}\,dt}<\infty.$$

Moreover we have the weak type estimate

$$|\{y\in(c,d):M_2F(y)>\lambda\}|\leq\frac{5}{\lambda^2}\|F\|^2_{L^2_{\mathcal{H}}((c,d))}\ ,\qquad \lambda>0.$$

Proof. This is proved exactly as in the classical case when the Hilbert space $\mathcal{H}$ is the scalar field $\mathbb{R}$. ∎

We now apply Lemma 1 in the case $\mathcal{H}=L^2((a,b))$, so that $F\in L^2_{\mathcal{H}}((c,d))$ can be realized as a real-valued function $f(x,y)$ defined on $\Omega_{a,b}^{c,d}\equiv(a,b)\times(c,d)$ with

$$\begin{aligned} F(y) &= f(x,y)\,, \qquad \text{for } (x,y)\in\Omega_{a,b}^{c,d},\\ \|F\|_{L^2_{\mathcal{H}}([a,b])} &= \sqrt{\int_c^d\left(\int_a^b |f(x,y)|^2\,dx\right)dy}=\|f\|_{L^2\left(\Omega_{a,b}^{c,d}\right)}\ . \end{aligned}$$

Conclusion (1) of the Trace Method Theorem can now be completed in three steps, and Conclusion (2) will require an additional five steps.

3.1.1. *Step one.* Suppose $u\in W_D^{1,2}(\Omega)$, where $D=\begin{bmatrix}1&0\\0&0\end{bmatrix}$, and set

$$f_1(x,y)\equiv u(x,y)\ \text{ and }\ f_2(x,y)\equiv\frac{\partial u}{\partial x}(x,y)\,.$$

Suppose $-1<a<b<1$, and $z\in(-1,1)$ satisfies

$$\begin{aligned}
&\lim_{I\searrow\{z\}}\frac{1}{|I|}\int_I\left(\int_{-1}^{1}|f_i(x,y)-f_i(x,z)|^2\,dx\right)dy=0,\\
&\lim_{j\to\infty}\int_a^b\left|\varphi_{\varepsilon_j}*f_i(x,z)-f_i(x,z)\right|^2dx=0,\\
&M_2F_i(z)\le\Gamma\ ,
\end{aligned}\tag{3.1}$$

for some positive sequence $\{\varepsilon_j\}_{j\ge1}$ converging to zero, where $F_i(y)\equiv f_i(\cdot,y)\in L^2_{\mathcal{H}}((-1,1))$ for $i=1,2$, and $\{\varphi_\varepsilon\}_{\varepsilon>0}$ denotes a smooth approximate identity in the plane. Then for $-1<a<b<1$, we have

$$\|u(\cdot,z)\|_{Lip_\gamma(a,b)}\le C_\gamma\Gamma\ \text{ for } 0<\gamma<\frac{1}{2}.\tag{3.2}$$

First note that (3.2) is equivalent to $\|F_1(z)\|_{Lip_\gamma(a,b)}\le C_\gamma\Gamma$. To prove it define

$$\Phi_\varepsilon(z)(x)\equiv\varphi_\varepsilon*f_1(x,z)=\varphi_\varepsilon*u(x,z)\ \text{ and }\widehat{\Phi}_\varepsilon(z)(x)\equiv\varphi_\varepsilon*f_2(x,z)=\varphi_\varepsilon*\frac{\partial u}{\partial x}(x,z)\,.$$

Then both $\Phi_\varepsilon(z)(x)$ and $\widehat{\Phi}_\varepsilon(z)(x)$ are smooth functions of x satisfying $\Phi_\varepsilon(z)'(x)=\widehat{\Phi}_\varepsilon(z)(x)$, and so by the Sobolev embedding theorem in dimension one,

$$\begin{aligned}
\|\Phi_\varepsilon(z)\|_{Lip_{\frac12}((a,b))}&\le C\left(\|\Phi_\varepsilon(z)\|_{L^2((a,b))}+\left\|\Phi_\varepsilon(z)'\right\|_{L^2((a,b))}\right)\\
&= C\left(|\Phi_\varepsilon(z)|_{\mathcal{H}}+\left|\widehat{\Phi}_\varepsilon(z)\right|_{\mathcal{H}}\right)\ .
\end{aligned}$$

Next, note that the second line in (3.1) is equivalent to

$$\lim_{j\to\infty}\left|\Phi_{\varepsilon_j}(z)-F_1(z)\right|_{\mathcal{H}}=0\text{ and }\lim_{j\to\infty}\left|\widehat{\Phi}_{\varepsilon_j}(z)-F_2(z)\right|_{\mathcal{H}}=0,$$

which together with the first inequality in (3.1) gives for j sufficiently large

$$\left\|\Phi_{\varepsilon_j}(z)\right\|_{Lip_{\frac12}((a,b))}\le C\left(\Gamma+|F_1(z)|+|F_2(z)|\right)\le C\left(\Gamma+M_2F_1(z)+M_2F_2(z)\right).$$

Finally, combining with the third inequality in (3.1) we obtain for j sufficiently large

$$\left\|\Phi_{\varepsilon_j}(z)\right\|_{Lip_{\frac12}((a,b))}\le C\Gamma.$$

Now from the uniform boundedness of the $Lip_{\frac12}((a,b))$ norms of $\Phi_{\varepsilon_j}(z)$, it follows that for $0<\gamma<\frac12$, there is a subsequence $\{\varepsilon_{j_k}\}$ such that the sequence of functions $\Phi_{\varepsilon_{j_k}}(z)$ converges in $Lip_\gamma((a,b))$ to $V(z)\in Lip_\gamma((a,b))$. Thus $\Phi_{\varepsilon_{j_k}}(z)$ also converges in $L^2((a,b))$ to $V(z)$, which by the second line in (3.1) coincides with the function $x\to u(x,z)$, i.e. the function $F_1(z)=u(\cdot,z)$. This completes the proof of (3.2).

3.1.2. *Step two.* Fix a smooth approximate identity $\{\varphi_\varepsilon\}_{\varepsilon>0}$ in the plane and consider the functions $f_i\in L^2(\Omega)$ in Step one. We claim there are points $c\in\left(-\frac78,-\frac38\right)$ and $d\in\left(\frac38,\frac78\right)$, and a sequence $\{\varepsilon_j\}_{j=1}^\infty$, such that for $z\in\{c,d\}$,

$$\begin{aligned}
&\lim_{I\searrow\{z\}}\frac{1}{|I|}\int_I\left(\int_{-1}^{1}|f_i(x,y)-f_i(x,z)|^2\,dx\right)dy=0,\\
&\lim_{j\to\infty}\int_q^r\left|\varphi_{\varepsilon_j}*f_i(x,z)-f_i(x,z)\right|^2dx=0,\qquad -1<q<r<1,\\
&M_2F_i(z)\le\sqrt{11}\,\|F_i\|_{L^2_{\mathcal{H}}((-1,1))}\ .
\end{aligned}\tag{3.3}$$

Indeed, the first two lines of (3.3) hold almost everywhere; the first line since the set of Lebesgue points have full measure, and the second line follows from

$$\lim_{\varepsilon\to0}\int_s^t\left\{\int_q^r|\varphi_\varepsilon*f_i(x,z)-f_i(x,z)|^2\,dx\right\}dz=0,\qquad -1<q<r<1,-1<s<t<1,$$

since the square root of the integral in braces is a function of z that converges to 0 in $L^2((s,t))$, and hence has an almost everywhere pointwise convergent to 0 sequence $\{\varepsilon_j\}_{j=1}^{\infty}$. The third line follows from the weak type estimate for the maximal operator M_2,

$$|\{y \in (-1,1) : M_2 F_i(y) > \lambda\}| \leq \frac{5}{\lambda^2} \|F_i\|^2_{L^2_{\mathcal{H}}((-1,1))} \ ,$$

since then

$$\left|\left\{y \in (-1,1) : M_2 F_i(y) > \sqrt{11}\, \|F_i\|_{L^2_{\mathcal{H}}((-1,1))}\right\}\right| < \frac{1}{2},$$

which shows there exist points $c \in \left(-\frac{7}{8}, -\frac{3}{8}\right)$ and $d \in \left(\frac{3}{8}, \frac{7}{8}\right)$ satisfying (3.3).

Before proceeding to Step three, we give a lemma which will play a crucial role in both Steps three and seven.

Lemma 2. *Given $v \in W^{1,2}_f(\Omega)$, $\Omega^{s,t}_{q,r} \Subset \Omega$, and a smooth approximate identity $\{\varphi_\varepsilon\}_{0<\varepsilon<1}$, we have*

$$(\varphi_\varepsilon * v)_+ \to v_+$$

in the norm of $W^{1,2}_f\left(\Omega^{s,t}_{q,r}\right)$ as $\varepsilon \to 0$.

Proof. Let φ^*_ε denote the convolution operator $v \mapsto \varphi_\varepsilon * v$, and let Y be a C^1 vector field on Ω. Since $\Omega^{s,t}_{q,r} \Subset \Omega$, we have by a result of Friedrichs [Fri], see also [GaNh, Lemma 2.6], that the commutator $\widetilde{\varphi}_\varepsilon \equiv [Y, \varphi^*_\varepsilon]$ is an integral operator from $L^2(\Omega)$ to $L^2\left(\Omega^{s,t}_{q,r}\right)$ such that $\|\widetilde{\varphi}_\varepsilon w\|_{L^2\left(\Omega^{s,t}_{q,r}\right)} \longrightarrow 0$ as $\varepsilon \longrightarrow 0$ for all $w \in L^2(\Omega)$. Using this with Y equal to each of the vector fields $\frac{\partial}{\partial x}$ and $f(x)\frac{\partial}{\partial y}$, we then obtain

$$\varphi_\varepsilon * v \longrightarrow v \text{ in } W^{1,2}_f\left(\Omega^{s,t}_{q,r}\right). \tag{3.4}$$

We must show that both

$$\begin{aligned} (\varphi_\varepsilon * v)_+ &\to v_+ \text{ in } L^2\left(\Omega^{s,t}_{q,r}\right), \\ \nabla_f\left[(\varphi_\varepsilon * v)_+\right] &\to \nabla_f[v_+] \text{ in } L^2\left(\Omega^{s,t}_{q,r}\right). \end{aligned} \tag{3.5}$$

We begin by using the dominated convergence theorem to prove the first line in (3.5). Indeed, pick any decreasing sequence $\{\varepsilon_k\}_{k=1}^{\infty} \subset (0,1)$ with $\lim_{k\to\infty} \varepsilon_k = 0$. From (3.4), we see that there is a subsequence converging pointwise almost everywhere, and we will continue to denote the subsequence by $\{\varepsilon_k\}_{k=1}^{\infty}$. Now let $\mathcal{L}[v]$ denote the set of Lebesgue points of v, and note that

$$\left\{x \in \Omega^{s,t}_{q,r} : \lim_{k\to\infty} \varphi_{\varepsilon_k} * v(x) = v(x)\right\} \subset \mathcal{L}[v],$$

and of course $\left|\Omega^{s,t}_{q,r} \setminus \mathcal{L}[v]\right| = 0$. On the set $\mathcal{L}[v]$ we have

$$\lim_{k\to\infty}\left[\varphi_{\varepsilon_k} * v(x)\right]_+ = [v(x)]_+ \ , \qquad x \in \mathcal{L}[v],$$

and by [Ste, Theorem 2 on page 62], the supremum over k satisfies

$$\sup_{1\leq k<\infty}\left[\varphi_{\varepsilon_k} * v(x)\right]_+ \leq CMv(x), \qquad x \in \Omega^{s,t}_{q,r} \ ,$$

where M is the maximal function. Since $Mv \in L^2(\Omega)$ by the maximal theorem, the dominated convergence theorem now yields the first line in (3.5).

To prove the second line in (3.5), we use identities from [SaW3, see (33) on page 1886], which the reader can easily verify translate into the following in our notation,

$$\nabla_f(v_+) = \mathbf{1}_{\{v>0\}}\nabla_f v \text{ and } \nabla_f\left[(\varphi_\varepsilon * v)_+\right] = \mathbf{1}_{\{\varphi_\varepsilon * v>0\}}\nabla_f(\varphi_\varepsilon * v). \tag{3.6}$$

We claim the same argument as above now yields the limit

$$\mathbf{1}_{\{\varphi_\varepsilon * v>0\}}\nabla_f(\varphi_\varepsilon * v) \to \mathbf{1}_{\{v>0\}}\nabla_f(v) \text{ in } L^2\left(\Omega^{s,t}_{q,r}\right). \tag{3.7}$$

Indeed, we see from (3.4) that

$$\varphi_\varepsilon * \nabla_f v \longrightarrow \nabla_f v \text{ in } L^2\left(\Omega^{s,t}_{q,r}\right).$$

Thus every decreasing sequence in $(0,1)$ with limit 0 at 0, has a subsequence $\{\varepsilon_k\}_{k=1}^{\infty}$ such that

$$\nabla_f\left(\varphi_{\varepsilon_k} * v\right) = \varphi_{\varepsilon_k} * \nabla_f v \longrightarrow \nabla_f v, \qquad \text{pointwise almost everywhere in } \Omega^{s,t}_{q,r} \ .$$

Then
$$\left\{x \in \Omega_{a,b}^{c,d} : \lim_{k\to\infty} \varphi_{\varepsilon_k} * \nabla_f v\left(x\right) = \nabla_f v\left(x\right)\right\} \subset \mathcal{L}\left[\nabla v\right],$$
where $\left|\Omega_{q,r}^{s,t} \setminus \mathcal{L}\left[v\right]\right| = 0$, and it is easily checked that
$$\mathbf{1}_{\{\varphi_\varepsilon * v>0\}}\nabla_f\left(\varphi_\varepsilon * v\right) \to \mathbf{1}_{\{v>0\}}\nabla_f v, \qquad \text{pointwise almost everywhere in } \Omega_{q,r}^{s,t}\ .$$
Thus from (3.6) we have $\nabla_f\left[\left(\varphi_\varepsilon * v\right)_+\right] \to \nabla_f\left(v_+\right)$ pointwise almost everywhere in $\Omega_{q,r}^{s,t}$, and then using $M\nabla_f v \in L^2\left(\Omega_{q,r}^{s,t}\right)$, the dominated convergence theorem yields the second line in (3.5). ∎

3.1.3. *Step three.* Under the hypotheses of Trace Method Theorem, we claim that
$$u \in L^\infty\left(\Omega_{-\frac{3}{4},\frac{3}{4}}^{-\frac{3}{8},\frac{3}{8}}\right),$$
which then completes the proof of Conclusion (1) of the Trace Method Theorem.

To prove this we begin with the following lemma.

Lemma 3. *Suppose that* $u \in W_f^{1,2}\left(\left(-1,1\right)^2\right)$ *satisfies* $\nabla^{\mathrm{tr}}A\nabla u = \phi$, *where* ϕ *is* f*-admissible, and where* $A\left(x,y\right) \approx D_f\left(x\right)$. *Choose* c,d *as in (3.3) and choose*
$$a = -\frac{3}{4} \text{ and } b = \frac{3}{4}.$$
Then if $\Omega_{a,b}^{c,d} \equiv (a,b)\times(c,d)$, *we have that* u *is bounded in the weak sense on* $\partial\Omega_{a,b}^{c,d}$, *i.e. there is a constant* ℓ *such that* $(u-\ell)^+ \in \left(W_f^{1,2}\left(\Omega_{a,b}^{c,d}\right)\right)_0$, *and in fact one can take*
$$\ell \equiv C' \max\left\{\left\|u\right\|_{W_f^{1,2}(\Omega)}, \left\|u\right\|_{L^2(\Omega)} + \left\|\phi\right\|_{L^q_{\mathrm{growth}}(\Omega)}\right\}.$$

Proof. By (3.2) and (3.3), we have for $z \in \{c,d\}$ that
$$\left\|\varphi_{\varepsilon_j} * u\left(x,z\right)\right\|_{L^\infty((a,b))} \leq C' \left\|u\right\|_{W_D^{1,2}(\Omega)} \leq C' \left\|u\right\|_{W_f^{1,2}(\Omega)}.$$
Then from ellipticity away from the y-axis, we have for $t \in \{a,b\}$ and $p \in (c+1/8, d-1/8)$
$$\begin{aligned}\left\|u\right\|_{L^\infty\left(B_{\mathrm{Euc}}\left((t,p),\frac{|t|}{6}\right)\right)} &\leq C\left(\left\|u\right\|_{L^2\left(B_{\mathrm{Euc}}\left((t,p),\frac{|t|}{3}\right)\right)} + \left\|\phi\right\|_{L^q\left(B_{\mathrm{Euc}}\left((t,p),\frac{|t|}{3}\right)\right)}\right)\\ &\leq C\left(\left\|u\right\|_{L^2(\Omega)} + \left\|\phi\right\|_{L^q_{\mathrm{growth}}(\Omega)}\right),\end{aligned}$$
where we note that $|t|/6 = 1/8$ and $c \in (-7/8,-3/8)$, $d \in (3/8,7/8)$, so $B_{\mathrm{Euc}}\left((t,p),\frac{|t|}{3}\right) \subset \Omega$ with $\left|B_{\mathrm{Euc}}\left((t,p),\frac{|t|}{3}\right)\right| = \frac{\pi}{16}$, and the constant C can be chosen independent of the point (t,p). Moreover, for every $y \in (c,d)$ there exists $p \in (c+1/8, d-1/8)$ such that $y \in B_{\mathrm{Euc}}\left((t,p),\frac{|t|}{6}\right)$. Thus we obtain for $t \in \{a,b\}$ that
$$\left\|\varphi_{\varepsilon_j} * u\left(t,y\right)\right\|_{L^\infty((c,d))} \leq \left\|u\left(t,y\right)\right\|_{L^\infty((c,d))} \leq C'\left(\left\|u\right\|_{L^2(\Omega)} + \left\|\phi\right\|_{L^q_{\mathrm{growth}}(\Omega)}\right).$$
Define $\ell \equiv C' \max\left\{\left\|u\right\|_{W_f^{1,2}(\Omega)}, \left\|u\right\|_{L^2(\Omega)} + \left\|\phi\right\|_{L^q_{\mathrm{growth}}(\Omega)}\right\}$, where now C' stands for the maximum of C' and C above. Since $\varphi_{\varepsilon_j} * u\left(x,y\right)$ is a smooth function in Ω provided $2\varepsilon < \min\{a+1, 1-b, c+1, 1-d\}$, the above inequalities imply
$$\left|\varphi_{\varepsilon_j} * u\left(x,y\right)\right| |_{\partial\Omega_{a,b}^{c,d}} \leq \frac{1}{2}\ell\ .$$
This gives
$$\left(\varphi_{\varepsilon_j} * u\left(x,y\right) - \frac{1}{2}\ell\right)_+ = 0 \quad \text{on} \quad \partial\Omega_{a,b}^{c,d},$$
and by continuity,
$$\operatorname{supp}\left(\varphi_{\varepsilon_j} * u\left(x,y\right) - \ell\right)_+ \Subset \Omega_{a,b}^{c,d}.$$

Thus we have

$$\left(\varphi_{\varepsilon_j} * u(x,y) - \ell\right)_+ \in \left(W_f^{1,2}\right)_0 \left(\Omega_{a,b}^{c,d}\right),$$

and it remains to show that

$$\left(\varphi_{\varepsilon_j} * u - \ell\right)_+ \to (u-\ell)_+ \tag{3.8}$$

in the norm of $W_f^{1,2}(\Omega)$ as $\varepsilon \to 0$. Indeed, since $\left(W_f^{1,2}(\Omega)\right)_0$ is closed in $W_f^{1,2}(\Omega)$, we would then conclude that $(u-l)^+ \in \left(W_f^{1,2}(\Omega)\right)_0$ as required. To prove (3.8), we note that since $\varphi_{\varepsilon_j} * u(x,y) - \ell = \varphi_{\varepsilon_j} * (u(x,y)-\ell)$, we may assume without loss of generality that $\ell = 0$, and then Lemma 2 applies to finish the proof of Lemma 3. ∎

We now want to use Lemma 3, together with Steps one and two, to obtain that u is bounded in the weak sense on $\partial\Omega_{-\frac{3}{4},\frac{3}{4}}^{c,d}$. Indeed, if u is a weak solution to $\nabla^{\mathrm{tr}}\mathcal{A}\nabla u = \phi$, then it is also a weak solution to the linear equation $\nabla^{\mathrm{tr}}\tilde{A}\nabla u = \phi$ with $\tilde{A}(x,y) = \mathcal{A}(x,y,u(x,y))$ which satisfies the assumptions of Lemma 3. Finally then, we apply the assumed $\mathcal{MPP}$ for the equation $\mathcal{L}u = \phi$ to conclude that u is bounded in $\Omega_{-\frac{3}{4},\frac{3}{4}}^{c,d}$, and in particular in $\Omega_{-\frac{3}{4},\frac{3}{4}}^{-\frac{3}{8},\frac{3}{8}}$

$$\sup_{\Omega_{-\frac{3}{4},\frac{3}{4}}^{-\frac{3}{8},\frac{3}{8}}} u \le C'\left(\|u\|_{W_f^{1,2}(\Omega)} + \|\phi\|_{L^q_{\mathrm{growth}}(\Omega)} + C_R\right),$$

where the constant C_R depends on the function ϕ.

3.2. **Continuity of weak solutions.** Here we will prove Conclusion (2) of the Trace Method Theorem in five additional steps. Steps five and seven are refinements of Steps two and three respectively.

3.2.1. *Step four.* Here we assume that ϕ is f-admissible in Ω, and in addition satisfies the following property:

$$\phi(x,y) \text{ has modulus of continuity } \rho \text{ in } y, \text{ uniformly in } x. \tag{3.9}$$

Let $\omega(\delta)$ be a modulus of continuity with $\omega \ge \rho$, and set $v_\delta(x,y) \equiv u(x,y+\delta)$. Consider the difference operators

$$D_{\mathbf{e}_2,\delta}^{\omega} u(x,y) \equiv \frac{u(x,y+\delta) - u(x,y)}{\omega(\delta)} = \frac{v_\delta(x,y) - u(x,y)}{\omega(\delta)}$$

in the y-variable. We claim that if u is a weak solution to $Lu = \phi$, then

$$w = D_{\mathbf{e}_2,\delta}^{\omega} u \text{ is a weak solution to } \left\{\begin{array}{ccc} Lw = 0 & \text{if} & \phi(x,y) \text{ is independent of } y \\ Lw = \eta \in L^\infty & \text{if} & \phi(x,\cdot) \in Lip_\rho \text{ uniformly in } x \end{array}\right. . \tag{3.10}$$

We first note that (3.10) will follow from the following equality

$$L\left(D_{\mathbf{e}_2,\delta}^{\omega} u\right)(x,y) = \left(D_{\mathbf{e}_2,\delta}^{\omega}\phi\right)(x,y),$$

which holds provided A is independent of y. To see this, note that

$$\begin{aligned} L\left(D_{\mathbf{e}_2,\delta}^{\omega} u\right)(x,y) &= \frac{Lv_\delta(x,y) - Lu(x,y)}{\omega(\delta)} \\ &= \frac{\left[\nabla^{\mathrm{tr}} A(x)\nabla\right] u(x,y+\delta) - Lu(x,y)}{\omega(\delta)} \\ &= \frac{\nabla^{\mathrm{tr}} A(x)\nabla u(x,y+\delta) - Lu(x,y)}{\omega(\delta)} \\ &= \frac{Lu(x,y+\delta) - Lu(x,y)}{\omega(\delta)} = \frac{\phi(x,y+\delta) - \phi(x,y)}{\omega(\delta)}, \end{aligned}$$

since $A(x)$ is independent of y. Then if $\phi(x,y)$ is independent of y, we have $L\left(D_{\mathbf{e}_2,\delta}^{\omega} u\right) \equiv 0$, while if $\phi(x,\cdot) \in Lip_\rho$ uniformly in x, then $L\left(D_{\mathbf{e}_2,\delta}^{\omega} u\right)$ is bounded.

3.2.2. *Step five (a refinement of Step two).* Fix $-1<a<-\frac{3}{4}$ and $\frac{3}{4}<b<1$. We claim that for every $0<\gamma<\frac{1}{2}$, there is a positive constant C_{γ} with the property that for every $0<\delta<\frac{1}{10}$, there is a set $\Theta_{\delta}\equiv\{c_{\delta},c_{\delta}+\delta,d_{\delta},d_{\delta}+\delta\}$ of four points with $-1+\delta<c_{\delta}<-\frac{1}{2}$ and $\frac{1}{2}<\ d_{\delta}<1-\delta$, such that $\Omega_{a,b}^{c_{\delta},d_{\delta}}\Subset\Omega$ and

$$\left\|u\left(\cdot,y\right)\right\|_{Lip_{\gamma}\left(a,b\right)}\leq C_{\gamma}\left\|u\right\|_{W_{D}^{1,2}\left(\Omega\right)}\ ,\qquad\text{for all }u\in W_{D}^{1,2}\left(\Omega\right)\text{ and all }y\in\Theta_{\delta}\ .\tag{3.11}$$

To prove this, we first note that if $F\in L_{\mathcal{H}}^{2}\left(\left(-1,1\right)\right)$, we can realize F as a real-valued function $f\left(x,y\right)$ defined on Ω with

$$\begin{aligned}F\left(y\right)&=f\left(x,y\right),\qquad\text{for }\left(x,y\right)\in\Omega,\\ \left\|F\right\|_{L_{\mathcal{H}}^{2}\left(\left(-1,1\right)\right)}&=\sqrt{\int_{-1}^{1}\left(\int_{-1}^{1}\left|f\left(x,y\right)\right|^{2}dx\right)dy}=\left\|f\right\|_{L^{2}\left(\Omega\right)}\ .\end{aligned}$$

Next fix $0<\delta<\frac{1}{10}$, a smooth approximate identity $\{\varphi_{\varepsilon}\}_{\varepsilon>0}$ in the plane, and points $c\in\left(-1,-\frac{1}{2}\right)$ and $d\in\left(\frac{1}{2},1\right)$. We now use Lemma 1 to choose $c_{\delta}\in\left(c,-\frac{1}{2}\right)$ and $d_{\delta}\in\left(\frac{1}{2},d\right)$ and a sequence $\{\varepsilon_{j}\}_{j=1}^{\infty}$ such that for

$$z\in\Theta_{\delta}\equiv\left\{c_{\delta},c_{\delta}+\delta,d_{\delta},d_{\delta}+\delta\right\},$$

we have

$$\begin{aligned}&\lim_{I\searrow\{z\}}\frac{1}{|I|}\int_{I}\left(\int_{-1}^{1}\left|f\left(x,y\right)-f\left(x,z\right)\right|^{2}dx\right)dy=0,\\ &\lim_{j\to\infty}\int_{a}^{b}\left|\varphi_{\varepsilon_{j}}*f\left(x,z\right)-f\left(x,z\right)\right|^{2}dx=0,\\ &M_{2}F\left(z\right)\leq\sqrt{401}\left\|F\right\|_{L_{\mathcal{H}}^{2}\left(\left(-1,1\right)\right)}\ .\end{aligned}\tag{3.12}$$

To this end, we first note that the set E of points z for which the first two lines of (3.12) hold is a set of full measure, since the first line holds for Lebesgue points, and since the second line follows from

$$\lim_{\varepsilon\to0}\int_{c}^{d}\left\{\int_{a}^{b}\left|\varphi_{\varepsilon}*f\left(x,z\right)-f\left(x,z\right)\right|^{2}dx\right\}dz=0.$$

This last assertion holds because the square root of the integral in braces is a function of z that converges to 0 in $L^{2}\left(\left(c,d\right)\right)$, and hence has an almost everywhere pointwise convergent to 0 sequence $\{\varepsilon_{j}\}_{j=1}^{\infty}$. Thus we can restrict our attention in what follows to points $z\in E$.

Now suppose, in order to derive a contradiction, that the third line in (3.12) fails. Then for each $j\in\mathbb{N}_{\mathrm{odd}}$ with $j\delta<\frac{1}{10}$, almost every pair of points $\left(c_{\delta}^{j},d_{\delta}^{j}\right)$ in $E\times E$ satisfying

$$\begin{aligned}-1+j\delta&\leq c_{\delta}^{j}<-1+\left(j+1\right)\delta,\\ 1-\left(j+2\right)\delta&\leq d_{\delta}^{j}<1-\left(j+1\right)\delta,\end{aligned}$$

has the property that

$$\max\left\{M_{2}F\left(c_{\delta}^{j}\right),M_{2}F\left(c_{\delta}^{j}+\delta\right),M_{2}F\left(d_{\delta}^{j}\right),M_{2}F\left(d_{\delta}^{j}+\delta\right)\right\}>\lambda\equiv\sqrt{401}\left\|F\right\|_{L_{\mathcal{H}}^{2}\left(\left(c,d\right)\right)}.\tag{3.13}$$

We claim that for each $j\in\mathbb{N}_{\mathrm{odd}}$ with $j\delta<\frac{1}{10}$, **at least one** of the pairwise disjoint sets

$$\begin{aligned}&\{M_{2}F>\lambda\}\cap I_{1};\quad\text{where }I_{1}\equiv\left[-1+j\delta,-1+\left(j+1\right)\delta\right),\\ &\{M_{2}F>\lambda\}\cap I_{2};\quad\text{where }I_{2}\equiv\left[-1+\left(j+1\right)\delta,-1+\left(j+2\right)\delta\right),\\ &\{M_{2}F>\lambda\}\cap I_{3};\quad\text{where }I_{3}\equiv\left[1-\left(j+2\right)\delta,1-\left(j+1\right)\delta\right),\\ &\{M_{2}F>\lambda\}\cap I_{4};\quad\text{where }I_{4}\equiv\left[1-\left(j+1\right)\delta,1-j\delta\right),\end{aligned}$$

has measure at least $\frac{1}{4}\delta$.

To see this, fix j and define for each pair of points $\left(c_{\delta}^{j},d_{\delta}^{j}\right)$ in $\left(E\cap I_{1}\right)\times\left(E\cap I_{3}\right)$, the corresponding four points $z_{k}=z_{k}\left(c_{\delta}^{j},d_{\delta}^{j}\right)$ given by $z_{1}=c_{\delta}^{j},z_{2}=c_{\delta}^{j}+\delta,z_{3}=d_{\delta}^{j},z_{4}=d_{\delta}^{j}+\delta$. Then we have $z_{k}\in I_{k}$ for

$1 \leq k \leq 4$. Set

$$\begin{aligned} G_k &\equiv I_k \cap \{M_2 F > \lambda\}, \\ \mathcal{P}_k &\equiv \left\{\left(c_\delta^j, d_\delta^j\right) \in (E \cap I_1) \times (E \cap I_3) : M_2 F\left(z_k\left(c_\delta^j, d_\delta^j\right)\right) > \lambda\right\}. \end{aligned}$$

Then from (3.13) we have $(E \cap I_1) \times (E \cap I_3) = \bigcup_{k=1}^{4} \mathcal{P}_k$, and so there is k_0 for which

$$|\mathcal{P}_{k_0}| \geq \frac{1}{4} |(E \cap I_1) \times (E \cap I_3)| = \frac{\delta^2}{4}.$$

Thus from the inclusions

$$\mathcal{P}_1 \subset G_1 \times I_3 \ , \quad \mathcal{P}_2 \subset (G_2 - \delta) \times I_3 \ , \quad \mathcal{P}_3 \subset I_1 \times G_3 \ , \quad \mathcal{P}_4 \subset I_1 \times (G_4 - \delta) \ ,$$

we obtain the desired inequality,

$$|G_{k_0}| \, \delta \geq |\mathcal{P}_{k_0}| \geq \frac{\delta^2}{4}.$$

Thus we have

$$|\{M_2 F > \lambda\}| \geq \sum_{j \in \mathbb{N}_{\mathrm{odd}}; j\delta < \frac{1}{10}} \frac{1}{4}\delta \geq \frac{1}{80},$$

by the pairwise disjointedness of these collections of sets in j, and together with the weak type estimate for M_2 in Lemma 1, we obtain

$$\frac{1}{80} \leq |\{y \in (c,d) : M_2 F(y) > \lambda\}| \leq \frac{5}{\lambda^2} \|F\|^2_{L^2_{\mathcal{H}}((c,d))} \ ,$$

which contradicts the choice $\lambda = \sqrt{401} \|F\|_{L^2_{\mathcal{H}}((c,d))}$. This completes the proof of (3.12).

We can also adapt the above argument to show that if, instead of a single function f, we have two functions $F_1, F_2 \in \mathcal{H} = L^2((-1,1))$, then we can choose $c_\delta \in \left(c, -\frac{1}{2}\right)$ and $d_\delta \in \left(\frac{1}{2}, d\right)$ such that

$$\begin{aligned} &\lim_{I \searrow \{z\}} \frac{1}{|I|} \int_I \left(\int_{-1}^{1} |f_i(x,y) - f_i(x,z)|^2 \, dx\right) dy = 0, \\ &\text{and } M_2 F_i(z) \leq 2\sqrt{1601} \|F_i\|_{L^2_{\mathcal{H}}((-1,1))}, \quad i = 1, 2, \\ &\text{for } z = c_\delta, c_\delta + \delta, d_\delta, d_\delta + \delta \ . \end{aligned} \tag{3.14}$$

Indeed, if $M_2 F_i(z) \leq \sqrt{1601} \|F_i\|_{L^2_{\mathcal{H}}((-1,1))}$ fails for the four choices of z and each $i = 1, 2$, then there will be *eight* pairwise disjoint sets instead of *four* in the above argument, and we obtain

$$\frac{1}{160} \leq |\{y \in (c,d) : \max\{M_2 F_1(y), M_2 F_2(y)\} > \lambda\}| \leq 2\frac{5}{\lambda^2} \|F_i\|^2_{L^2_{\mathcal{H}}((-1,1))} .$$

Now we use (3.14) to apply Step one with

$$\Gamma = \sqrt{1601}\left(\|u\|_{L^2(\Omega)} + \left\|\frac{\partial u}{\partial x}\right\|_{L^2(\Omega)}\right) = \sqrt{1601} \|u\|_{W_D^{1,2}(\Omega)} ,$$

in order to obtain the uniform boundedness of the $Lip_{\frac{1}{2}}((a,b))$ norms of $\Phi_\varepsilon(z)$. Then it follows that for $0 < \gamma < \frac{1}{2}$, there is a sequence of functions $\Phi_{\varepsilon_j}(z)$ that converges in $Lip_\gamma((a,b))$ to $V(z) \in Lip_\gamma((a,b))$. Thus $\Phi_{\varepsilon_j}(z)$ also converges in $L^2((a,b))$ to $V(z)$, which by the second line in (3.12) coincides with the function $x \to u(x,z)$, i.e. $U(z) = u(\cdot, z)$. This completes the proof of (3.11).

3.2.3. *Step six.* Recall that for a modulus of continuity ω, we defined the corresponding difference operator $D^{\omega}_{\mathbf{e}_2,\delta}$ in the direction $\mathbf{e}_2$ by

$$D^{\omega}_{\mathbf{e}_2,\delta} u(x,y) \equiv \frac{u(x,y+\delta)-u(x,y)}{\omega(\delta)}.$$

Then given L as above and $-1<a<-\frac{3}{4}$, $\frac{3}{4}<b<1$, we claim there is a modulus of continuity $\omega_f(\delta)$ defined for $0<\delta<\frac{1}{10}$ (see (3.19) below for an explicit formula) such that

$$\left\| D^{\omega}_{\mathbf{e}_2,\delta} u(\cdot,z)\right\|_{L^\infty(a,b)} \leq C_0 = C_0\left(\|u\|_{W^{1,2}_f(\Omega)}, \|\phi\|_{L^q_{\text{growth}}(\Omega)}, C_R\right), \tag{3.15}$$
$$\text{for all weak solutions } u \text{ to } Lu=\phi \text{ and points } z\in\Theta_\delta\ ,$$

where the constant C_R is that arising in the $\mathcal{MPP}$.

To prove this we will apply a classical Hölder estimate for elliptic equations in small balls arbitrarily close to the singular y -axis, which we now describe. Without loss of generality we may assume $x\geq 0$, and then fix $\beta>0$. Consider a Euclidean ball $B_{\text{Euc}}\equiv B_{\text{Euc}}((x+2\beta,y),\beta)$ and a Euclidean ball $RB_{\text{Euc}}\equiv B_{\text{Euc}}((x+2\beta,y),R)$ concentric with B_{Euc} and having radius $0<R\leq\beta$. Let ϕ be f-admissible. Then in particular, $\phi\in L^q_{\text{growth}}(\Omega)$ for some $q>\frac{n}{2}=1$. Fix such a q and set

$$M\equiv\|\phi\|_{L^q_{\text{growth}}(\Omega)}\,.$$

For elliptic operators, we have from Theorem 8.22 in [GiTr] with notation used there, the following classical Hölder estimate for weak solutions,

$$\operatorname*{osc}_{RB_{\text{Euc}}} u \leq \frac{1}{\gamma}\left(\left(\frac{R}{\beta}\right)^{\alpha}\sup_{B_{\text{Euc}}}|u| + kR^{\alpha}\right),\quad R<\beta. \tag{3.16}$$

We now derive bounds for both γ and α from an examination of the proofs in [GiTr]. Theorems 8.17 and 8.18 in [GiTr] immediately yield the following Harnack inequality:

$$\begin{aligned}\sup_{B_{\text{Euc}}} u &\leq C_H\left(\inf_{B_{\text{Euc}}} u + k\beta^{2-\frac{n}{q}}\right),\\ \text{with } C_H &\equiv C(n)^{\frac{\Lambda}{\lambda}},\quad k=\frac{C\,\|\phi\|_{L^q(B_{\text{Euc}})}}{\lambda},\end{aligned}$$

and so k can be bounded above by $\frac{CM}{\lambda}$. For homogeneous equations, where $k=0$, this is recorded as Theorem 8.20 in [GiTr]. An inspection of the inequality at the top of page 202 in [GiTr] now shows that we can take $\gamma=1-\frac{1}{C_H}\geq\frac{1}{2}$ if $C_H\geq 2$, and

$$\alpha\approx-\ln\left(1-\frac{1}{2C_H}\right),$$

since $\alpha=(1-\mu)\frac{\log\gamma}{\log\tau}$ in the notation of [GiTr], and μ can be chosen to satisfy $\dfrac{2}{3-n/(2q)}=\dfrac{2}{3-1/q}<\mu<1$.

With the Hölder estimate (3.16) in hand, we now note that from Conclusion (1) of the Trace Method Theorem, already proved in Step three, it follows that for any Euclidean ball B_{Euc},

$$\sup_{B_{\text{Euc}}} u\leq C\left\{||u||_{W^{1,2}_f(\Omega)}+\|\phi\|_{L^q_{\text{growth}}(\Omega)}+C_R\right\}\ ,$$

with C independent of λ. Combining this with (3.16) and the lower estimate for γ gives

$$\operatorname*{osc}_{RB_{\text{Euc}}} u\leq C\left(\left(\frac{R}{\beta}\right)^{\alpha}\left(||u||_{W^{1,2}_f(\Omega)}+\|\phi\|_{L^q_{\text{growth}}(\Omega)}+C_R\right)+kR^{\alpha}\right)\ ,$$

with the constant C independent of λ.

Our operator is elliptic in B_{Euc} with the ellipticity constant satisfying

$$\lambda\geq f(\beta)^2.$$

Therefore, for $0<\delta\leq\beta$, and using $k\leq\frac{CM}{\lambda}\leq\frac{CM}{f(\beta)^2}$, we have

$$\begin{aligned}
(3.17)\qquad & |u(x+2\beta,y+\delta)-u(x+2\beta,y)| \leq \operatorname*{osc}_{B_{\mathrm{Euc}}(x+2\beta,y)} u \\
&\leq C\left(\frac{\delta}{\beta}\right)^{\alpha(\beta)}\left(||u||_{W_f^{1,2}(\Omega)}+\|\phi\|_{L^q_{\mathrm{growth}}(\Omega)}+C_R\right)+C\,\|\phi\|_{L^q_{\mathrm{growth}}(\Omega)}\frac{\delta^{\alpha(\beta)}}{f(\beta)^2} \\
&\leq C_\beta\delta^{\alpha(\beta)}\left(||u||_{W_f^{1,2}(\Omega)}+\|\phi\|_{L^q_{\mathrm{growth}}(\Omega)}+C_R\right),
\end{aligned}$$

where

$$\alpha(\beta)=-C'\ln\left(1-C^{-\frac{1}{f(\beta)^2}}\right).$$

We now need to choose $\delta=\delta(\beta)$ such that

$$\left(\frac{\delta}{\beta}\right)^{\alpha(\beta)}\to 0\quad\text{and}\quad\frac{\delta^{\alpha(\beta)}}{f(\beta)^2}\to 0\quad\text{as}\quad\beta\to 0.$$

To satisfy the first condition it is sufficient to require

$$\begin{aligned}
-\ln\left(1-C^{-\frac{1}{f(\beta)^2}}\right)\ln\frac{\delta}{\beta}&\to-\infty\quad\text{as}\quad\beta\to 0,\\
C^{-\frac{1}{f(\beta)^2}}\ln\frac{\delta}{\beta}&\to-\infty\quad\text{as}\quad\beta\to 0,\\
\ln\frac{\delta}{\beta}&=-C^{\frac{2}{f(\beta)^2}},
\end{aligned}$$

which holds for $\delta=\Gamma_f(\beta)$ where

$$\Gamma_f(\beta)\equiv\beta\exp\left(-\exp\left(\frac{C'}{f(\beta)^2}\right)\right). \tag{3.18}$$

Note that the function $\Gamma_f(\beta)$ is invertible for small β since f is strictly increasing on $(0,R)$ for some $R>0$.

We also note that as $\beta\to 0$ we have $\delta\to 0$ **very quickly**. Thus, as $\delta\to 0$, $\beta\to 0$ **very slowly**. We now verify that with δ as above we also have

$$\frac{\delta^{\alpha(\beta)}}{f(\beta)^2}\to 0\quad\text{as}\quad\beta\to 0.$$

Passing to logarithms again we have

$$\begin{aligned}
\ln\left(\frac{\delta^{\alpha(\beta)}}{f(\beta)^2}\right)=\alpha(\beta)\ln\delta+2\ln\frac{1}{f(\beta)}&=C'\ln\left(1-C^{-\frac{1}{f(\beta)^2}}\right)\left(\exp\left(\frac{C'}{f(\beta)^2}\right)+\ln\frac{1}{\beta}\right)+2\ln\frac{1}{f(\beta)}\\
&\approx-\exp\left(-\frac{C}{f(\beta)^2}\right)\left(\exp\left(\frac{C'}{f(\beta)^2}\right)+\ln\frac{1}{\beta}\right)+2\ln\frac{1}{f(\beta)}
\end{aligned}$$

and the expression converges to $-\infty$ as $\beta\to 0$ provided we choose C' sufficiently large.

We now calculate a modulus of continuity $\omega(\delta)$ that ensures the function $D^{\omega}_{\mathbf{e}_2,\delta}u(x,y)$ is bounded uniformly for $y\in\{c_\delta,d_\delta\}$. Using (3.17) on the second term of line 2, and (3.11) on the first and third terms of line 2, we have for $0<x<x+2\beta<b$ and $0<\delta<\frac{1}{2}\beta$ (and similarly for $x<0$),

$$\begin{aligned}
&\left|D^{\omega}_{\mathbf{e}_2,\delta}u(x,y)\right|=\frac{1}{\omega(\delta)}\left|u(x,y+\delta)-u(x,y)\right|\\
\leq\;&\frac{|u(x,y+\delta)-u(x+2\beta,y+\delta)|}{\omega(\delta)}+\frac{|u(x+2\beta,y+\delta)-u(x+2\beta,y)|}{\omega(\delta)}+\frac{|u(x+2\beta,y)-u(x,y)|}{\omega(\delta)}\\
\leq\;&\frac{C_0}{\omega(\delta)}\left\{\beta^\gamma+\left(\frac{\delta}{\beta}\right)^{\alpha(\beta)}+\frac{\delta^{\alpha(\beta)}}{f(\beta)^2}+\beta^\gamma\right\}\leq C_0,
\end{aligned}$$

where C_0 depends on $\|u\|_{W_D^{1,2}(\Omega)}$, $\|\phi\|_{f-\mathrm{adm}(\Omega)}$ and C_R, provided we choose $\omega(\delta)=\omega_f(\delta)$ where

$$\omega_f(\delta)\equiv\Gamma_f^{-1}(\delta)^\gamma+\left(\frac{\delta}{\Gamma_f^{-1}(\delta)}\right)^{\alpha(\Gamma_f^{-1}(\delta))}+\frac{\delta^{\alpha(\Gamma_f^{-1}(\delta))}}{f(\Gamma_f^{-1}(\delta))^2}. \tag{3.19}$$

Note that the modulus of continuity $\omega_f(\delta)$ is increasing on $(0,1)$ and satisfies $\lim_{\delta\searrow 0}\omega_f(\delta)=0$. We also remark here that from the definitions of Γ_f and α above, it follows that

$$\omega_f(\delta)\approx\Gamma_f^{-1}(\delta)^{\gamma},$$

where Γ_f is defined in (3.18) and it converges to zero very quickly as its argument approaches zero. Estimates for Γ_f^{-1} give

$$\omega_f(\delta)\approx\left[f^{-1}\left(\left(\frac{C}{\ln\ln\frac{1}{\delta}}\right)^{\frac{1}{2}}\right)\right]^{\gamma}.$$

Already in the case of finite type degeneracy $(f(x)\approx x^k)$ one can see that the modulus of continuity goes to zero as a power of a double logarithm. When the degeneracy is of infinite type, the convergence is even slower.

Remark 2. *The constant C_R in the Step six arguments above can be replaced*

1. *by $\|\phi\|_{X_f(\Omega)}$ if we are using the maximum principle in Theorem 1,*
2. *by 0 if we are using the homogeneous maximum principle in Theorem 2.*

3.2.4. *Step seven (a refinement of Step three).* We claim that with $\omega(\delta)\equiv\max\{\omega_f(\delta),\rho\}$, where ρ is the modulus of continuity of $\phi(x,\cdot)$ as in (3.9), we have

$$\left\|D^{\omega}_{\mathbf{e}_2,\delta}u\right\|_{L^\infty\left(\Omega^{-\frac{1}{3},\frac{1}{3}}_{-\frac{1}{3},\frac{1}{3}}\right)}\leq C,\qquad\text{with a constant } C \text{ independent of } 0<\delta<\frac{1}{10}.$$

To prove this we will apply the assumed *Maximum Principle Property in* Ω to the function $D^{\omega}_{\mathbf{e}_2,\delta}u(x,y)$, and for this in turn we will need the following lemma.

Lemma 4. *Suppose that $u\in W^{1,2}_f\left((-1,1)^2\right)\cap C^\infty\left((-1,1)^2\setminus\ y\text{-axis}\right)$ satisfies $\nabla A^{\mathrm{tr}}\nabla u=\phi$, where ϕ is f-admissible and satisfies (3.9), and $A(x)\approx D_f(x)$. Let $0<\delta<\frac{1}{10}$, and choose $c_\delta,c_\delta+\delta,d_\delta,d_\delta+\delta$ as in (3.12) and choose $a=-\frac{3}{4}$ and $b=\frac{3}{4}$. Then with $\Omega^{c_\delta,d_\delta}_{-\frac{3}{4},\frac{3}{4}}=\left(-\frac{3}{4},\frac{3}{4}\right)\times(c_\delta,d_\delta)$, we have that $v_\delta\equiv D^{\omega}_{\mathbf{e}_2,\delta}u$ is bounded in the weak sense on $\partial\Omega^{c_\delta,d_\delta}_{a,b}$, i.e. there is a constant ℓ such that $(v_\delta-\ell)^+\in\left(W^{1,2}_f\left(\Omega^{c_\delta,d_\delta}_{a,b}\right)\right)_0$.*

Proof. We fix δ and write $v=v_\delta$. From (3.15), we have for $z\in\{c_\delta,d_\delta\}$ that

$$\|\varphi_\varepsilon*v(\cdot,z)\|_{L^\infty((a,b))}\leq C_0,$$

and from (3.10) and ellipticity away from the y-axis, we have for $t\in\{-\frac{3}{4},\frac{3}{4}\}$ that

$$\begin{aligned}\|\varphi_\varepsilon*v(t,y)\|_{L^\infty((c,d))}&\leq C'\left(\|v\|_{L^2\left(B_{\mathrm{Euc}}\left((\pm\frac{3}{4},y),\frac{1}{10}\right)\right)}+\left\|D^{\omega}_{\mathbf{e}_2,\delta}\phi\right\|_{L^q(B_{\mathrm{Euc}})}\right)\\&\leq C'\left(\|u\|_{W^{1,2}_f(\Omega)}+\|\phi\|_{\mathrm{Lip}_\omega}+\|\phi\|_{f-\mathrm{adm}(\Omega)}+C_R\right)\\&\leq C'\left(\|u\|_{W^{1,2}_f(\Omega)}+\|\phi\|_{\mathrm{Lip}_\omega}+C_R\right),\end{aligned}$$

since (3.17) shows that

$$\begin{aligned}\|v\|_{L^2\left(B_{\mathrm{Euc}}\left((\pm\frac{3}{4},y),\frac{1}{10}\right)\right)}&\leq\left\|D^{\omega}_{\mathbf{e}_2}u\right\|_{L^\infty\left(B_{\mathrm{Euc}}\left((\pm\frac{3}{4},y),\frac{1}{10}\right)\right)}\\&\leq C'\frac{\delta^{\alpha\left(\pm\frac{3}{4}\right)}}{\omega\left(\pm\frac{3}{4}\right)}\left(\|u\|_{W^{1,2}_f(\Omega)}+\|\phi\|_{f-\mathrm{adm}(\Omega)}\right).\end{aligned}$$

Define

$$\ell\equiv 2\max\left\{C_0,\|u\|_{W^{1,2}_f(\Omega)}+\|\phi\|_{\mathrm{Lip}_\omega}\right\}.$$

Since $\varphi_\varepsilon*v(x,y)$ is a smooth function in Ω provided $2\varepsilon<\min\{a+1,1-b,c_\delta+1,1-d_\delta\}$, the above inequalities imply

$$\left|\varphi_\varepsilon*v(x,y)\right|\big|_{\partial\Omega^{c_\delta,d_\delta}_{a,b}}\leq\frac{1}{2}\ell\ .$$

This gives

$$\left(\varphi_\varepsilon*v(x,y)-\frac{1}{2}\ell\right)_+=0\quad\text{on}\quad\partial\Omega^{c_\delta,d_\delta}_{a,b},$$

and by continuity,

$$\operatorname{supp}\left(\varphi_{\varepsilon}*v\left(x,y\right)-\ell\right)_{+}\Subset\Omega_{a,b}^{c_{\delta},d_{\delta}}.$$

Thus we have

$$\left(\varphi_{\varepsilon}*v\left(x,y\right)-\ell\right)_{+}\in\left(W_{f}^{1,2}\right)_{0}\left(\Omega_{a,b}^{c_{\delta},d_{\delta}}\right),$$

and it remains to show that

$$\left(\varphi_{\varepsilon}*v-\ell\right)_{+}\rightarrow\left(v-\ell\right)_{+} \tag{3.20}$$

in the norm of $W_{f}^{1,2}\left(\Omega\right)$ as $\varepsilon\rightarrow0$. Indeed, since $\left(W_{f}^{1,2}\left(\Omega\right)\right)_{0}$ is closed in $W_{f}^{1,2}\left(\Omega\right)$, we would then conclude that $u_{\varepsilon}^{+}=\left(v-l\right)^{+}\in\left(W_{f}^{1,2}\left(\Omega\right)\right)_{0}$ as required. So it remains to prove (3.20), and since $\varphi_{\varepsilon}*v\left(x,y\right)-\ell=\varphi_{\varepsilon}*\left(v\left(x,y\right)-\ell\right)$, we may assume without loss of generality that $\ell=0$. Lemma 2 now completes the proof of Lemma 4. ∎

With Lemma 4 in hand, we can now apply the assumed $\mathcal{MPP}$ for the equation $\mathcal{L}v=D_{\mathbf{e}_{2},\delta}^{\omega}\phi$ in Ω to conclude that $D_{\mathbf{e}_{2},\delta}^{\omega}u\in L^{\infty}\left(\Omega_{-\frac{1}{3},\frac{1}{3}}^{-\frac{1}{3},\frac{1}{3}}\right)$ uniformly in $0<\delta<\frac{1}{10}$.

3.2.5. *Step eight.* In order to complete the proof of Conclusion (2) of the Trace Method Theorem, it remains to show that

$$u\in Lip_{\omega}\left(\Omega_{-\frac{1}{3},\frac{1}{3}}^{-\frac{1}{3},\frac{1}{3}}\right) \tag{3.21}$$

for the modulus of continuity $\omega\left(\delta\right)$ in Steps six and seven.

For this, suppose we are given points $P=\left(x,y\right)$ and $P+\left(\delta_{1},\delta_{2}\right)=\left(x+\delta_{1},y+\delta_{2}\right)$, both near the origin, and set

$$\delta\equiv\max\left\{\sqrt{\delta_{1}},\delta_{2}\right\}.$$

Then choose a 'δ-good' point z near y, i.e. such that

$$\left|z-y\right|\leq\delta\qquad\text{and}\qquad\int_{0}^{1}\left(\left|u\left(t,z\right)\right|^{2}+\left|\frac{\partial u}{\partial x}\left(t,z\right)\right|^{2}\right)dt\leq\frac{C^{2}}{\delta}. \tag{3.22}$$

Indeed, this is possible since if we take $\lambda=\frac{C}{\sqrt{\delta}}$ with $C=\frac{1}{\sqrt{10}\left\Vert u\right\Vert_{W_{f}^{1,2}}}$ in the weak type estimate in Lemma 1, we obtain

$$\begin{aligned}
&\left|\left\{y\in\left(c,d\right):M_{2}\sqrt{\left|u\right|^{2}+\left|\frac{\partial u}{\partial x}\right|^{2}}\left(y\right)>\frac{C}{\sqrt{\delta}}\right\}\right|\\
\leq\ &\frac{5\left(\left\Vert u\right\Vert_{L_{\mathcal{H}}^{2}\left(\left(c,d\right)\right)}^{2}+\left\Vert\frac{\partial u}{\partial x}\right\Vert_{L_{\mathcal{H}}^{2}\left(\left(c,d\right)\right)}^{2}\right)}{C^{2}}\delta=\frac{\delta}{2}\ ,\qquad0<\delta<\frac{1}{10},
\end{aligned}$$

and hence conclude that there is $z\in\left(y,y+\delta\right)$ with $M_{2}\sqrt{\left|u\right|^{2}+\left|\frac{\partial u}{\partial x}\right|^{2}}\left(z\right)\leq\frac{C}{\sqrt{\delta}}$.

Now we apply (3.2) in Step one with $\frac{1}{4}<\gamma<\frac{1}{2}$ to obtain the inequality

$$\left|u\left(x+\delta_{1},z\right)-u\left(x,z\right)\right|\leq C_{\gamma}\delta_{1}^{\gamma}\frac{C}{\sqrt{\delta}}\leq C_{\gamma}\delta^{2\gamma-\frac{1}{2}}.$$

Altogether then we have using Step seven that

$$\begin{aligned}
&\left|u\left(x+\delta_{1},y+\delta_{2}\right)-u\left(x,y\right)\right|\\
\leq\ &\left|u\left(x+\delta_{1},y+\delta_{2}\right)-u\left(x+\delta_{1},z\right)\right|+\left|u\left(x+\delta_{1},z\right)-u\left(x,z\right)\right|+\left|u\left(x,z\right)-u\left(x,y\right)\right|\\
\leq\ &\omega\left(\left|y+\delta_{2}-z\right|\right)+C_{\gamma}\delta^{2\gamma-\frac{1}{2}}+\omega\left(\left|y-z\right|\right)\leq2\omega\left(\delta\right)+C_{\gamma}\delta^{2\gamma-\frac{1}{2}},
\end{aligned}$$

which completes the proof of Step eight since for $\frac{1}{4}<\gamma<\frac{1}{2}$, we have $2\omega\left(\delta\right)+C_{\gamma}\delta^{2\gamma-\frac{1}{2}}\leq C'\omega\left(\delta\right)$ for $\delta>0$ sufficiently small. The proof of the Trace Method Theorem 3 is now complete.

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